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Inverse Scattering in 60 Minutes

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Abstract

This lecture reports on joint work with Robert Jenkins, Jiaqi Liu, and Catherine Sulem. We illustrate the strengths of the inverse scattering method for addressing large-time behavior of completely integrable dispersive PDE's by proving global well-posedness and determining large-time asymptotic behavior for the Derivative Nonlinear Schrödinger equation (DNLS) for soliton-free initial data. Our work uses techniques from the work of Deift and Zhou on the defocussing NLS together with further developments due to Dieng and McLaughlin.

1. Introduction and Results

It is a remarkable fact that a number of canonical dispersive PDE's that describe nonlinear waves possess the property of *complete integrability*: that is, there is an invertible nonlinear mapping that linearizes the flow defined by the PDE. This transformation can be calculated with sufficient precision to compute large-time asymptotic behavior of solutions to the PDE. Let us begin by recalling some well-known examples:

Example 1.1. (the Korteweg-de Vries equation [19]) The KdV equation

$$q_t - 6qq_x + q_{xxx} = 0$$

gives the amplitude $q(x,t)$ of long waves propagating in a shallow channel. Gardner, Greene, Kruskal and Miura [11] showed that this flow may be integrated using the scattering theory of the Schrödinger operator

$$L(t) = -\frac{d^2}{dx^2} + q(x,t).$$

Fix t and write $q(x)$ for $q(x,t)$. For simplicity, we assume that the Schrödinger operator has no L^2 eigenvalues. The *Jost solutions* $f_\pm(x,k)$ are the unique solutions of the problem $-f'' + q(x)f(x) = k^2 f(x)$ with respective asymptotics

$$f^+(x,k) \sim e^{ikx}, \quad x \to +\infty$$

and

$$f^-(x,k) \sim e^{-ikx}, \quad x \to -\infty$$

where we write $f \sim g, \quad x \to \pm\infty$ if

$$\lim_{x\to\pm\infty} \left(|f(x) - g(x)| + |f'(x) - g'(x)|\right) = 0$$

with a single choice of sign throughout. The scattering data $a(k)$ and $b(k)$ are defined via the relations

$$f_+(x,k) = a(k)f_-(x,-k) + b(k)f_-(x,k)$$

This paper reports on joint work with Robert Jenkins (University of Arizona), Jiaqi Liu (University of Kentucky) and Catherine Sulem (University of Toronto).

and the *reflection coefficient* is defined by $r(k) = b(k)/a(k)$. If $q = q(x,t)$ solves KdV then

$$\dot{a}(k,t) = 0, \quad \dot{b}(k,t) = 8ik^3 b(k,t)$$

so that the reflection coefficient obeys

$$\dot{r}(k,t) = 8ik^3 r(k,t).$$

On the other hand, the scattering solutions, and hence the potential q, may be reconstructed from r using the Gelfand-Levitan-Marcenko [12] or the Riemann-Hilbert method, giving an inverse to the scattering transform.

Example 1.2. (the defocussing NLS equation [31, 30]) The defocussing nonlinear Schrödinger equation

$$iq_t + q_{xx} - 2|q|^2 q = 0$$

arises in the study of weakly nonlinear waves. Zakharov and Shabat [31] showed that this equation may be integrated using the spectral theory of the ZS-AKNS operator [1, 31]

$$L(t) = -\frac{i}{2}\sigma\frac{d}{dx} + \frac{i}{2}\sigma\begin{pmatrix} 0 & q(x,t) \\ \overline{q(x,t)} & 0 \end{pmatrix}$$

where

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The operator L operates on 2×2 matrix-valued functions and is formally self-adjoint. For each $\lambda \in \mathbb{R}$, there exist unique solutions $\Psi^{\pm}$, of $L\Psi = \lambda\Psi$ with $\lim_{x\to\pm\infty} \Psi^{\pm} e^{-ix\lambda\sigma} = I$, where I denotes the identity matrix. It is easy to see that there is a matrix $T(\lambda)$ of determinant 1 so that $\Psi^+(x,\lambda) = \Psi^-(x,\lambda)T(\lambda)$. By the symmetries of the equation $L\Psi = \lambda\Psi$ we have

$$T(\lambda) = \begin{pmatrix} a(\lambda) & \overline{b(\lambda)} \\ b(\lambda) & \overline{a(\lambda)} \end{pmatrix}.$$

If $q = q(x,t)$ solves the NLS, one can show that

$$\dot{a}(\lambda,t) = 0, \quad \dot{b}(\lambda,t) = -i\lambda^2 b(\lambda,t)$$

The reflection coefficient

$$r(\lambda) = -\overline{b(\lambda)}/\overline{a(\lambda)}$$

obeys

$$\dot{r}(\lambda,t) = -i\lambda^2 r(\lambda,t).$$

On the other hand, the potential $q(x)$ may be reconstructed from the reflection coefficient ([31]; see [8] for a self-contained treatment by the Riemann-Hilbert method).

Example 1.3. (the Derivative Nonlinear Schrödinger Equation [17]) The derivative nonlinear Schrödinger equation (DNLS)

$$iu_t + u_{xx} - i\left(|u|^2 u\right)_x = 0 \tag{1.1}$$

describes the propagation of Alfvén wave in plasmas. It is convenient to make an (invertible) gauge transformation

$$q(x,t) = u(x,t)\exp\left(i\int_{-\infty}^{x} |u(x,t)|^2\,dy\right) \tag{1.2}$$

and instead study the equation

$$iq_t + q_{xx} + i(|q|^2\overline{q}_x) + \frac{1}{2}|q|^4 q = 0. \tag{1.3}$$

Kaup and Newell [17] showed that the (1.3) can be integrated by considering the 'spectral' problem (technically a quadratic operator pencil rather than a linear spectral problem)

$$\frac{d}{dx}\Psi = -i\zeta^2\sigma\Psi + \zeta Q(x,t)\Psi + P(x,t)\Psi \tag{1.4}$$

where

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ \overline{q(x,t)} & 0 \end{pmatrix}, \quad P(x,t) = -\frac{i}{2}\begin{pmatrix} |q(x,t)|^2 & 0 \\ 0 & -|q(x,t)|^2 \end{pmatrix}.$$

Here Ψ is a 2×2 matrix-valued function and ζ, the spectral parameter, lives on the region $\operatorname{Im}(\zeta^2) = 0$. If we introduce Jost solutions $\Psi^\pm$ with

$$\lim_{x\to\pm\infty} \Psi^\pm e^{ix\zeta^2\sigma} = I$$

then one has the relation

$$\Psi^+(x,\zeta) = \Psi^-(x,\zeta) \begin{pmatrix} a(\zeta) & \overline{b(\zeta)} \\ b(\zeta) & \overline{a(\zeta)} \end{pmatrix}$$

If $q(x,t)$ solves (1.3) and if $a(\zeta,t)$ and $b(\zeta,t)$ are the associated scattering data, then

$$\dot{a}(\zeta,t) = 0, \quad \dot{b}(\zeta,t) = 4i\zeta^4 b(\zeta,t).$$

It turns out the function $\zeta^{-1}\overline{b(\zeta)}/a(\zeta)$ is even, so that we may define a function ρ on $\mathbb{R}$ by $\rho(\zeta^2) = \zeta^{-1}\overline{b(\zeta)}/a(\zeta)$. The direct and inverse scattering problems are most usefully formulated in terms of this function ρ on the real line. Note that ρ evolves in time according to

$$\dot{\rho}(\lambda,t) = -4i\lambda^2\rho(\lambda,t).$$

Lee [20] studied the inverse spectral problem for Schwartz class potentials q with "generic" spectral properties and proved global well-posedness for the DNLS equation with generic Schwartz class initial data [21].

In each of these cases, there is a 'direct scattering map' $\mathcal{R}$ from $q(\,\cdot\,,t)$ to a reflection coefficient $r(\,\cdot\,,t)$ (the scattering data) which linearizes the flow in the sense that, if $q(x,t)$ solves the given dispersive PDE, $r(k,t) = e^{it\phi(k)}r(k,0)$ for a polynomial phase function ϕ. There is also an *inverse scattering map* $\mathcal{I}$ from $r(\,\cdot\,)$ to $q(\,\cdot\,)$ (for any fixed t) that recovers the scattering data.

Both the direct and inverse scattering maps are defined via *scattering solutions* to the linear PDE. These scattering solutions may be computed either directly from q or recovered from the scattering data r. In the case of the NLS and the DNLS, a key role is played by the *Beals-Coifman solutions* which, on the one hand, solve the direct scattering problem, and, on the other, may be recovered from the scattering data through a Riemann-Hilbert problem.

In order to use the inverse scattering method rigorously, one needs to show that the maps $\mathcal{R}$ and $\mathcal{I}$ are well-defined, locally Lipschitz continuous maps from a 'good' function space X to itself. In the case of Examples 1.2 and 1.3, the linearizations of the maps $\mathcal{R}$ and $\mathcal{I}$ at 0 are respectively the Fourier transform $\mathcal{F}$, normalized as

$$(\mathcal{F}f)(\lambda) = \int e^{-2i\lambda x} f(x)\,dx$$

and its inverse

$$\left(\mathcal{F}^{-1}f\right)(x) = \frac{1}{\pi}\int e^{2i\lambda x} f(x)\,dx.$$

These linearizations suggest that a 'good' space X should have the following properties: (1) The Fourier transform $\mathcal{F}$ should be a continuous map from X to itself and (2) the map $r \mapsto e^{-ik^2 t}r$ (NLS) (resp. $r \mapsto e^{4i\zeta^4 t}r$ (DNLS)) should be continuous. Given a good continuity result, one then recovers global well-posedness on X via the solution formula

$$q(x,t) = \mathcal{I}\left(e^{it\varphi(\diamond)}\left(\mathcal{R}q_0\right)(\diamond)\right)(x) \tag{1.5}$$

where $\varphi(k) = -ik^2$ (NLS) (resp. $\varphi(\lambda) = -4i\lambda^2$ (DNLS)) is the appropriate polynomial phase function.

The inverse map is formulated as a Riemann-Hilbert problem in which x and t enter as parameters. Deift and Zhou [6] pioneered the technique of 'nonlinear steepest descent' for extracting large-t asymptotic behavior of solutions to completely integrable equations through transformations of the Riemann-Hilbert problem. Applying these methods to Example 1.3, we will obtain the following results. To state them, let

$$H^{2,2}(\mathbb{R}) = \left\{ f \in L^2 : x^2 f, f'' \in L^2 \right\}.$$

Theorem 1.4. *There is a spectrally determined open set U in $H^{2,2}(\mathbb{R})$ containing 0 so that the map $\mathcal{R} : q \to \rho$ is locally Lipschitz continuous from U into $H^{2,2}(\mathbb{R})$. Moreover, the inverse $\mathcal{I} : \rho \to q$ is locally Lipschitz continuous on $\mathcal{R}(U)$.*

We give a more precise characterization of this open set in Assumption 3.1 of what follows.

With these continuity results, we can appeal to the solution formula (1.5) and prove:

Corollary 1.5. *There is a spectrally determined open set U in $H^{2,2}$ so that the initial value problem for* (1.3) *is globally-well posed for initial data in U.*

Remark 1.6. Hayashi [13] proved global existence for solutions of (1.1) assuming $u(x,0) \in H^1(\mathbb{R})$ and $\|u(\,\cdot\,,0)\|_{L^2}$ is small. Simultaneously and independently of our work, Pelinovsky and Shimabukuro [25] used inverse scattering methods to prove a global well-posedness result very similar to ours, but with slightly less restrictive spaces. To see that there are large L^2 data, one can use ideas of Tovbis and Venakides [26] to construct explicit functions q_0 which have large L^2 norm and no soliton singularities. We solve the spectral problem for DNLS with a potential q of the form

$$q(x) = A(x)e^{iS(x)}$$

where $A(x) = \nu \operatorname{sech} x$ for a parameter ν and S is a phase function chosen so that the change of variables $\Psi = e^{iS(x)\sigma/2}\varphi$ in (1.4) yields an explicitly solvable equation for the Jost solutions. By direct computation one can show that, for suitable S and any $\nu > 0$, q has scattering data with no spectral or soliton singularities. As $\|q\|_2 = \sqrt{2}\nu$ this shows that there exist spectral data of arbitrary L^2 norm which obey the hypotheses of Corollary 1.5. This calculation will appear in [16].

A careful asymptotic analysis of the inverse map allows one to prove the following asymptotic formulas for the solution of (1.1). In what follows we denote by $\Gamma(\,\cdot\,)$ the usual Gamma function. We set

$$\kappa(\lambda) = -\frac{1}{2\pi}\log\left(1 - \lambda|\rho(\lambda)|^2\right) \tag{1.6}$$

and

$$\theta(z;x,t) = -\left(z\frac{x}{t} + 2z^2\right). \tag{1.7}$$

We define $\alpha_1(\xi)$ and $\alpha_2(\xi)$ as follows. First, the amplitudes are given by

$$|\alpha_1(\xi)|^2 = |\alpha_2(\xi)|^2 = \frac{\kappa(\xi)}{2\xi}.$$

The phases are given by

$$\begin{aligned}\arg\alpha_1(\xi) &= \frac{\pi}{4} + \arg\Gamma(i\kappa(\xi)) + \arg\rho(\xi)\\ &\quad + \frac{1}{\pi}\int_{-\infty}^{\xi}\log|s-\xi|\,d\log\left(1 - s|\rho(s)|^2\right),\\ \arg\alpha_2(\xi) &= \arg\alpha_1(\xi) - \pi\end{aligned}$$

while for $t < 0$,

$$\begin{aligned}\arg\alpha_1(\xi) &= -\frac{\pi}{4} - \arg\Gamma(i\kappa(\xi)) + \arg\rho(\xi)\\ &\quad + \frac{1}{\pi}\int_{\xi}^{\infty}\log|s-\xi|\,d\log\left(1 - s|\rho(s)|^2\right),\\ \arg\alpha_2(\xi) &= \arg\alpha_1(\xi) + \pi.\end{aligned}$$

We can now give sharp formulas for the asymptotic behavior of a solution $u(x,t)$ of (1.1) that does not support solitons.

Theorem 1.7. *Suppose that $u_0 \in H^{2,2}(\mathbb{R})$ and let*

$$q_0(x) = u_0(x)\exp\left(-i\int_{-\infty}^{x}|u_0(y)|^2\,dy\right).$$

Let ρ be the reflection coefficient associated to q_0 by the direct scattering map and κ defined by (1.6)*. Assume also that $c = \inf_{z\in\mathbb{R}}\left(1 - z|\rho(z)|^2\right) > 0$. Denote by $\xi = -x/4t$ the stationary phase point of the phase function* (1.7) *and fix $\xi \neq 0$. Then:*

(i) *As $t \to +\infty$,*

$$u(x,t) \sim \begin{cases} \dfrac{1}{\sqrt{t}}\alpha_3(\xi)e^{-i\kappa(\xi)\log(8t)+ix^2/(4t)} + \mathcal{O}_\xi\left(t^{-3/4}\right), & x>0 \\ \\ \dfrac{1}{\sqrt{t}}\alpha_4(\xi)e^{-i\kappa(\xi)\log(8t)+ix^2/(4t)} + \mathcal{O}_\xi\left(t^{-3/4}\right), & x<0 \end{cases} \tag{1.8}$$

(ii) *As $t \to -\infty$,*

$$u(x,t) \sim \begin{cases} \dfrac{1}{\sqrt{-t}}\alpha_4(\xi)e^{i\kappa(\xi)\log(-8t)+ix^2/(4t)} + \mathcal{O}_\xi\left((-t)^{-3/4}\right) & x>0 \\ \\ \dfrac{1}{\sqrt{-t}}\alpha_3(\xi)e^{i\kappa(\xi)\log(-8t)+ix^2/(4t)} + \mathcal{O}_\xi\left((-t)^{-3/4}\right) & x<0 \end{cases} \tag{1.9}$$

Here,

$$|\alpha_3(\xi)|^2 = |\alpha_4(\xi)|^2 = \frac{\kappa(\xi)}{2\xi} \tag{1.10}$$

For $t>0$,

$$\arg\alpha_3(\xi) = \arg\alpha_1(\xi) - \frac{1}{\pi}\int_\xi^\infty \frac{\log(1-s|\rho(s)|^2)}{s}\,ds \tag{1.11}$$

$$\arg\alpha_4(\xi) = \arg\alpha_2(\xi) - \frac{1}{\pi}\int_\xi^\infty \frac{\log(1-s|\rho(s)|^2)}{s}\,ds. \tag{1.12}$$

while for $t<0$,

$$\arg\alpha_3(\xi) = \arg\alpha_1(\xi) - \frac{1}{\pi}\int_{-\infty}^\xi \frac{\log(1-s|\rho(s)|^2)}{s}\,ds \tag{1.13}$$

$$\arg\alpha_4(\xi) = \arg\alpha_2(\xi) - \frac{1}{\pi}\int_{-\infty}^\xi \frac{\log(1-s|\rho(s)|^2)}{s}\,ds. \tag{1.14}$$

Remark 1.8. For the linear Schrödinger equation $iu_t + u_{xx} = 0$, the analogous result is

$$u(x,t) \sim \frac{1}{\sqrt{2\pi t}}\widehat{u_0}(\xi)e^{ix^2/(4t)} + \mathcal{O}\left(t^{-1}\right).$$

The effect of the nonlinearity is encoded in the logarithmic phase shift and shows that solutions of DNLS do not approach solutions of the free linear Schrödinger equation asymptotically. A similar logarithmic phase shift was computed for the NLS equation by Zakharov-Manakov [30] (without an error estimate) and derived, with error estimates, by Deift-Its-Zhou [5]. A self-contained analysis of NLS asymptotics may be found in Deift-Zhou's classic paper [8], and a more recent approach to the steepest descent method which we follow in our work may be found in the paper of Dieng and McLaughlin [9].

Remark 1.9. Kitaev and Vartanian [18] obtained these asymptotic formulae assuming "small data" in the Schwarz class. Xu and Fan [29] also obtained asymptotic formulae with similar assumptions. Although we do make spectral assumptions on the initial data, we do not need a small data condition; the examples discussed in Remark 1.6 show that there exist functions q with large L^2 norm which satisfy our spectral assumptions.

In what follows, we will first discuss the formal direct and inverse scattering theory for DNLS, describe the estimates needed to prove Theorem 1.4 and Corollary 1.5, and then describe the steepest descent method used to prove Theorem 1.7.

2. DNLS: Formal Theory

We now consider the formal inverse scattering theory for (1.3). Kaup and Newell [17] discovered that the linear spectral problem (1.4) defines action-angle variables (1.3). These action-angle variables are determined by scattering data for (1.4) that we now describe.

First, observe that for $P = Q = 0$, the spectral problem (1.4) has solutions $\exp(-ix\zeta^2\sigma)$. These solutions are bounded provided $\zeta \in \Sigma$ where

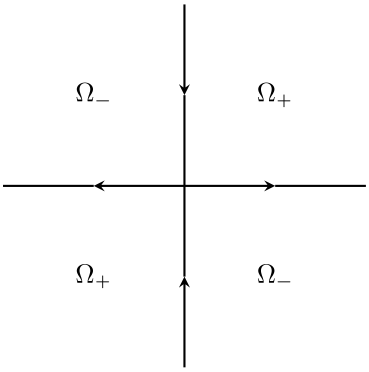
$$\Sigma := \left\{ \mathrm{Im}(\zeta^2) = 0 \right\}.$$

Note that Σ divides $\mathbb{C}$ into regions

$$\Omega_\pm = \left\{ \zeta \in \mathbb{C} : \pm \mathrm{Im}(\zeta^2) > 0 \right\}$$

as shown below.

Figure 2.1: The Oriented Contour Σ



It will be important to note, for later use, that the quadratic map $\zeta \mapsto \zeta^2$ is an orientation-preserving map from Σ to $\mathbb{R}$ with its usual orientation.

Since σ, P and Q are traceless, $\det \Psi$ is independent of x for any solution $\Psi(x,\zeta)$ of (1.4). Moreover, if Ψ_1 and Ψ_2 are nonsingular solutions, it is not difficult to see that $\Psi_1 = \Psi_2 A$ for a constant matrix A These observations lead to the definition of scattering data through the Jost solutions.

2.1. Jost Solutions

If $q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\Im(\zeta^2) = 0$, there exist unique, bounded solutions $\Psi^\pm$ with $\Psi^\pm(x,\zeta) \sim \exp(-ix\zeta^2\sigma)$ as $x \to \pm\infty$. Moreover

$$\Psi^+(x,\zeta) = \Psi^-(x,\zeta) \begin{pmatrix} a(\zeta) & \overline{b(\overline{\zeta})} \\ b(\zeta) & \overline{a(\overline{\zeta})} \end{pmatrix}$$

The data a and b are *action-angle* variables for the flow: if

$$q(x,t) \mapsto (a(\zeta,t), b(\zeta,t))$$

then

$$\dot{a}(\zeta,t) = 0, \quad \dot{b}(\zeta,t) = -4i\zeta^4 b(\zeta,t) \tag{2.1}$$

2.2. Beals-Coifman Solutions

In his 1983 thesis [20], Lee showed how to invert the map $q \mapsto (a,b)$ by introducing the *Beals-Coifman (BC) solutions* which solve a Riemann-Hilbert problem determined by (a,b).

To define the BC solutions, set $\Psi^\pm(x,\zeta) = m^\pm(x,\zeta)e^{-ix\zeta^2\sigma}$ then

$$\frac{d}{dx}m^\pm = -i\zeta^2 \,\mathrm{ad}(\sigma)m^\pm + \zeta Q(x)m^\pm + P(x)m^\pm \tag{2.2a}$$

$$\lim_{x\to\pm\infty} m^\pm(x,\zeta) = I \tag{2.2b}$$

where I is the identity matrix and

$$\operatorname{ad}(\sigma)A = \sigma A - A\sigma = \begin{pmatrix} 0 & 2a_{12} \\ -2a_{21} & 0 \end{pmatrix}$$

Write $m^{\pm}_{(1)}$, $m^{\pm}_{(2)}$ for the first and second columns of $m^{\pm}$. For $q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, one can show:

(i) $m^{+}_{(1)}$ and $m^{-}_{(2)}$ have analytic extensions to $\Omega^- := \Im(\zeta^2) < 0$

(ii) $m^{-}_{(1)}$ and $m^{+}_{(2)}$ have analytic extensions to $\Omega^+ := \Im(\zeta^2) > 0$

(iii) $a(\zeta)$ has an analytic extension to Ω^-

We now define the Beals-Coifman (BC) solution $M(x,\zeta)$.

$$M(x,\zeta) = \begin{cases} \left(\dfrac{m^{-}_{(1)}(x,\zeta)}{\overline{a(\overline{\zeta})}}, m^{+}_{(2)}(x,\zeta) \right), & \zeta \in \Omega^+ \\ \\ \left(m^{+}_{(1)}(x,\zeta), \dfrac{m^{-}_{(2)}(x,\zeta)}{a(\zeta)} \right), & \zeta \in \Omega^- \end{cases} \tag{2.3}$$

Lemma 2.1. *The Beals-Coifman solution* (2.3) *has the following properties:*

(i) $M(x,z)$ *solves* (2.2a)*,*

(ii) $M(x,z)$ *is analytic away from* Σ*,*

(iii) $M(x,z)$ *satisfies* $\lim_{x\to+\infty} M(x,\zeta) = I$ *if* $\Im(\zeta^2) \neq 0$

(iv) $M(x,z)$ *is bounded as* $x \to -\infty$

Morever, $M(x,z)$ *is the unique such function with these properties.*

Proof. We'll prove the uniqueness part of the lemma. Suppose that M and $M^\sharp$ are two solutions. From properties of the solution space to (2.2a) we may conclude that

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} M^\sharp_{11} & M^\sharp_{12} \\ M^\sharp_{21} & M^\sharp_{22} \end{pmatrix} \begin{pmatrix} A_{11} & e^{-2ix\zeta^2} A_{12} \\ e^{2ix\zeta^2} A_{21} & A_{22} \end{pmatrix}$$

for A_{ij} depending only on ζ. Hence, fixing $\Im(\zeta^2) > 0$ say we deduce the following. First, $A_{12} = 0$ by considering the limit as $x \to +\infty$. Second, $A_{21} = 0$ by considering the limit as $x \to -\infty$. Finally, $A_{11} = A_{22} = 1$ by the normalization as $x \to +\infty$. This shows that $M = M^\sharp$ as claimed. ☐

The BC solutions also satisfy the following *Riemann-Hilbert Problem* (RHP):

Problem 2.2. Given scattering data (a,b) and $x \in \mathbb{R}$, find $M(x,\zeta)$ satisfying:

(i) $M(x,z)$ is analytic in Ω_+ and Ω_- with boundary values $M_\pm(x,\zeta)$ as $z \to \zeta \in \Sigma$ from $\Omega^\pm$.

(ii) $M(x,\zeta) = 1 + \mathcal{O}\left(\zeta^{-1}\right)$ as $\zeta \to \infty$

(iii) $M_+(x,\zeta) = M_-(x,\zeta) e^{-ix\zeta^2 \operatorname{ad}(\sigma)} V(\zeta)$ where

$$V(\zeta) = \begin{pmatrix} 1 - b(\zeta)\overline{b(\overline{\zeta})}/a(\zeta)\overline{a(\overline{\zeta})} & \overline{b(\overline{\zeta})}/a(\zeta) \\ \\ -b(\zeta)/\overline{a(\overline{\zeta})} & 1 \end{pmatrix}$$

This problem also has a unique solution. One can show by direct differentiation of the jump relation with respect to x that the solution of the RHP satisfies (2.2a). It is also easy to obtain q from the large-z asymptotics of M. Substituting the asymptotic expansion

$$M(x,\zeta) = 1 + \frac{1}{\zeta} M_1(x) + o\left(\zeta^{-1}\right)$$

into the differential equation (2.2a), we obtain

$$Q(x) = -i \operatorname{ad}(\sigma) M_1(x).$$

Taking into account the time evolution of the scattering data (see (2.1)), we get a new RHP

$$M_+(x,t,\zeta) = M_-(x,t,\zeta) e^{it\theta(x,t,\zeta) \operatorname{ad}(\sigma)} V(\zeta)$$

where

$$\theta(x,t,\zeta) = -\left(\zeta^2 \frac{x}{t} + 2\zeta^4\right)$$

so that

$$Q(x,t) = -i \operatorname{ad}(\sigma) M_1(x,t).$$

Thus, formally at least, we have derived a solution formula for (1.3) in terms of the scattering data associated to the initial data associated to the Cauchy problem.

3. Global Well-Posedness

To obtain global well-posedness we need to show that $q \mapsto (a,b)$ is continuous and that $(a,b) \mapsto q$ is continuous. In order to do so, we must make a spectral assumption.

Assumption 3.1 (Spectral Assumption).

$$a(\zeta) \text{ has no zeros in } \Omega^- \cup \Sigma$$

This assumption has an important physical meaning.

(i) $a(\zeta) \neq 0$ on Σ rules out "algebraic solitons," i.e., solutions of (1.3) which translate in time and have algebraic decay, and

(ii) $a(\zeta) \neq 0$ on Ω_- rules out "bright solitons," i.e., solutions of (1.3) which translate in time and have exponential decay.

Next, we will carry out a symmetry reduction based on the symmetries $a(-\zeta) = a(\zeta), \quad b(-\zeta) = -b(\zeta)$ and the observation that an even function f on $\Sigma = \{\Im \zeta^2 \neq 0\}$ induces a uniquely defined function $g(\zeta^2)$ on $\mathbb{R}$.

The function

$$\rho(\zeta^2) = \frac{1}{\xi} \frac{\overline{b(\overline{\zeta})}}{a(\zeta)}$$

defines a function on $\mathbb{R}$. Moreover, the function $\rho(\lambda)$ determines a and b uniquely. We will study the *direct scattering map* $q \mapsto \rho$ and the *inverse scattering map* $\rho \mapsto q$ for q in a spectrally determined open set containing 0.

Theorem 3.2. *There is a spectrally determined open subset U of $H^{2,2}(\mathbb{R})$ containing 0 so that:*

- *The map $\mathcal{R} : q \to \rho$ is locally Lipschitz continuous from U to $H^{2,2}(\mathbb{R})$.*
- *The map $\mathcal{I} : \rho \to q$ is locally Lipschitz continuous from $\mathcal{R}(U)$ to U.*

Thus $\mathcal{R}$ and $\mathcal{I}$ behave much like their linearizations at 0, which are the Fourier transform and the inverse Fourier transform. Global well-posedness follows from this result and the formula

$$q(x,t) = \mathcal{I}\left(e^{-4i(\diamond)^2 t} \mathcal{R}(q(\,\cdot\,,0))(\diamond)\right)(x)$$

Let us now describe the analysis required to prove Theorem 3.2.

3.1. Direct Scattering Map

We make the changes of variable $\lambda = \zeta^2$ and set

$$N(x,\zeta^2) = \begin{pmatrix} M_{11}(x,\zeta) & \zeta^{-1}M_{12}(x,\zeta) \\ \zeta M_{21}(x,\zeta) & M_{22}(x,\zeta) \end{pmatrix}.$$

We study normalized Jost solutions $N^{\pm}(x,\lambda)$ satisfying

$$N^{\pm}(x,\lambda) = I + \int_{\pm\infty}^{x} e^{-i\lambda(x-y)\,\mathrm{ad}(\sigma)} \left[\begin{pmatrix} 0 & q \\ \lambda\overline{q} & 0 \end{pmatrix} N + PN \right] (y,\lambda)\, dy$$

and define scattering data

$$N^{+}(x,\lambda) = N^{-}(x,\lambda) e^{-ix\lambda\,\mathrm{ad}(\sigma)} \begin{pmatrix} \alpha(\lambda) & \overline{\beta(\lambda)} \\ \lambda\beta(\lambda) & \overline{\alpha(\lambda)} \end{pmatrix}.$$

We recover α and β, and hence $\rho = \beta/\alpha$ from the Wronskian formulae

$$\alpha(\lambda) = \begin{vmatrix} N_{11}^{+}(0,\lambda) & N_{12}^{-}(0,\lambda) \\ N_{21}^{+}(0,\lambda) & N_{22}^{-}(0,\lambda) \end{vmatrix}, \quad \beta(\lambda) = \begin{vmatrix} N_{12}^{+}(0,\lambda) & N_{12}^{-}(0,\lambda) \\ N_{22}^{+}(0,\lambda) & N_{22}^{-}(0,\lambda) \end{vmatrix}.$$

One uses the Volterra integral equations for $N^{\pm}$ to show that the maps $q \mapsto N^{\pm}(0,\lambda)$ are Lipschitz continuous from $H^{2,2}(\mathbb{R})$ to suitable weighted Sobolev spaces. From this continuity one deduces that the map $q \mapsto \beta/\alpha$ is Lipschitz continuous from $H^{2,2}(\mathbb{R})$ to itself.

3.2. Inverse Scattering Map

Making the same change of variables from ζ to $\lambda = \zeta^2$ and from $M(x,\zeta)$ to $N(x,\lambda)$, we obtain the following RHP with contour $\mathbb{R}$ and reconstruction formula involving $\mathbf{N}(x,\lambda)$, the *first row* of $N(x,\lambda)$:

$$\mathbf{N}_{+}(x,\lambda) = \mathbf{N}_{-}(x,\lambda)(x,\lambda) e^{-i\lambda x\,\mathrm{ad}(\sigma)} \begin{pmatrix} 1-\lambda|\rho(\lambda)|^2 & \rho(\lambda) \\ -\lambda\overline{\rho(\lambda)} & 1 \end{pmatrix},$$

$$\mathbf{N}(x,z) \sim (1,0) + \mathcal{O}\left(\frac{1}{z}\right), \quad |z| \to \infty,$$

$$q(x) = \lim_{z\to\infty} -2iz\mathbf{N}_{12}(x,z).$$

Beals and Coifman showed how to reduce RHP's to a singular integral equation. We factor the jump matrix

$$\begin{pmatrix} 1-\lambda|\rho(\lambda)|^2 & \rho(\lambda) \\ -\lambda\overline{\rho(\lambda)} & 1 \end{pmatrix} = (I - w_{-}(\lambda))^{-1}(I + w_{+}(\lambda)),$$

$$w_{-}(\lambda) = \begin{pmatrix} 0 & \rho(\lambda) \\ 0 & 0 \end{pmatrix}, \quad w_{+}(\lambda) = \begin{pmatrix} 0 & 0 \\ -\lambda\overline{\rho(\lambda)} & 0 \end{pmatrix}$$

and set

$$\mu = \mathbf{N}_{+}(I + w_{+})^{-1} = \mathbf{N}_{-}(I - w_{-})^{-1}.$$

Then

$$\mathbf{N}_{+} - \mathbf{N}_{-} = \mu(w_{+} + w_{-})$$

so that $\mathbf{N}(x,z)$ is given by the Cauchy integral

$$\mathbf{N}(x,z) = (1,0) + \int_{-\infty}^{\infty} \frac{\mu(x,s)(w_{+}(s,x) + w_{-}(s,x))}{s-z} \frac{ds}{2\pi i}.$$

By taking limits as $\Im(z) \downarrow 0$ and using the Sokhotski-Plemelj Theorem, one obtains the *Beals-Coifman Integral Equation*

$$\mu = (1,0) + \mathcal{C}_w \mu$$

where

$$\mathcal{C}_w h = C_{-}(hw_{+}) + C_{+}(hw_{-})$$

and $C_{\pm}$ are the Cauchy projectors onto boundary values in L^2 of functions analytic in $\mathbb{C}^{+}$ and $\mathbb{C}^{-}$.

To recover q from ρ, solve the Fredholm-type integral equation

$$\mu = (1,0) + C_-(\mu w_+) + C_+(\mu w_-)$$

for $\mu - (1,0) \in L^2$. Note that

$$\|C_-(hw_+) + C_+(hw_-)\|_{L^2} \leqslant \|\rho\|_\infty \|h\|_{L^2}.$$

and use the reconstruction formula

$$\begin{aligned} q(x) &= -\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-2i\lambda x}\rho(\lambda)\mu_{11}(x,\lambda)\,d\lambda \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-2i\lambda x}\rho(\lambda)\left(1+[\mu_{11}-1]\right)(x,\lambda)\,d\lambda. \end{aligned}$$

The first term, an inverse Fourier transform, has the correct mapping properties from ρ to q. For the second term, we need estimates on $\mu_{11} - 1$ and its derivatives in x and λ.

Let us describe the estimates required to check that the inverse map takes $\rho \in H^{2,2}(\mathbb{R})$ to a $q \in H^{2,2}(\mathbb{R})$. Denote by $\breve{\rho}$ the inverse Fourier transform of ρ. Rewriting the reconstruction formula as

$$q(x) - \breve{\rho}(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-2i\lambda x}\rho(\lambda)\left[\mu_{11}-1\right](x,\lambda)\,d\lambda,$$

we see that we need estimates on $\|q\|_2$, $\|q''\|_2$, $\|x^2 q\|_2$. Calculations such as

$$\begin{aligned} q'(x) - \breve{\rho}'(x) &= -\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-2i\lambda x}(-i\lambda\rho(\lambda))\left[\mu_{11}-1\right]\,d\lambda \\ &\quad -\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-2i\lambda x}\frac{\partial\mu_{11}}{\partial x}\,d\lambda \\ xq(x) - x\breve{\rho}(x) &= \frac{2i}{\pi}\int_{-\infty}^{\infty} e^{-2i\lambda x}\rho'(\lambda)\left[\mu_{11}-1\right]\,d\lambda \\ &\quad -\frac{2i}{\pi}\int_{-\infty}^{\infty} e^{-2i\lambda x}\rho(\lambda)\frac{\partial\mu_{11}}{\partial\lambda}\,d\lambda \end{aligned}$$

show that we need $L^2_x L^2_\lambda$ estimates on $\mu_{11} - 1$, $(\mu_{11})_x$, $(\mu_{11})_\lambda$, ...

Let $\mu^\sharp = \mu - (1,0)$ so

$$\mu^\sharp = \mathcal{C}_w((1,0)) + \mathcal{C}_w(\mu^\sharp)$$

where

$$\mathcal{C}_w h = C^+(hw_-) + C^-(hw_+),$$

and

$$w_+ = \begin{pmatrix} 0 & \rho \\ 0 & 0 \end{pmatrix}, \quad w_+ = \begin{pmatrix} 0 & 0 \\ -\lambda\overline{\rho} & 0 \end{pmatrix}.$$

Proposition 3.3. *If $\rho \in H^{2,2}(\mathbb{R})$ and $\mu^\sharp = \mu - (1,0)$ then*

$$\mu^\sharp, \quad (\mu^\sharp)_x, \quad (\mu^\sharp)_{xx}, \quad (\mu^\sharp)_\lambda, \quad \langle\lambda\rangle^{-1}\mu^\sharp_{\lambda\lambda}$$

belong to $L^2((-a,\infty)\times\mathbb{R}, dx\,d\lambda)$ with bounds uniform for ρ in a bounded subset of $H^{2,2}(\mathbb{R})$.

Proof of Theorem 3.2. The continuity of direct map $\mathcal{R}$ and the inverse map $\mathcal{I}$ follow from the integral equations and estimates above together with the second resolvent formula

$$(I-A)^{-1} - (I-B)^{-1} = (I-A)^{-1}(A-B)(I-B)^{-1}$$

applied for the direct map to the Volterra integral equations that define scattering solutions, and for the inverse map to the Beals-Coifman integral equations.

The fact that $\mathcal{R}$ and $\mathcal{I}$ are mutual inverses follows from the uniqueness of the Beals-Coifman solutions. ☐

4. Large-Time Asymptotics

To compute the large-t asymptotics of the solution $q(x,t)$, we first note that the solution formula

$$q(x,t) = \mathcal{I}\left(e^{-4i(\diamond)^2 t}\mathcal{R}(q(\,\cdot\,,0))(\diamond)\right)(x)$$

translates to an RHP with contour $\mathbb{R}$:

$$\begin{aligned} M_+(x,t,\lambda) &= M_-(x,t,\lambda)\begin{pmatrix} 1-\lambda|\rho(\lambda)|^2 & \rho(\lambda)e^{2it\theta} \\ -\lambda\overline{\rho(\lambda)}e^{-2it\theta} & 1 \end{pmatrix} \\ \theta(x,t,\lambda) &= -\left(\lambda\frac{x}{t} + 2\lambda^2\right) \\ M(x,t,z) &\sim (1,0) + \mathcal{O}\left(\frac{1}{z}\right), \quad z\to\infty \text{ in } \mathbb{C}\backslash\mathbb{R} \\ q(x,t) &= \lim_{z\to\infty}(2iz)M_{12}(x,t,z) \end{aligned}$$

The key point is that the solution $M(x,t,z)$ depends on x and t only through the oscillatory phase θ.

4.1. Riemann-Hilbert Warm-Up

To appreciate what the analysis of such a Riemann-Hilbert problem entails, we consider an illuminating "toy model" problem that appears in the paper of Do [10]. In what follows it will be useful to recall:

Theorem 4.1. *There exists a unique solution to the problem*

(i) $F(z)$ *is analytic in* $\mathbb{C}\backslash\mathbb{R}$,

(ii) $\lim_{|z|\to\infty} F(z) = 1$,

(iii) $\lim_{\varepsilon\downarrow 0} F(\lambda \pm i\varepsilon) = f_\pm(\lambda) \in H^1(\mathbb{R})$

given by

$$F(z) = 1 + \int_{-\infty}^{\infty}\frac{1}{s-z}(f_+(s) - f_-(s))\frac{ds}{2\pi i}.$$

Do's model problem is the following:

Problem 4.2. Find a function $M(x,t,z)$ analytic for $z\in\mathbb{C}\backslash\mathbb{R}$ with

(i) $M(x,t,z)\to I$ as $|z|\to\infty$

(ii) $M_+(x,t,\lambda) = M_-(x,t,\lambda)\begin{pmatrix} 1 & p(\lambda)e^{it\theta} \\ 0 & 1\end{pmatrix}$

Writing out the jump relation component-by-component, we have

$$\begin{aligned} (M_+)_{11}(x,t,z) &= (M_-)_{11}(x,t,z) \\ (M_+)_{12}(x,t,z) &= p(z)e^{it\theta}(M_-)_{11}(x,t,z) + (M_-)_{12}(x,t,z) \\ (M_+)_{21}(x,t,z) &= (M_-)_{21}(x,t,z) \\ (M_+)_{22}(x,t,z) &= (M_-)_{22}(x,t,z) \end{aligned}$$

It is easy to see that $(M_\pm)_{11}\equiv 1$, $(M_\pm)_{21}\equiv 0$, and

$$M_{12}(z) = \int_{-\infty}^{\infty}\frac{p(s)e^{it\theta(x,t,s)}}{s-z}\frac{ds}{2\pi i}$$

This solution formula leads to the 'recovered potential'

$$\lim_{z\to\infty} zM_{12}(z) = \int_{-\infty}^{\infty} e^{it\theta(x,t,s)}p(s)\,ds$$

as would be expected for a linear PDE.

4.2. Nonlinear Steepest Descent

We now consider the full nonlinear problem

$$M_+(x,t,\lambda) = M_-(x,t,\lambda)\begin{pmatrix} 1-\lambda|\rho(\lambda)|^2 & \rho(\lambda)e^{2it\theta} \\ -\lambda\overline{\rho(\lambda)}e^{-2it\theta} & 1 \end{pmatrix} \tag{4.1a}$$

$$\theta(x,t,\lambda) = -\left(\lambda\frac{x}{t} + 2\lambda^2\right) \tag{4.1b}$$

$$M(x,t,z) \sim I + \mathcal{O}\left(\frac{1}{z}\right), \quad z \to \infty \text{ in } \mathbb{C}\backslash\mathbb{R} \tag{4.1c}$$

The phase function (4.1b) has a single stationary point at $\xi = -x/4t$. We reduce to a model problem frozen at $\lambda = \xi$ by:

(1) 'Preparing' for steepest descent by a change of dependent variable,

(2) Deforming the contour $\mathbb{R}$ to a new contour Γ which captures the decay of the complexified phase function, and

(3) Reducing to a model problem.

4.2.1. Preparation for Steepest Descent

Write $\xi = -x/4t$ so

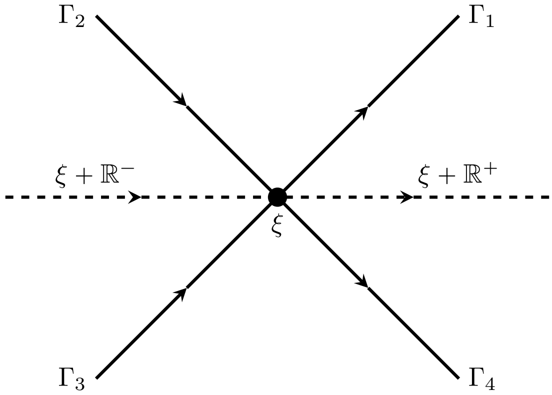
$$\theta(x,t,\lambda) = -2(\lambda-\xi)^2 + 2\xi^2$$

and let

$$\Gamma_i = \xi + \mathbb{R}^+ e^{(2i-1)\pi i/4}, \quad 1 \leqslant i \leqslant 4.$$

Diagrammatically:

Figure 4.1: The Deformed Contour Γ



Observe that, in the jump matrix (4.1a), the oscillatory factor $e^{2it\theta}$ decays along the rays Γ_2 and Γ_4, while the oscillatory factor $e^{-2it\theta}$ decays along the rays Γ_1 and Γ_3. On the other hand, the jump matrix factorizes as

$$\begin{pmatrix} 1 & \rho e^{-2it\theta} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\lambda\overline{\rho}e^{2it\theta} & 1 \end{pmatrix}$$

which has 'good' behavior for deforming $\xi+\mathbb{R}^+$ to $\Gamma_1 \cup \Gamma_4$, but 'bad' behavior for deforming $\xi+\mathbb{R}^-$ to $\Gamma_2 \cup \Gamma_3$. For this reason, we make a change of dependent variable which is trivial on $\xi+\mathbb{R}^+$ but introduces an alternative factorization of the jump matrix on $\xi+\mathbb{R}^-$.

Let $\delta(z)$ solve the scalar RHP

$$\delta_+(\lambda) = \begin{cases} (1-\lambda|\rho(\lambda)|^2)\delta_-(\lambda) & \lambda < \xi, \\ \delta_-(\lambda) & \lambda > \xi. \end{cases} \tag{4.2}$$

The explicit solution is

$$\delta(z;\xi) = \exp\left(\int_{-\infty}^{\xi} \frac{\log(1-s|\rho(s)|^2)}{s-z} \frac{ds}{2\pi i} \right). \tag{4.3}$$

Define a new unknown by $\mathbf{N}^{(1)}(x,t,z) = \mathbf{N}(x,t,z)\delta(z;\xi)^{\sigma}$. Then, the potential $q(x,t) = \lim_{z\to\infty} -2iz\mathbf{N}^{(1)}(x,t,z)$ is unchanged since $\delta(z;\xi)^{\sigma}$ is diagonal and $\delta(z;\xi) = 1 + \mathcal{O}(1/z)$. On the other hand, the new RHP for $\mathbf{N}^{(1)}$ has separate jump matrices for $\lambda < \xi$ and $\lambda > \xi$ which allow deformation to a problem with good decay properties.

The new jump matrix $V^{(1)}$ is

$$V^{(1)}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{-\delta_-^{-2} z\overline{\rho}}{1-z|\rho|^2} e^{-2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\delta_+^2 \rho}{1-z|\rho|^2} e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in (-\infty, \xi) \\ \\ \begin{pmatrix} 1 & \rho\delta^2 e^{2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z\overline{\rho}\delta^{-2} e^{-2it\theta} & 1 \end{pmatrix}, & z \in (\xi, \infty). \end{cases}$$

Next we want to deform to a cross-shaped contour centered at ξ and show that the RHP "concentrates" at $z = \xi$, reducing to a model RHP with constant jump matrix after rescaling.

4.2.2. Deforming the Contour

We'll change variables again to $\mathbf{N}^{(2)} = \mathbf{N}^{(1)} R(z;x,t)$ where

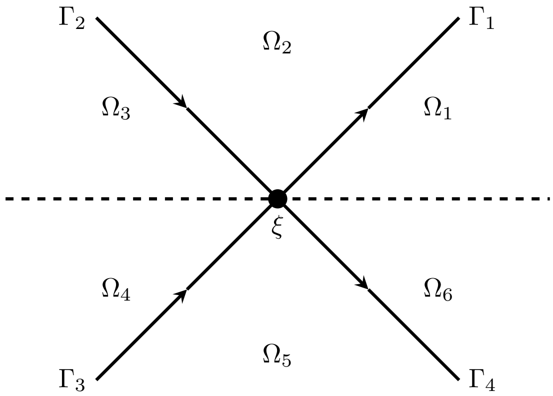
(i) $R(z;x,t)$ is 'almost' analytic (i.e., has a small $\overline{\partial}$-derivative) in the sectors Ω_1 through Ω_6 (see Figure 4.2),

(ii) $R(z;x,t)$ is the identity matrix in Ω_2 and Ω_5,

(iii) $R(z;x,t)$ removes the jump on the real axis, and

(iv) $R(z;x,t)$ introduces jumps across the contours Γ_1, Γ_2, Γ_3, and Γ_4.

Figure 4.2: The Contour Γ and Regions $\Omega_1 - \Omega_6$



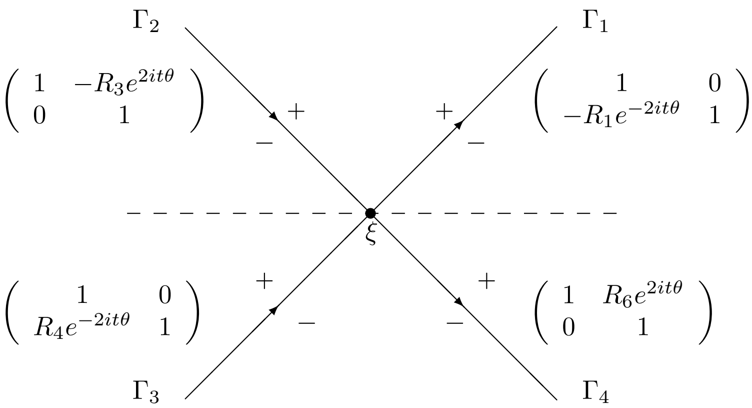
We change to $\mathbf{N}^{(2)} = \mathbf{N}^{(1)}R$ where R is a piecewise continuous matrix-valued function, $R = I$ in $\Omega_2 \cup \Omega_5$, and

$$\begin{pmatrix} 1 & 0 \\ R_1(z)e^{-2it\theta} & 1 \end{pmatrix}, \quad z \in \Omega_1, \qquad \begin{pmatrix} 1 & R_3(z) \\ 0 & 1 \end{pmatrix} \quad z \in \Omega_3$$
$$\begin{pmatrix} 1 & 0 \\ R_4(z)e^{-2it\theta} & 1 \end{pmatrix} \quad z \in \Omega_4, \qquad \begin{pmatrix} 1 & R_6(z)e^{2it\theta} \\ 0 & 1 \end{pmatrix} \quad z \in \Omega_6.$$

The functions R_1, R_3, R_4, R_6 interpolate between values on the real axis (removing the jumps there) and 'frozen coefficient' values of the jump matrix on $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

This process introduces non-analyticity that can be 'solved away' using a $\overline{\partial}$-problem; see [9] for the application to NLS and [23] for their application to DNLS.

Figure 4.3: Jump Matrices for $\mathbf{N}^{(2)}$



4.2.3. Reduction to a Model Problem

If

$$\kappa(\xi) = -\frac{1}{2\pi}\log(1 - \xi|\rho(\xi)|^2),$$

then the function $\delta(z)$ in (4.3) solving (4.2) has behavior

$$\delta(z) = \delta_0(\xi)(\xi - z)^{i\kappa(\xi)} + \mathcal{O}\left(|z-\xi|\log|z-\xi|\right)$$

as $z \to \xi$ so that, setting $\eta(z;\xi) = (z-\xi)^{i\kappa(\xi)}$ we have as $z \to \xi$

$$\begin{aligned} R_1(z) &\sim \xi\overline{\rho(\xi)}\delta_0(\xi)^{-2}\eta(z;\xi)^{-2}, \\ R_3(z) &\sim \frac{\rho(\xi)}{1-\xi|\rho(\xi)|^2}\delta_0(\xi)^2\eta(z;\xi)^2, \\ R_4(z) &\sim -\frac{\xi\overline{\rho(\xi)}}{1-\xi|\rho(\xi)|^2}, \delta_0(\xi)^{-2}\eta(z;\xi)^{-2}, \\ R_6(z) &\sim \rho(\xi)\delta_0(\xi)^2\eta(z;\xi)^2. \end{aligned}$$

We can now rescale

$$\zeta(z) = \sqrt{8t}(z - \zeta)$$

so that

$$e^{2it\theta} \Rightarrow e^{-i\zeta^2/2}e^{ix^2/(4t)}, \quad e^{-2it\theta} \Rightarrow e^{i\zeta^2/2}e^{-ix^2/(4t)}$$

and get a new RHP with jump matrix (a, b, c, d constants)

$$V|_{\Gamma_1} = \begin{pmatrix} 1 & 0 \\ a\zeta^{-2i\kappa}e^{i\zeta^2/2} & 1 \end{pmatrix}, \quad V|_{\Gamma_2} = \begin{pmatrix} 1 & b\zeta^{2i\kappa}e^{-i\zeta^2/2} \\ 0 & 1 \end{pmatrix},$$

$$V|_{\Gamma_3} = \begin{pmatrix} 1 & 0 \\ c\zeta^{-2i\kappa}e^{i\zeta^2/2} & 1 \end{pmatrix}, \quad V|_{\Gamma_4} = \begin{pmatrix} 1 & d\zeta^{2i\kappa}e^{-i\zeta^2/2} \\ 0 & 1 \end{pmatrix}.$$

The model problem may be solved explicitly using parabolic cylinder functions. Denote its solution by $\mathbf{N}^{\text{RHP}}(\zeta;\xi)$. Then

$$\mathbf{N}^{\text{RHP}}(\zeta;\xi) = \left(1 + \frac{m^{(0)}}{\zeta} + o\left(\frac{1}{\zeta}\right)\right)$$

and we can compute the leading asymptotics from $m^{(0)}$.

Note that the leading $t^{-1/2}$ behavior comes naturally from the scaling as

$$1/\zeta \sim 1/(\sqrt{8t}z).$$

Long-Time Asymptotics

From the model problem and the $\overline{\partial}$-problem, we obtain

$$q(x,t) = \frac{1}{\sqrt{t}}\alpha_0(\xi)e^{-i\kappa(\xi)\log(8t)+ix^2/(4t)} + \mathcal{O}\left(t^{-3/4}\right)$$

where $|\alpha_0(\xi)|^2 = |\kappa(\xi)|/2|\xi|$ and

$$\begin{aligned} \arg\alpha_0(\xi) &= \frac{\pi}{4} + \arg(\Gamma(i\kappa(\xi)) \\ &\quad + \frac{1}{\pi}\int_{-\infty}^{\xi}\log(\xi - s)d\log(1 - s|\rho(s)|^2). \end{aligned}$$

The gauge transformation adds an additional term to the phase of

$$u(x,t) = q(x,t)\exp\left(-i\int_{-\infty}^{x}|q(y,t)|^2\,dy\right)$$

through the asymptotic formula

$$\int_{-\infty}^{x}|q(y,t)|^2\,dy \sim -\int_{\xi}^{\infty}\frac{\log(1-s|\rho(s)|^2)}{s}\,ds + \mathcal{O}_{\xi}\left(\frac{1}{\sqrt{t}}\right)$$

which comes from analyzing the RHP and using the identity

$$N_{11}^{\pm}(x,t,0) = \exp\left(-\frac{i}{2}\int_{\pm\infty}^{x}|q(y,t)|^2\,dy\right).$$

5. Further Results

Recently, Jenkins, Liu, Perry, and Sulem [16] have extended some of these results to initial data that support finitely many "bright" (exponentially decaying) solitons. We can show that the set of initial data with finitely many "bright" solitons and no spectral singularities constitute an open and dense set of $H^{2,2}(\mathbb{R})$, and show that the Cauchy problem is well-posed on this open and dense set. Finally, we are able to use steepest descent methods to obtain soliton resolution for our class of Cauchy data. Initial data with spectral singularities are not as yet within the reach of inverse scattering methods.

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