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Zaher Hani

Abstract

The purpose of this note is to report on some recent advances in the study of out-of-equilibrium behavior of dispersive PDE. One can address this problematic from two different perspectives: a *dynamical systems* one, and a *statistical physics* one. The dynamical systems perspective corresponds to constructing solutions exhibiting "energy cascade" between scales, whereas the statistical physics perspective corresponds to deriving effective equations for the dynamics under some "macroscopic limits" in what is often called *wave turbulence theory.* The rigorous justification of this theory is an outstanding open problem from a rigorous mathematical point of view, and we will touch on it here. We shall discuss some recent attempts to better understand both of the above perspectives.

1. Introduction

Broadly speaking, the main problematic that we discuss here is out-of-equilibrium dynamics and statistics of Hamiltonian systems. There are systems that can be written as

$$\dot{p}_n = -\frac{\partial H}{\partial q_n}, \quad \dot{q}_n = \frac{\partial H}{\partial p_n}; \quad 1 \leq n \leq N.$$

Here, $(p_n(t), q_n(t))_n$ are functions of time and $H(\mathbf{p}, \mathbf{q})$ is the Hamiltonian function. N stands for the number of degrees of freedom that could well be infinite. Indeed, we will be mostly interested in the infininte dimensional case corresponding to Hamiltonian nonlinear PDE, but we shall start our discussion by drawing some insight from a particularly illuminating finite dimensional problem.

1.1. Baby turbulence: The FPU Paradox

In the early 1950's, around the time when computers started to be used for numerical scientific experimentation, Nobel Laureate Enrico Fermi, joined by the mathematician John Pasta, and computer scientist Stan Ulam, decided to investigate numerically how a crystal evolves towards thermal equilibrium. For this, they took a one dimensional model of a crystal given by a chain of N-particles connected by springs. Denoting by $(p_n, q_n) \in \mathbb{R} \times \mathbb{R}$ the momentum-displacement vector of the $n - th$ particle and identifying $(p_{N+1}, q_{N+1}) \equiv (p_1, q_1)$ (chain), this system is given by Hamilton's equations of motion:

$$\begin{cases} H(\mathbf{p}, \mathbf{q}) = \frac{1}{2}\sum_{n=1}^N p_n^2 + V(q_{n+1} - q_n); \qquad V(q) = \frac{1}{2}q^2 + \frac{\alpha}{3}q^3 \\ \dot{q}_n = \frac{\partial H}{\partial p_n} = p_n; \qquad \dot{p}_n = -\frac{\partial H}{\partial q_n} = \nabla V(q_{n+1} - q_n) - \nabla V(q_n - q_{n-1}); \quad 1 \leq n \leq N. \end{cases} \tag{1.1}$$

Here $\alpha \geq 0$. Notice that the interaction is a nearest-neighbor interaction in the sense that each particle only interacts with the two particles on its left and right.

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When $\alpha = 0$, the system is linear and can be solved by using Fourier series for example: Defining $A_n = \frac{2}{N+1}\sum_{k=1}^N q_k \sin(\frac{\pi k n}{N+1})$, then the system satisfied by $\{A_n\}_{n=1}^N$ is given by:

$$\ddot{A}_n + \omega_n^2 A_n = 0, \qquad 1 \le n \le N; \qquad \omega_n = 2\sin\frac{n\pi}{2(N+1)}. \tag{1.2}$$

In words, the system gets decoupled, and each mode A_n oscillates independently with its *harmonic frequency* ω_n. So if energy is initially present *only* in the first mode A_1, it will never migrate to the rest of the modes. Systems that can be written as such are called *completely integrable*.

When $\alpha > 0$, the modes get coupled to each other, and the right-hand side of (1.2) becomes quadratic in $\{A_n\}_{n=1}^N$. Thus, the FPU system is said to be a *perturbation of the completely integrable system* corresponding to $\alpha = 0$. Statistical mechanics asserts that such nonlinear systems should approach, as t becomes large, an equilibrium configuration in which the initial (conserved) energy is *essentially* equally distributed (at least if α is small enough) among all modes $\{A_n\}_{n=1}^N$.

This is what Fermi, Pasta, and Ulam set on verifying in their numerical investigation. They started by exciting only the first mode A_1 at time $t = 0$ and left $A_n(0) = 0$ for $n \neq 0$ (with $N = 32$). Much to their surprise, as time passed by and became very large, the energy did not get equidistributed amongst all the modes A_n, but rather kept concentrated on the first five modes. Not only did this not comply with the conclusions of statistical physics, but also it contradicted Fermi's own "ergodicity theorem" concerning the ergodicity of nonlinear systems. This became to be known the FPU paradox, and Fermi is said to have remarked that these results might be one of the most significant discoveries of his career [12].

So, what's going on?! This leads us to the broad question under which falls all the problems we discuss below:

Main Problematic Q.) *How does energy get transferred and redistributed among the degrees of freedom in Hamiltonian systems?*

The FPU paradox arises due to a clash between two perspectives to address this question:

I) *The dynamical systems perspective*: which studies individual orbits and their long-time behaviour, in order to answer whether energy gets redistributed or not;

II) *The statistical mechanics perspective*: which studies, instead of individual orbits, invariant measures for the dynamics, (i.e. ones for which $\mu(A) = \mu(S_t A)$ for any measurable set A, where S_t is the solution operator for any $t \in \mathbb{R}$).

For the Hamiltonian system (1.1), the invariant measure is non-other than the Liouville measure (i.e. restricted Lebesgue measure on the level set $H(\mathbf{p}, \mathbf{q}) = H_0$) which gives essentially equal weight to each harmonic mode A_n. However, the orbits that the FPU team found kept all the weight in the first five modes and not the rest!

It should be mentioned here that statistical mechanics derives from considerations *extrinsic* to dynamical principles (compare the reversibility of Hamilton's equations (1.1) to the irreversibility of the second law of thermodynamics pretaining to entropy *increase* in the *forward* direction of time). The link between the statistical mechanics and dynamical systems perspectives is usually provided by ergodic theorems (or laws of large numbers) that typically look like: Suppose that f is a smooth test function then, given an orbit X_t

$$\lim_{t\to\infty} \underbrace{\frac{1}{t}\int_0^t f(X_s)\,ds}_{\text{Temporal average along an orbit}} = \underbrace{\int f(x) d\mu}_{\text{Spatial average w.r.t the invariant measure}} \tag{1.3}$$

Definitely, this ergodicity did not hold for the orbit that FPU found (simply take f to supported on the last modes A_n with $n > 5$).

The first key to the puzzle came in 1954 in Kolmogorov's address to the International Congress of Mathematicians, in which he showed that for a small perturbation of a completely integrable system (small α here), "many" (in a measure theoretic sense) of the decoupled (highly non-ergodic!) orbits of the unperturbed system (1.2) persist under the perturbation. This (which evolved later to what is now known as Kolmogorov-Arnold-Moser (KAM) theory) obviously presents an obstruction to energy transfer between the degrees of freedom in Hamiltonian systems. In particular, statistical physics does not seem to give the right conclusions at low energies for systems with a small number of degrees of freedom.

But this begs the question: *When are the laws of statistical physics upheld?* The following is now believed to be true: Suppose that $M_{E,N}$ is the set of initial data with energy E for which the energy does not eventually get equidistributed amongst the modes (i.e. for which (1.3) does not hold): Then, in an appropriate sense of measure, the measure of $M_{E,N}$ should go to $\to 0$ as $N \to \infty$. In other words, the laws of statistical physics are expected to hold *only* in the limit as $N \to \infty$.

Despite attracting a huge amount of research over the past 60 years (cf. [14]), the above "conjecture" is still not proved rigorously. As we shall emphasize later, this is rather common for questions bridging statistical physics and dynamics, at least in deterministic settings, and apart from very few exceptions (see [39, 33, 9, 13] and references therein). That being said, we should point out that such problems becomes considerably more tractable in the presence of a stochastic element in the system. For instance, if one attaches the chain system given by (1.1) to a heat bath on the left with temperature T_1 and one on the right with temperature T_N, which is equivalent to only replacing the equations for p_1 and p_N in (1.1) by

$$\dot{p}_1 = \nabla V(q_2 - q_1) - \gamma p_1 + \sqrt{2\gamma T_1}dB_1; \qquad \dot{p}_N = -\nabla V(q_N - q_{N-1}) - \gamma p_N + \sqrt{2\gamma T_N}dB_N$$

where dB_1 and dB_2 are independent white noise terms, then one can prove ergodicity statements like (1.3) as well as *dynamical* convergence towards the invariant measure of the system (cf. [10, 21, 20]).

1.2 Out-of-equilibrium behavior for dispersive PDE

The takeaway message from "baby turbulence" above can be summarized as follows: 1) Even in finite dimensions, questions of energy transfer and redistribution in Hamiltonian systems are highly non-trivial; 2) There are two perspectives to address such questions: a dynamical systems one, and a statistical physics one; 3) Reconciling the two perspectives, or equivalently giving a rigorous justification of statistical physics from dynamical systems principles, is a very deep and hard problem in the deterministic setting; 4) However, this problem can be more tractable if a stochastic element is present in the system.

Now we turn to our main focus, which is addressing the question of non-equilibrium behavior and energy transfer for dispersive and wave PDE. In this setting, the main question of interest, both mathematically and physically, is how energy (which can refer to the *"kinetic energy"* $\int |\nabla u(t)|^2\, dx$ or the *"mass"* $\int |u(t)|^2\, dx$) changes its concentration zones in *frequency space*. In fact, the frequency (or Fourier) modes play here the role of the coordinates A_n for FPU: they are completely decoupled at the linear level, where they satisfy the same kind of equation as (1.2) (with ω_n being the dispersion relation); Moreover, they indicate the spatial oscillations of the solution. Hence, the Main Problematic **Q.)** posed in Section 1.1 translates here to:

Question (Energy Cascade): *Suppose the energy of the dispersive system is concentrated at time $t = 0$ in a certain region of Fourier space (say at low frequencies (a.k.a. large scales)), how will this energy be redistributed as time evolves? Will it keep its original concentration zones, or will it* cascade *to characteristically different scales?*

This question is central to many areas of physics like oceanography, plasma physics, and superfluidity[1]. The relevant mathematical models here are nonlinear dispersive equations posed on compact domains (possibly very large, like the ocean), as opposed to making the infinite volume approximation from the start and posing them on $\mathbb{R}^d$. Indeed, as we will explain in Section 1.4, one has to take particular care in passing to the right infinite-volume approximation (when working on very large domains) in a way that captures correctly the energy transfer phenomenon.

[1]The ocean is excited by the wind at large scales (small frequencies) and this energy is dissipated at much smaller scales through wave breaking or dissipation. This energy transfer between scales is the main driving force of ocean wave dynamics. In plasma physics, studying energy dynamics is central to control plasma inside a tokamak, or even understanding the feasibility of controlled nuclear fusion. In superfluidity, reverse (mass) cascade is the mechanism to forming Bose-Einstein condensates.

Remark 1.1 (Dispersive equations on $\mathbb{R}^d$). *Out-of-equilibrium dynamics is not a particularly generic phenomenon for dispersive PDE on $\mathbb{R}^d$. There, linear solutions decay in L^∞ norm, and, as a result, the typical behavior of small nonlinear solutions is scattering (asymptotically linear behavior). This precludes any significant energy cascade between scales (e.g. all high Sobolev norms are uniformly bounded). While it is interesting to study the possibility of energy cascade for large initial data on $\mathbb{R}^d$, our main interest here is* energy dynamics of solutions with *small* initial data on compact domains. *There, unlike $\mathbb{R}^d$, one cannot make any uniform asymptotic statements (like scattering) that would hold for all small solution. The system can sustain a zoo of different behaviors, even starting from arbitrary small neighborhoods of zero.*

Just like for other Hamiltonian systems, there are two perspectives to address such questions of long-time behavior and energy cascade: a *dynamical systems perspective* based on constructing orbits exhibiting the energy cascade phenomena, and a *statistical physics perspective* which is based on deriving effective equations for the dynamics, and often goes by the name of "*wave turbulence theory*". We will split our discussion for the rest of this note accordingly.

2. Dynamical systems perspective: Energy cascade and Sobolev norm growth

2.1. Background

The question here is to construct solutions to nonlinear dispersive PDE that exhibit a strong form of energy cascade. We will restrict attention to the forward cascade phenomenon in which an energy (here the kinetic energy $\int |\nabla u|^2 dx$) moves its concentration zones from low frequencies to high frequencies, while remaining bounded in time. Similar questions and results hold for the backward cascade phenomenon in which an energy moves its concentration zone from high to low frequencies. For instance, for the cubic NLS equation that we will consider below, it is expected that the kinetic energy cascades towards high frequencies, whereas the mass would cascade towards low frequencies.

A good way to capture the forward movement of kinetic energy is to look at the behavior of the Sobolev norms H^s of the solution. On the torus $\mathbb{T}^d$, one has

$$\|u(t)\|_{H^s(\mathbb{T}^d)}^2 = \sum_{|\alpha|\leq s} \|\nabla^\alpha u\|_{L^2(\mathbb{T}^d)}^2 \sim \sum_{n\in\mathbb{Z}^d} (1+|n|^2)^s |\widehat{u}(n)|^2,$$

so we directly notice that these norms penalize large frequencies much more than they penalize low frequencies. This implies that if the energy of the solution moves to high frequencies, then this should be accompanied by a growth of such norms. The question then becomes whether we can exhibit solutions whose Sobolev norms grow in time.

We shall discuss this problem for the nonlinear Schrödinger equation (NLS) for simplicity, but the same questions could be asked (but are still mostly open) for any other dispersive or wave PDE. The (NLS) equation is given by

$$(i\partial_t + \Delta)u(t,x) = \lambda |u(t,x)|^2 u(t,x); \qquad u(0) = u_0(x), \lambda \in \{+1,-1\}, \tag{NLS}$$

posed first on the unit torus $\mathbb{T}^d \ni x$. The problem of energy cascade for this equation was well-known for a long time, but its Sobolev norm formulation started with the works of Bourgain [2, 3] who asked in [4]: *Does there exist global solutions to the cubic NLS equation whose Sobolev norm H^s (with some $s > 1$) grows indefinitely in time, i.e. for which*

$$\limsup_{t\to\infty} \|u(t)\|_{H^s} = +\infty \qquad \text{for some } s > 0, s \neq 1 \ ? \tag{2.1}$$

We will refer to such solutions as *unbounded or infinite-cascade orbits*. Such growth cannot happen for $s = 0$ or 1 due to the conservation laws of the equation which control the H^1 norm in the defocusing case, and for small data in the focusing case. Upper bounds on the rate of any possible growth of H^s norms were studied by Bourgain[2], Staffilani[41], Sohinger[40], among others.

2.2. Recent results

The starting point of all recent progress on this problem (cf. [2, 3, 32] for earlier work) is a celebrated result of Colliander, Keel, Staffilani, Takaoka, and Tao [8] who constructed solutions exhibiting *large but finite* growth of Sobolev norms. To state precisely their theorem, let us start by introducing a notion that will be useful for us below:

Definition 2.1. *We say that the flow exhibits* <u>long-time strong instability</u> *near some* $\phi \in H^s(\mathbb{T}^d)$ *if for any* $0 < \delta < 1$ *and any* $K > 1$*, there exists a solution* $u(t)$ *to* (NLS) *and a time* T *such that*

$$\|u(0) - \phi\|_{H^s(\mathbb{T}^d)} \leq \delta \quad \text{and} \quad \|u(T)\|_{H^s(\mathbb{T}^d)} \geq K. \tag{2.2}$$

With this definition in hand, the result of CKSTT [8] can be stated as follows: The nonlinear Schrödinger flow exhibits long-time strong instability near 0 in $H^s(\mathbb{T}^d)$ with $s > 1$. Guardia and Kaloshin revisited the proof in [18] and obtained quantitative upper bounds on the time T required for the growth to happen. Those results were generalized to the higher-order analytic nonlinearities in [29, 19].

Despite being key to all the later progress, the above mentioned result in [8], corresponding to $\phi = 0$ in the above definition, is still considerably weaker than Bourgain's question. Indeed, giving a positive answer to this question on the torus $\mathbb{T}^d$ remains one of the biggest outstanding problems in the field.

That being said, it was noticed in [23, 24] that a sufficient extension of this result *does* yield the existence, and possible genericness, of unbounded orbits as in (2.1). Indeed, the key observation can be vaguely stated as follows:

Proposition 2.2 (H. [23, 24])**.** *If a Hamiltonian flow exhibits long-time strong instability near "sufficiently many" initial data in* H^s*, then unbounded orbits as in* (2.1) *exist and are "generic" in an appropriate topological sense.*

For instance, if "sufficiently many" means dense, then "generic" would mean co-meager in the sense of Baire-Category; but less restrictive versions can also be formulated [24]. This proposition suggests a program to prove infinite cascade results like (2.1) by proving long-time strong instability near "sufficiently many" data. This program witnessed at least two instances of success: A) In [24], the author constructed infinite cascade solutions as in (2.1) for a family of equations (E_n) that converge to cubic (NLS) as $n \to \infty$. In particular, this family includes the important resonant (NLS) system (defined below in (3.2)). This was crucial for the progress that we report on next. B) More recently, P. Gérard and S. Grellier used this program to obtain *generic* unbounded orbits for the cubic Szegö equation [15].

Nonetheless, a remarkable fact is that much more can be said if one adds one non-compact direction to $\mathbb{T}^d$, and studies the cubic NLS equation on the spatial domain $M = \mathbb{R} \times \mathbb{T}^d$. In this case, one can take advantage of the decay of linear solutions on M (coming from the $\mathbb{R}$-direction) to get *global* control on nonlinear solutions. Indeed, we were able (with B. Pausader, N. Tzvetkov, and N. Visciglia [26]) to give a positive answer to Bourgain's question (2.1) in this case:

Theorem 2.3 ([26, 27])**.** *For any* $d \geq 2$*, there exists global* $H^s(s > 30)$ *solutions to the cubic* (NLS) *equation on* $\mathbb{R} \times \mathbb{T}^d$ *satisfying:* $\limsup_{t\to\infty} \|u(t)\|_{H^s} = +\infty$*. Indeed, there exists a sequence* $t_k \to \infty$ *of times, such that* $\|u(t_k)\|_{H^s} \geq c(\log\log t_k)^C$ *for some constants* $c, C > 0$*.*

It is worth mentioning here why such a result is possible on $\mathbb{R}\times\mathbb{T}^d$, but remains quite challenging on the $\mathbb{T}^d$. This is surprising because adding a non-compact direction to $\mathbb{T}^d$ leads to the decay of solutions, which is a mechanism of stability rather than non-equilibrium. In fact, if one adds more than one non-compact direction, and studies the equation on $\mathbb{R}^n \times \mathbb{T}^d$ with $n \geq 2$, then small solutions scatter in H^s [44]; in particular if they start at size ϵ, they would stay of size $O(\epsilon)$ for all later times. The reason why Theorem 2.3 holds is because of a combination of the following two facts which also constitute the two main components of the proof:

1. *Modified Scattering towards resonant dynamics:* This means that solution of (NLS) on $\mathbb{R}\times\mathbb{T}^d$ starting from small (in appropriate norms) initial data converge as $t \to \infty$ to solutions of the *resonant cubic system* on $\mathbb{R} \times \mathbb{T}^d$. This system is given by

$$\begin{aligned} i\partial_t G(t) = &\ \mathcal{R}[G(t), G(t), G(t)], \quad \text{defined via} \\ i\partial_t \widehat{G}(\xi, p) = &\sum_{(p_1,p_2,p_3)\in\mathcal{R}(p)} \widehat{G}(\xi, p_1)\overline{\widehat{G}(\xi, p_2)}\widehat{G}(\xi, p_3). \end{aligned} \tag{RS}$$

where for $(\xi, p) \in \mathbb{R} \times \mathbb{Z}^d$, $\widehat{G}(\xi, p)$ denotes the Fourier transform $\mathcal{F}_{\mathbb{R}\times\mathbb{T}^d} G(\xi, p)$. Notice that for each ξ this system is an infinite system of ODE, and it is nothing but the *resonant NLS system* on $\mathbb{T}^d$! The latter is given below in (3.2) and is obtained from the cubic NLS equation by removing all non-resonant interactions. This steps constitutes the bulk of the proof, and effectively gives that the asymptotic dynamics of (NLS) on $\mathbb{R} \times \mathbb{T}^d$ is dictated by (RS) above.

2. *Infinite cascade solutions for* (RS) The second component is to show that there exists solutions of (RS) that exhibit infinite growth in time. This is done by noticing that since the dependence on ξ is merely parametric in (RS), one only needs to have solutions to system (3.2) that exhibit infinite growth. This was already done in [23, 24] as we discussed before (relying on the CKSTT work [8]), but without any estimate on the growth rate. In [26], we revisit this question relying on the more precise work [18] instead, which allows to obtain the $\log \log t_k$ growth in Theorem 2.3.

3. Statistical physics perspective: Kinetic formalism of wave turbulence

3.1. Background

We now turn to the statistical physics perspective to addressing out-of-equilibrium behavior of dispersive PDE. This is presented by *wave turbulence theory*, which is the theory of non-equilibrium statistical mechanics for dispersive systems. It was many similarities in its conclusions with Kolmogorov's theory of fluid turbulence (see [36] for a comparison), but takes a different formalism that appears more systematic at some points as we shall see below.

The standard setting in this theory is to start by posing a dispersive equation, such as (NLS), on the box $\mathbb{T}^d_L = [0, L]^d$ of size L with periodic boundary conditions. The solution is assumed to have characteristic size ϵ (say in the conserved L^2 norm). To emphasize the size of the solution, we shall adopt the ansatz $u(t, x) = \epsilon v(t, x)$, so that v satisfies:

$$(i\partial_t + \Delta)v(t, x) = \lambda\epsilon^2 |v(t, x)|^2 v(t, x); \qquad x \in \mathbb{T}^d_L. \tag{3.1}$$

The smallness of the data is now reflected in the "weakness" of the nonlinearity, which explains why the theory is sometimes called *weak turbulence*. From now on, we set $\lambda = 1$: The sign of the nonlinearity plays little role in this theory, partly because it deals with small solutions.

The fundamental equations of wave turbulence aim to describe the effective dynamics and energy distribution in *frequency space*. As such, we start by Fourier expanding

$$v(t, x) = \sum_{K\in\mathbb{Z}^d_L} \widehat{v}(t, K) e^{i(2\pi i K.x)}$$

where $\mathbb{Z}^d_L := (L^{-1}\mathbb{Z})^d$ is the Fourier dual of $\mathbb{T}^d_L$. Setting $a_K(t) = e^{4\pi^2 i|K|^2 t}\widehat{v}(t, K)$, the equation satisfied by the modes $\{\widehat{v}(t, K)\}_{K\in\mathbb{Z}^d_L}$ is equivalent to the following infinite system of ODE for $a_K(t)$:

$$i\partial_t a_K(t) = \epsilon^2 \sum_{\mathcal{S}(K)} a_{K_1}(t)\overline{a_{K_2}(t)} a_{K_3}(t) e^{-4\pi^2 i\Omega t}; \qquad K \in \mathbb{Z}^d_L := (L^{-1}\mathbb{Z})^d \tag{$\mathcal{F}$NLS}$$

where $\mathcal{S}(K) = \{(K_1, K_2, K_3) \in \mathbb{Z}^d_L : K_1 - K_2 + K_3 = K\}$ and $\Omega(K) \overset{def}{=} |K_1|^2 - |K_2|^2 + |K_3|^2 - |K|^2$. Equation ($\mathcal{F}$NLS) describes how the mode a_K is excited by other modes through the nonlinear interactions included in $\mathcal{S}(K)$. Of all those interactions, a particular sub-family stands out in its crucial effect on the dynamics, namely *resonant interactions* corresponding to $(K_1, K_2, K_3) \in \mathcal{S}(K)$ such that $\Omega(K) = 0$. Restricting ($\mathcal{F}$NLS) to those interactions we arrive at the *resonant NLS system*

$$i\partial_t r_K(t) = \epsilon^2 \sum_{\mathcal{R}(K)} r_{K_1}(t)\overline{r_{K_2}(t)} r_{K_3}(t) \tag{3.2}$$

where $\mathcal{R}(K) = \{(K_1, K_2, K_3) \in \mathcal{S}(K) : \Omega(K) = |K_1|^2 - |K_2|^2 + |K_3|^2 - |K|^2 = 0\}$. The fundamental importance of system (3.2) in *approximating* the long-time behavior of (NLS) *for small* ϵ has been pointed out in Section 2.2 for constructing energy cascade solutions [24, 26, 27], and has been exploited in many works (see [34, 8, 7, 25]).

Starting from a random distribution of initial data (taking each $a_K(0)$ as an independent random variable), one would now like to write an effective equation for the *mass density* defined by $n^L(K,t) = \mathbb{E}(|a_K(t)|^2)$, where $a_K(t)$ solves ($\mathcal{F}$NLS) with those random initial data. A direct computation shows that

$$\partial_t n^L(K,t) = 2\epsilon^2 \mathrm{Re}\, \mathbb{E}(\partial_t a_K(t)\overline{a_K(t)}) = 2\epsilon^2 \sum_{\mathcal{S}(K)} \mathrm{Im}\left[e^{-4\pi^2 i\Omega t}\, \mathbb{E}(a_{K_1}(t)\overline{a_{K_2}(t)}a_{K_3}(t)\overline{a_K(t)})\right]. \tag{3.3}$$

In other words, the only way to study the dynamics of $n^L(K,t)$ is to understand that of the forth-order correlations $\mathbb{E}(a_{K_1}(t)\overline{a_{K_2}(t)}a_{K_3}(t)\overline{a_{K_4}(t)})$. Those correlations satisfy themselves an equation involving the sixth-order correlations $\mathbb{E}(a_{K_1}(t)\dots\overline{a_{K_6}(t)})$ and so on. This gives an infinite hierarchy of equations for the $N-$th order correlations, instead of a closed scalar equation describing the effective dynamics of $n^L(K,t)$. Going from this hierarchy to a closed equation describing the effective dynamics of $n^L(K,t)$ is called a *closure problem*, and it is reminiscent of the derivation of Boltzmann's kinetic equation for an ideal gas of particles (cf [33, 13]). To get to this closed equation, physicists perform several, rather cavalier and formal, manipulations or limiting arguments, which we summarize in the following three steps:

(S1): *Statistical averaging*: These are particularly non-rigorous manipulations that involve mathematically unjustified (at least not a priori) ergodicity assumption on the dynamics, particularly on the phases of $a_K(t)$. Formally, such assumptions allow replacing higher order correlations by products of lower order ones and error terms.

(S2): *Weak-nonlinearity limit* ($\epsilon \to 0$): This is essentially a time-averaging step and, in effect, restricts the dynamics to resonant (or near-resonant) interactions, in a way similar to the approximation of (3.2) to ($\mathcal{F}$NLS) when ϵ is small.

(S3): *Large-box limit* ($L \to \infty$): in which L is sent to ∞, and hence $n^L(K)$ which was previously a function on $\mathbb{Z}^d_L = (L^{-1}\mathbb{Z})^d$ becomes approximated by a function $n(K)$, *defined on* $\mathbb{R}^d$, that describes the effective behavior of $n^L(K) := \mathbb{E}(|a_K(t)|^2)$ for large enough domains.

After all these formal manipulations, one obtains the following equation for $n(K,t)$:

$$\begin{aligned}\partial_t n(K) =& \epsilon^4 \iiint n(K_1)n(K_2)n(K_3)n(K)\left(\frac{1}{n(K_1)} - \frac{1}{n(K_2)} + \frac{1}{n(K_3)} - \frac{1}{n(K)}\right) \\ & \delta_{\mathbb{R}^d}(K_1 - K_2 + K_3 - K)\delta_{\mathbb{R}}(|K_1|^2 - |K_2|^2 + |K_3|^2 - |K|^2)dK_1 dK_2 dK_3.\end{aligned} \tag{WKE}$$

It is called the *wave-kinetic equation* (WKE) and represents the wave-analog of Boltzmann's equation of particle interactions ("wave collisions" in a dispersive system).

Even though the derivation of this equation is not rigorous yet, its analysis suggests very strong and important implications on the out-of-equilibrium dynamics of NLS. This is reflected by its stationary solutions, called *Kolmogorov-Zakharov spectra*[2], which can be understood as corresponding to invariant steady states or measures for NLS dynamics. We won't go into the details of these all-important solutions here due to lack of space, but they play a big role in many fields like oceanography and plasma physics.

Unfortunately, the very formal nature of the manipulations used to derive (WKE) in the physics literature breaks any backward logical implication (i.e. pulling information from the wave-kinetic equation back to (NLS)). This causes problems even at the level of physics in some models [35, 36], and emphasizes the desperate need of putting this theory on more solid mathematical foundation.

[2] They are the analogs of the all-important Kolmogorov spectra in hydrodynamic turbulence.

3.2. Recent progress

3.2.1. The Continuous Resonant Equation

In [11] and [5], we take the first step towards understanding the limits involved in the derivation of the fundamental equation (WKE) of wave turbulence. More precisely, we dispense with the statistical averaging step **(S1)**, but still perform steps **(S2)** and **(S3)** corresponding to the *weak-nonlinearity* and the *large-box limit* for the NLS equation posed on the periodic box $\mathbb{T}_L^d$ of size L. In [11] we treated the 2D case using elementary number theoretic techniques, while in [5] we generalized, streamlined, and sharpened the proof and results. This allowed us to treat any dimension and any analytic power nonlinearity replacing the cubic one in (NLS). The outcome of the analysis is a new equation that describes 1) the effective dynamics of $\{a_K(t)\}_{K\in\mathbb{Z}_L^2}$ when L is large enough[3], and 2) The effective dynamics of high-frequncy envelopes of NLS solutions on the unit torus $\mathbb{T}^d$. We will mostly elaborate on the first description here.

The weak-nonlinearity limit **(S2)** corresponds to approximating the dynamics ($\mathcal{F}$NLS) of NLS with that of the resonant system (3.2). This is done rigorously using a *normal forms transformation*, which is a change of coordinate transformation that allows regarding the contribution of non-resonant interactions as errors. Up to logarithmic corrections in L, this approximation holds for small enough ϵ and for times up to $L^2\epsilon^{-2}$, where ϵ is the characteristic size of the initial data.

The next step is to take the large-box limit of (3.2), which amounts to thinking of the sum in (3.2) as a Riemann sum, that should be approximated by an appropriate integral if L is large enough. Unfortunately, the sum in (3.2) is not a regular Riemann sum due to the nonlinear restriction to the set $\{\Omega(K)=0\}$. This necessitates studying *quantitatively* the distribution of lattice points on the algebraic variety defined by $\{\Omega(K)=0\}$. More precisely, we would like to be able to say that for a sufficiently smooth and decaying function $F(K_1,K_2,K_3)$, it holds that

$$\sum_{\mathcal{R}(K)} F(K_1,K_2,K_3) \overset{?}{=} Z_d(L)\int_{\substack{(K_1,K_2,K_3)\in\mathbb{R}^{3d}\\ \Omega(K)=0}} F(K_1,K_2,K_3)d\nu + \text{lower order errors} \tag{3.4}$$

for some constant $Z_d(L)$ and a measure $d\nu$ supported on $\{\Omega(K)=0\}$ to be determined, where we recall that $\mathcal{R}(K)=\{(K_1,K_2,K_3)\in\mathcal{S}(K):\Omega(K)\overset{def}{=}|K_1|^2-|K_2|^2+|K_3|^2-|K|^2=0\}$.

This type of questions turns out to be a deep problem in analytic number theory. The main method to handle such problems is the *Hardy-Littlewood circle method* which was first developed to answer the classical *Waring's problem* (i.e. identifying $Z_d(L)$ when F is the characteristic function of a unit ball). In [5], we adapt (and improve in some aspects) a relatively new version of this method due to Heath-Brown [31], and obtain that (3.4) holds with $d\nu$ being the *Dirac measure* on the (singular) variety $\Omega(K)=0$ and

$$Z_d(L)=\begin{cases}\frac{1}{\zeta(2)}L^2\log L & \text{if } d=2\\ \frac{\zeta(d-1)}{\zeta(d)}L^{2d-2} & \text{if } d\geq 3\end{cases} \qquad \text{where } \zeta(\cdot) \text{ is the Riemann } \zeta \text{ function.}$$

As a result of all this analysis, we can show that the effective dynamics of $a_K(t)$ (for large L and small ϵ) is given by the following nonlinear equation, now set on $\mathbb{R}^d$, given by

$$i\partial_t g(\xi,t)=\int_{(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^{3d}} g(\xi_1)\overline{g}(\xi_2)g(\xi_3)\delta(\xi_1-\xi_2+\xi_3-\xi)\delta(|\xi_1|^2-|\xi_2|^2+|\xi_3|^2-|\xi|^2)d\xi_1 d\xi_2 d\xi_3 \tag{CR}$$

We call this the *continuous resonant* (CR) equation. It is new to the best of our knowledge, and can be thought of as a deterministic analog of the wave kinetic equations (WKE).

(CR) has a very unique structure in terms of its surprising symmetries and dynamics. Such dynamics inform on the long-time behavior of (NLS) thanks to *rigorous approximation results* that allow to approximate $a_K(t)$ solving ($\mathcal{F}$NLS) with (a rescaled version of) $g(K,t)$ for any $K\in\mathbb{Z}_L^d$, when L is large enough, and ϵ us small enough (provided of course that $a_K(0)=g(K,0)$) [11, 5]. This can be roughly stated as follows:

Theorem 3.1 (Buckmaster, Germain, H., Shatah '16 [5])**.** *Let $d\geq 2$ and $M,\gamma>0$ be arbitrary. Suppose that $g(t,\xi)$ is a sufficiently "nice" solution of the* (CR) *equation on an interval* $[0,M]$.

[3] Compared to the effective dynamics of $|a_K|^2$ given by the wave kinetic equation (WKE).

Suppose we start with an NLS solution such that $a_K(0) = g_0(K)$. If L is large enough, and if $\epsilon < L^{-\gamma}$, then

$$\|a_K(t) - g(\frac{t}{T_R}, K)\|_{L^2 \cap L^\infty} = o(1)_{L\to\infty}, \qquad \textit{i.e. it } \to 0 \textit{ as } L \to \infty$$

for all $0 \leq t \leq MT_R$ where $T_R = \frac{\zeta(d)}{\zeta(d-1)} \left(\frac{L^2}{\epsilon^2}\right)$.

It is worth mentioning that the first result in this direction was obtained in 2D by E. Faou, P. Germain, and the author [11] giving a logarithmic decay in L for the $o(1)_{L\to\infty}$ term (and requiring a more restrictive $\epsilon - L$ relationship). The result in [5] generalizes the result to higher dimensions and higher-order nonlinearities, as well as refines this $o(1)$ decay into polynomial decay in L when $d \geq 3$. When $d = 2$, it also identifies the logarithmically decaying terms. The less restrictive $\epsilon - L$ relationship is possible thanks to a normal form of very high order [5].

Finally, we remark that the approximation theorem is strong enough to project essentially any nonlinear dynamics observed for (CR) into (NLS). Moreover, analogous results hold on the unit torus $\mathbb{T}^d$, where (CR) describes the dynamics of high-frequency envelopes of solutions [11, 5].

3.2.2. Analysis of the (CR) equation

Upon analyzing it, one soon realizes that (CR) enjoys many interesting properties, symmetries, and even explicit solutions [11, 6]. This is particularly the case in the special dimension $d = 2$ (or $d = 1$ with the quintic nonlinearity), in which the (NLS) nonlinearity that we start with is mass-critical. We mention some of those properties from [11, 16, 17]:

- (CR) is *invariant under the Fourier transform*! In other words, if $g(t)$ is a solution to (CR), then so is its Fourier transform $\widehat{g}(t)$.

- (CR) leaves invariant each of the eigen-spaces of the harmonic oscillator $-\Delta + |x|^2$ (the union of which spans $L^2(\mathbb{R}^2)$).

- (CR) is Hamiltonian, with Hamiltonian functional given by

$$\mathcal{H}(g) = \int_{\mathbb{R}} \int_{\mathbb{R}^2} |e^{is\Delta_{\mathbb{R}^2}} g(x)|^4 ds dx,$$

  a.k.a. the $L^4_{t,x}$ Strichartz norm on $\mathbb{R}^2$.

- (CR) enjoys many stationary solutions, including (up to a phase factor $e^{i\omega t}$) the Gaussian $e^{-\frac{1}{2}\xi^2}$ and $|\xi|^{-1}$. The latter corresponds exactly to the stationary solution $n(\xi) = |\xi|^{-2}$ of the wave kinetic equation (WKE)! (Recall that $n(\xi)$ corresponds to $|g(\xi)|^2$).

The dynamics of equation (CR) in 2D were further analyzed by P. Germain, L. Thomann, and the author from the deterministic [16] and probabilistic [17] point of views. Finally, in a joint work with Laurent Thomann[28], we exhibit the dynamics of (CR) in a completely independent fashion, as an *asymptotic system* for NLS with partial harmonic trapping[4]:

$$(i\partial_t + H)u(t,x) = |u(t,x)|^2 u(t,x); \quad H := -\Delta_{\mathbb{R}^3} + |x'|^2; \quad x = (x_1, x'), x_1 \in \mathbb{R}, x' \in \mathbb{R}^2. \tag{3.5}$$

More precisely, we show that solutions of (3.5) with small initial data (in appropriate spaces) exhibit modified scattering towards (CR) dynamics: i.e. they converge as $t \to \infty$ to solutions of (CR) in an appropriate sense. The proof follows the modified scattering result we described in Section 2.2 for the cubic NLS equation on $\mathbb{R} \times \mathbb{T}^d$ [26]. This result allowed the justification, *and extension*, of some heuristic multiple time-scale approximations in the theory of Bose-Einstein condensation.

[4]This one of the main models in Bose-Einstein condensation used to describe cigar-shaped magnetic traps, and was at theoretical basis of the first observations of *dark solitons*, recognized with a Nobel Prize in 2001.

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