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Abstract

This is the written version of a talk given at the *Journées Équations aux Dérivées Partielles 2016* at Roscoff. We present in this note recent results on the asymptotic behaviour of the Landau equation with Coulomb potential, in both spatially homogeneous and inhomogeneous cases. These results have been obtained in joint works with L. Desvillettes and L. He in [6], and with S. Mischler in [7].

1. Introduction

The Landau equation is a fundamental model in kinetic theory that describes the evolution of a plasma, taking into account collisions between the charged particles. The unknown is the distribution $F = F(t,x,v) \geq 0$ of particles that at time $t \in \mathbf{R}^+$ and position $x \in \mathbf{T}^3$ possess velocity $v \in \mathbf{R}^3$. The Landau equation reads

$$\partial_t F + v \cdot \nabla_x F = Q(F,F), \tag{1.1}$$

with initial condition $F_0 = F_0(x,v) \geq 0$. When the initial condition only depends on the velocity variable, that is $F_0 = F_0(v)$, then the solution $F = F(t,v)$ also depends only on v and verifies the *spatially homogeneous* Landau equation

$$\partial_t F = Q(F,F). \tag{1.2}$$

Equation (1.1) is then referred to as the *spatially inhomogeneous* Landau equation.

The Landau collision operator Q acts only on the velocity variable v and is given by

$$Q(F,G)(v) = \nabla \cdot \int_{\mathbf{R}^3} a(v-v_*)\{F(v_*)\nabla G(v) - \nabla F(v_*)G(v)\}\,dv_*, \tag{1.3}$$

where a is a matrix-valued function that is symmetric, nonnegative and depends on the interaction between particles. One usually assumes that particles interact by an inverse power law potential, in which case a is given by (for $i,j = 1,2,3$)

$$a_{ij}(z) = |z|^{\gamma+2}\,\Pi_{ij}(z), \quad \Pi_{ij}(z) = \delta_{ij} - \frac{z_i z_j}{|z|^2}, \quad -3 \leq \gamma \leq 1. \tag{1.4}$$

Observe that $\Pi(z) := (\Pi_{ij}(z))_{i,j=1,2,3}$ is the orthogonal projection onto $z^\perp$. One usually classifies the different cases as follows

- $0 < \gamma \leq 1$: hard potentials;
- $\gamma = 0$: Maxwellian molecules;
- $-2 \leq \gamma < 0$: moderately soft potentials;

- $-3 < \gamma < -2$: very soft potentials;
- $\gamma = -3$: Coulomb potential.

It is worth mentioning that the Coulomb potential is the most physically interesting case, and hereafter we will be mainly interested in this case (except in Section 1 where we present some fundamental properties of the Landau equation valid to all cases $-3 \le \gamma \le 1$).

1.1. Fundamental properties

We introduce the following quantities

$$b_i(z) = \partial_j a_{ij}(z) = -2\, z_i\, |z|^\gamma,$$

and

$$c(z) = \partial_{ij} a_{ij}(z) = \begin{cases} -2(\gamma+3)|z|^\gamma & \text{if } -3 < \gamma \le 1, \\ -8\pi\, \delta_0(z) & \text{if } \gamma = -3, \end{cases}$$

where here and below we use the convention of implicit summation of repeated indices and the usual shorthands $\partial_i = \partial_{v_i}$, $\partial_{ij} = \partial_{v_i,v_j}$. In this way the Landau operator can be rewritten into two other forms

$$Q(f,g) = \partial_i\{(a_{ij} * f)\, \partial_j g - (b_i * f)\, g\} \tag{1.5}$$

and

$$Q(f,g) = (a_{ij} * f)\, \partial_{ij} g - (c * f)\, g. \tag{1.6}$$

At the formal level, we can write a weak formulation of the Landau operator Q, thanks to (1.3), in the following way: for any smooth test function $\varphi = \varphi(v)$,

$$\int_{\mathbf{R}^3} Q(F,F)(v)\varphi(v)\, dv = -\frac{1}{2} \iint_{\mathbf{R}^3\times\mathbf{R}^3} a_{ij}(v-v_*) \left\{ \frac{\partial_i F}{F}(v) - \frac{\partial_i F}{F}(v_*) \right\} \times \{\partial_j \varphi(v) - \partial_j\varphi(v_*)\}\, F(v_*) F(v)\, dv_*\, dv. \tag{1.7}$$

Furthermore, based on the equations (1.5) or (1.6), another weak formulation also holds at the formal level, namely

$$\int_{\mathbf{R}^3} Q(F,F)(v)\varphi(v)\, dv = \iint_{\mathbf{R}^3\times\mathbf{R}^3} \mathfrak{L}\varphi(v,v_*)\, F(v)\, F(v_*)\, dv_*\, dv, \tag{1.8}$$

with

$$\mathfrak{L}\varphi(v,v_*) = \frac{1}{2} a_{ij}(v-v_*)\big\{\partial_{ij}\varphi(v) + \partial_{ij}\varphi(v_*)\big\} + b_i(v-v_*)\big\{\partial_i\varphi(v) - \partial_i\varphi(v_*)\big\}.$$

Coming back to the weak formulation (1.7), we are now able to deduce two fundamental properties of the Landau collision operator Q, which hold at least formally :

- the conservation of mass, momentum and energy;
- and the (Landau's version of) Boltzmann's H-Theorem;

which we present in details below.

1.1.1. Conservation laws

Taking $\varphi = 1$ or $\varphi(v) = v_\alpha$, for $\alpha \in \{1,2,3\}$, we easily observe that $\partial_j\varphi(v) - \partial_j\varphi(v_*) = 0$ so that (1.7) vanishes. Moreover, for $\varphi(v) = |v|^2$ we get $\partial_j\varphi(v) - \partial_j\varphi(v_*) = 2(v_j - v_{*j})$ and $a_{ij}(v-v_*)(v_j - v_{*j}) = 0$ thanks to (1.4). We hence deduce that the operator conserves (at the formal level) mass, momentum and energy, more precisely

$$\int_{\mathbf{R}^3} Q(F,F)(v)\varphi(v)\, dv = 0 \quad \text{for} \quad \varphi(v) = 1, v_\alpha, |v|^2. \tag{1.9}$$

From this last estimate we obtain that the Landau equation (1.1) (or (1.2)) conserves the mass, momentum and energy, that is, for $\varphi(v) = 1, v_\alpha, |v|^2$ there hold

$$\frac{d}{dt}\int_{\mathbf{T}^3\times\mathbf{R}^3} F(t,x,v)\varphi(v)\,dx\,dv = \int_{\mathbf{T}^3\times\mathbf{R}^3} \big\{Q\big(F(t,x,\cdot),F(t,x,\cdot)\big)(v) - v\cdot\nabla_x F(t,x,v)\big\}\,\varphi(v)\,dx\,dv = 0,$$

or, in the spatially homogeneous case (1.2),

$$\frac{d}{dt}\int_{\mathbf{R}^3} F(t,v)\varphi(v)\,dv = \int_{\mathbf{R}^3} Q\big(F(t,\cdot),F(t,\cdot)\big)(v)\,\varphi(v)\,dv = 0.$$

1.1.2. Boltzmann's H-Theorem

Still from the weak formulation (1.7) (and at a formal level), choosing now the test function $\varphi(v) = \log F(v)$ we deduce the Landau's version of the celebrated Boltzmann's H-Theorem:

- The entropy functional

$$H(F) := \int_{\mathbf{T}^3\times\mathbf{R}^3} F(x,v)\log F(x,v)\,dx\,dv \tag{1.10}$$

is non-increasing along time, more precisely we obtain the following identity

$$\begin{aligned}\frac{d}{dt}H(F) &= -\int D(F)\,dx := \int Q(F,F)(v)\log F(v)\,dx\,dv\\ &= -\frac{1}{2}\int a_{ij}(v-v_*)\left\{\frac{\partial_i F}{F}(v) - \frac{\partial_i F}{F}(v_*)\right\}\left\{\frac{\partial_j F}{F}(v) - \frac{\partial_j F}{F}(v_*)\right\} F(v)\,F(v_*)\,dx\,dv_*\,dv\\ &\le 0,\end{aligned} \tag{1.11}$$

since the matrix a is nonnegative, and where we drop the dependancy on t and x for simplicity. The functional

$$D(F) = -\int Q(F,F)(v)\log F(v)\,dv \tag{1.12}$$

is called the entropy dissipation.

- The global equilibria of (1.1) are global Maxwellian distributions in the velocity variable v (i.e. Gaussian distributions) that are independent of time t and position x.

In the spatially homogeneous case (1.2), we also have the entropy inequality (1.11) (dropping the integral in x), and the equilibria are Maxwellian distributions independent of time.

1.2. Trend to equilibrium

Let us normalise the initial data as (without loss of generality)

$$\int_{\mathbf{T}^3\times\mathbf{R}^3} F_0(x,v)\,dx\,dv = 1,\quad \int_{\mathbf{T}^3\times\mathbf{R}^3} vF_0(x,v)\,dx\,dv = 0,\quad \int_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2F_0(x,v)\,dx\,dv = 3, \tag{1.13}$$

and therefore we consider the associated global Maxwellian equilibrium (centred reduced Gaussian)

$$\mu(v) = (2\pi)^{-3/2}e^{-|v|^2/2}, \tag{1.14}$$

with same mass, momentum and energy of the initial data (normalising the volume of the torus to $|\mathbf{T}^3_x| = 1$). In the spatially homogeneous case, the initial condition $F_0 = F_0(v)$ satisfies the same normalisation as in (2.3) (just dropping the integral in x).

From the H-Theorem presented above in Section 1.1.2, we then expect that solutions $F(t)$ to the Landau equation (in both spatially homogeneous (1.2) and spatially inhomogeneous (1.1) cases) converge to the associated Maxwellian equilibrium μ when time goes to infinity. We are then interested in the following questions:

1. Does $F(t) \to \mu$ as $t \to \infty$? (in some sense to be precised)

2. If yes, at which rate?

Our aim in this note is to present two different results that prove this convergence and give explicit estimates of the rate of convergence in the case of Coulomb potential $\gamma = -3$. For a brief description of known results concerning well-posedness and asymptotic behaviour in the cases $-3 < \gamma \le 1$ we refer to [6, 7] and the references therein.

In Section 2 we present the main result of [6] on the spatially homogeneous equation (1.2) with Coulomb potential ($\gamma = -3$), in which we prove that any global weak solution converges to the associated equilibrium with explicit rates.

In Section 3 we present the main result of [7], in which we study the spatially inhomogeneous equation (1.1) with Coulomb potential ($\gamma = -3$) in a close-to-equilibrium framework (or perturbative regime) and establish new well-posedness and quantitative trend to the equilibrium results.

2. The spatially homogeneous equation

We consider the *spatially homogeneous* Landau equation with Coulomb potential ($\gamma = -3$ in (1.4))

$$\begin{cases} \partial_t F = Q(F,F) \\ F_{|t=0} = F_0. \end{cases} \tag{2.1}$$

We shall always suppose that the initial datum F_0 satisfies the natural physical assumptions: F_0 is nonnegative and has finite mass, energy and entropy, that is

$$F_0 \ge 0, \qquad \int_{\mathbf{R}^3} (1 + |v|^2 + \log F_0(v))\, F_0(v)\, dv < +\infty.$$

From this last bound it is standard to obtain that $F_0 \in L^1(\langle v \rangle^2) \cap L \log L$, $\langle v \rangle := (1+|v|^2)^{1/2}$, more precisely that

$$\int_{\mathbf{R}^3} \left(1 + |v|^2 + |\log(F_0(v))|\right) F_0(v)\, dv < +\infty, \tag{2.2}$$

which we assume hereafter. We also suppose, without loss of generality, that F_0 satisfies the normalisation

$$\int_{\mathbf{R}^3} F_0(v)\, dv = 1, \quad \int_{\mathbf{R}^3} v F_0(v)\, dv = 0, \quad \int_{\mathbf{R}^3} |v|^2 F(v)\, dv = 3, \tag{2.3}$$

and denote by $\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$ the Maxwellian equilibrium with same mass, momentum and energy than F_0.

Existence and uniqueness. In the case of Coulomb potential, the existence of global weak solutions has been established by Arsenev-Penskov [2], Villani [29] and Desvillettes [12] for any initial condition with finite mass, energy an entropy; and Fournier [17] obtained local uniqueness of strong solutions.

Convergence to equilibrium. Let us briefly mention some results concerning quantitative convergence to equilibrium of the Landau equation in the spatially homogeneous case. The case of hard potentials $0 \le \gamma \le 1$ has been addressed in [14, 4], soft potentials $-3 < \gamma < 0$ with bounded kernel (i.e. truncating the singularity of (1.4) at the origin) in [27], and moderately soft potentials $-2 < \gamma < 0$ in [5].

We now describe the main result of [6]. Consider weight functions $\omega = \omega(v) : \mathbf{R}^3 \to \mathbf{R}_+$ of the form

$$\begin{cases} \omega = \langle v \rangle^{\ell}, & \text{with } \ell > 19/2; \\ \omega = \exp(\alpha \langle v \rangle^{\sigma}), & \text{with } 0 < \sigma < 1/2 \text{ and } \alpha > 0, \text{ or } \sigma = 1/2 \text{ and } 0 < \alpha < 2/e. \end{cases} \tag{2.4}$$

We denote the weighted Lebesgue space $L^p(\omega)$, with $1 \le p \le \infty$, as the space associated to the norm

$$\|f\|_{L^p(\omega)} := \|\omega f\|_{L^p}.$$

We define the relative entropy of $F \in L^1$ with respect to μ by

$$H(F|\mu) := \int_{\mathbf{R}^3} F(v) \log\left(\frac{F(v)}{\mu(v)}\right) dv \tag{2.5}$$

and observe that, thanks to the conservation laws and the identity (1.11), we get the following entropy-entropy dissipation identity

$$\frac{d}{dt} H(F(t)|\mu) = -D(F(t)). \tag{2.6}$$

It is also worth mentioning that the relative entropy $H(F|\mu)$ is a strong way to measure the distance between F and μ, more precisely there holds, thanks to the Cziszar-Kullback-Pinsker inequality ([10, 20]),

$$\|F - \mu\|_{L^1}^2 \le 2H(F|\mu).$$

We obtain the following result on the convergence to equilibrium.

Theorem 2.1 ([6, Theorem 2])**.** *Let $F_0 \in L^1(\langle v\rangle^2) \cap L\log L$ satisfy the normalisation* (2.3)*, and consider any global weak solution F to the spatially homogeneous Landau equation* (2.1) *with Coulomb potential ($\gamma = -3$) and with initial data F_0. Let ω satisfy* (2.4) *and assume moreover that $F_0 \in L^1(\omega)$. Then there holds*

$$H(F(t)|\mu) \lesssim \Gamma_\omega(t), \quad \forall\, t \ge 0, \tag{2.7}$$

where

$$\Gamma_\omega(t) = \langle t\rangle^{-\beta}, \quad \text{if } \omega = \langle v\rangle^\ell \tag{2.8}$$

for any $\beta \in (0, \beta_\ell)$ with $\beta_\ell := (2\ell^2 - 25\ell + 57)/(9(\ell - 2))$, or

$$\Gamma_\omega(t) = \exp\left(-\lambda\, \frac{\langle t\rangle^{\frac{\sigma}{3+\sigma}}}{(\log\langle t\rangle)^{\frac{3}{3+\sigma}}}\right), \quad \text{if } \omega = e^{\alpha\langle v\rangle^\sigma}, \tag{2.9}$$

for some constant $\lambda > 0$.

2.1. Overview of the proof of Theorem 2.1

The proof of Theorem 2.1 is based on a variant of the entropy-entropy dissipation method.

We shall briefly present below the entropy-entropy dissipation method, which has been widely used to investigate the large time behaviour of several models in kinetic theory and also other evolution PDEs.

Consider an abstract evolution equation given by

$$\partial_t f = \mathcal{Q}(f), \quad f_{|t=0} = f_0,$$

and suppose that this equation possesses a Lyapunov functional $\mathcal{H}$, usually called entropy, that is the functional $\mathcal{H}$ satisfies

$$\frac{d}{dt}\mathcal{H}(f) \le 0.$$

We then define the associated dissipation functional, usually called entropy dissipation, by

$$\mathcal{D}(f) := -\frac{d}{dt}\mathcal{H}(f).$$

Furthermore, we suppose that there exists an unique equilibrium f_∞ in the sense that

$$\mathcal{Q}(f) = 0 \quad \Leftrightarrow \quad \mathcal{D}(f) = 0 \quad \Leftrightarrow \quad \mathcal{H}(f) = \mathcal{H}(f_\infty) \quad \Leftrightarrow \quad f = f_\infty.$$

We then investigated the existence of functional inequalities relating the entropy dissipation $\mathcal{D}(f)$ to the entropy $\mathcal{H}(f)$ itself. If the method is successful, these inequalities enable us to close a differential inequality for the entropy and hence imply the large time behaviour of the solutions f.

For example, if we are able to obtain an inequality of the form

$$\mathcal{D}(u) \ge \lambda(\mathcal{H}(u) - \mathcal{H}(u_\infty)), \quad \lambda > 0,$$

we hence deduce that solutions $f = (f_t)_{t\geq 0}$ satisfy

$$\frac{d}{dt}(\mathcal{H}(f_t) - \mathcal{H}(f_\infty)) \leq -\lambda(\mathcal{H}(f_t) - \mathcal{H}(f_\infty)),$$

which, by Gronwall's lemma, yields an exponential rate of convergence

$$0 \leq (\mathcal{H}(f_t) - \mathcal{H}(f_\infty)) \leq e^{-\lambda t}\,(\mathcal{H}(u_0) - \mathcal{H}(u_\infty)).$$

In the particular case of Boltzmann and Landau equations of kinetic theory, Cercignani [9] suggested the functional inequality that hopefully links the entropy dissipation and the entropy, and this is known since then as Cercignani's conjecture (see [13] for a detailed description and a review on results).

Let us now describe the main ideas of the proof of Theorem 2.1.

2.1.1. Entropy dissipation estimate and convergence to equilibrium

Consider a solution F to (2.1). We recall that the relative entropy $H(F|\mu)$ was defined in (2.5), the entropy dissipation $D(F)$ in (1.12) and that they satisfy the entropy-entropy dissipation identity (2.6).

Our first step is then to look for inequalities relating $H(F|\mu)$ and $D(F)$. Inspired by some arguments developed by Desvillettes in [12], we obtain a new estimate that bounds from below the entropy dissipation $D(F)$ by a weighted relative Fisher information of F with respect to the associated Maxwellian distribution μ in the following way:

Theorem 2.2. *Let $F \in L^1(\langle v\rangle^2) \cap L\log L$ satisfy the normalisation* (2.3)*. Then there exists a constant $C > 0$ such that*

$$D(F) \geq C\,(M_5(F))^{-1} \int_{\mathbf{R}^3} F(v) \left| \frac{\nabla F(v)}{F(v)} + v \right|^2 \langle v\rangle^{-3}\,dv,$$

where, for any $k \geq 0$, $M_k(F) := \int_{\mathbf{R}^3} (1+|v|^2)^{k/2} F(v)\,dv$, denotes the moment of order k of F.

As a consequence of this last estimate, using the logarithmic Sobolev inequality, we shall prove a variant of the so-called weak Cercignani's conjecture for the Landau equation with Coulomb potential:

Corollary 2.3. *Under the same conditions of Theorem 2.2, there exists $C > 0$ such that*

$$D(F) \geq C\,(M_5(F))^{-1} \int_{\mathbf{R}^3} \left\{ F\log\left(\frac{Z_1}{Z_2}\frac{F}{\mu}\right) + \frac{Z_2}{Z_1}\mu - F \right\} \langle v\rangle^{-3}\,dv,$$

with $Z_1 = \int \langle v\rangle^{-3}\mu$ and $Z_2 = \int \langle v\rangle^{-3} F$.

We finally obtain that, for any $R > 0$, there holds

$$\begin{aligned} D(F) \geq C\,(M_5(F))^{-1}\,R^{-3} \Bigg(H(F|\mu) &- \int_{\langle v\rangle \geq R} F\log F\,dv \\ &- C\int_{\langle v\rangle\geq R} \langle v\rangle^2\,F\,dv - C\int_{\langle v\rangle \geq R} \mu\,dv \Bigg). \end{aligned} \tag{2.10}$$

We then write equation (2.6) and use (2.10) with some $R = R(t)$ (to be chosen later) depending on time. We use the propagation of moments of the solution F (established in [6, Lemma 8 and Corollary 8.1]), that is a control in time of quantities of the form $\|\omega F(t)\|_{L^1}$ for weight functions satisfying (2.4).

Gathering all these estimates together with the following "regularity" type estimate established in [12]:

$$\|\langle v\rangle^{-3} F(t)\|_{L^3} \leq D(F(t)) + C,$$

for some constant $C > 0$ independent of time, we are then able to complete the proof of Theorem 2.1 with some interpolation inequalities and choosing the function $R = R(t)$ in a specific way.

3. The spatially inhomogeneous equation

We consider now the spatially inhomogeneous Landau equation (1.1) with Coulomb potential ($\gamma = -3$ in (1.4)), in a perturbative framework, that is, in a close-to-equilibrium setting.

Consider initial condition F_0 satisfying the normalisation

$$\int F_0(x,v)\,dx\,dv = 1, \qquad \int vF_0(x,v)\,dx\,dv = 0, \qquad \int |v|^2 F_0(x,v)\,dx\,dv = 3,$$

and denote $\mu = (2\pi)^{-3/2} e^{-|v|^2/2}$ the global Maxwellian equilibrium with same mass, momentum and energy than F_0 (considering the normalisation $|\mathbf{T}^3_x| = 1$).

We then define the perturbation

$$f := F - \mu$$

that satisfies

$$\begin{cases} \partial_t f = \Lambda f + Q(f,f), \\ f_0 = F_0 - \mu, \end{cases} \tag{3.1}$$

where

$$\Lambda f := \mathcal{L} f - v \cdot \nabla_x \tag{3.2}$$

is the inhomogeneous linearised operator, and

$$\mathcal{L} f := Q(\mu, f) + Q(f, \mu) \tag{3.3}$$

is the (homogeneous) linearised collision operator.

We consider weight functions $m = m(v) : \mathbf{R}^3 \to \mathbf{R}_+$ satisfying

$$\begin{cases} m = \langle v\rangle^k, & \text{with } k > 2 + 3/2; \\ m = \exp(\kappa\langle v\rangle^s), & \text{with } 0 < s < 2 \text{ and } \kappa > 0, \text{ or } s = 2 \text{ and } 0 < \kappa < 1/2; \end{cases} \tag{3.4}$$

and we denote $\sigma = 0$ when $m = \langle v\rangle^k$ and $\sigma = s$ when $m = e^{\kappa\langle v\rangle^s}$. We denote by $H^2_x L^2_v$ the Sobolev space associated to the norm

$$\|f\|^2_{H^2_x L^2_v} := \|f\|^2_{L^2_{x,v}} + \|\nabla_x f\|^2_{L^2_{x,v}} + \|\nabla^2_x f\|^2_{L^2_{x,v}}$$

where $L^2_{x,v} = L^2(\mathbf{T}^3_x \times \mathbf{R}^3_v)$ is the usual Lebesgue space in $\mathbf{T}^3_x \times \mathbf{R}^3_v$.

The main result of [7] is the following result on the existence, uniqueness and convergence to equilibrium in a close-to-equilibrium framework.

Theorem 3.1 ([7, Theorem 1.1])**.** *Let m be a weight function satisfying* (3.4)*. There exists $\epsilon_0 > 0$ small enough so that, if $\|mf_0\|_{H^2_x L^2_v} \le \epsilon_0$, there exists a unique global weak solution f to* (3.1) *such that*

$$\sup_{t\ge 0} \|mf(t)\|^2_{H^2_x L^2_v} + \int_0^\infty \|\langle v\rangle^{\frac{\sigma-3}{2}} mf(t)\|^2_{H^2_x L^2_v}\,dt + \int_0^\infty \|\langle v\rangle^{-\frac{3}{2}}\, \widetilde{\nabla}_v \{mf(t)\}\|^2_{H^2_x L^2_v}\,dt \lesssim \epsilon_0^2, \tag{3.5}$$

where $\widetilde{\nabla}_v$ is the anisotropic gradient

$$\widetilde{\nabla}_v f = P_v \nabla_v f + \langle v\rangle (I - P_v)\nabla_v f, \quad P_v \nabla_v f = \left(\frac{v}{|v|}\cdot \nabla_v f\right)\frac{v}{|v|}.$$

This solution verifies the decay estimate

$$\|f(t)\|_{H^2_x L^2_v} \lesssim \Theta_m(t)\, \|mf_0\|_{H^2_x L^2_v}, \quad \forall\, t \ge 0, \tag{3.6}$$

where

$$\Theta_m(t) = \langle t\rangle^{-\frac{(k-k_*)}{3}}, \quad \forall\, k_* \in (2 + 3/2, k), \quad \text{if } m = \langle v\rangle^k,$$

or

$$\Theta_m(t) = \exp\left(-\lambda\,\langle t\rangle^{s/3}\right), \quad \text{if } m = e^{\kappa\langle v\rangle^s},$$

for some constant $\lambda > 0$.

Let us mention some known results for the Landau equation with Coulomb potential in the spatially inhomogeneous case. For large data, that is, in a non perturbative setting, based on the theory of renormalised solutions developed by DiPerna-Lions [16], the existence of global renormalised solutions with a defect measure was established by Villani [28] and Alexandre-Villani [1], for any initial datum with finite mass, energy and entropy. Desvillettes and Villani [15] proved algebraic convergence to the equilibrium for a priori smooth solutions with uniform-in-time bounds.

On the other hand, in a perturbative regime, Guo [19] proved well-posedness in the high-order Sobolev space with fast decay in velocity $H^8_{x,v}(\mu^{-1/2}) := \{f \mid \mu^{-1/2} f \in H^8_{x,v}\}$, and Guo and Strain [25, 26] proved sub-exponential convergence to equilibrium also in the same type of space $H^8_{x,v}(\mu^{-\theta})$, $\theta \in (1/2, 1)$.

Our result thus improves the well-posedness theory of Guo [19] to larger spaces $H^2_x L^2_v(m) := \{f \mid mf \in H^2_x L^2_v\}$ as well as the convergence to equilibrium of Guo and Strain [25, 26] to larger spaces and with more accurate rate.

As a corollary of Theorem 3.1, we are able to improve the rate of convergence to equilibrium established in [15] in a non perturbative setting assuming a priori bounds on the solution, in the following way:

Corollary 3.2 ([7, Corollary 1.4])**.** *Consider a global strong solution F to the spatially inhomogeneous Landau equation* (1.1) *such that*

$$\sup_{t\geq 0} \left(\|F(t)\|_{H^\ell_{x,v}} + \|mF(t)\|_{L^1_{x,v}} \right) < +\infty,$$

for some explicit $\ell \geq 3$ large enough and some exponential weight function m satisfying (3.4)*. Assume further that the spatial density is uniformly positive on the torus, that is*

$$\forall\, t \geq 0,\ x \in \mathbf{T}^3, \quad \int_{\mathbf{R}^3} f(t,x,v)\, dv > 0.$$

Then this solution satisfies

$$\|F(t) - \mu\|_{H^2_x L^2_v} \lesssim \Theta_m(t), \quad \forall\, t \geq 0, \tag{3.7}$$

where Θ_m is defined in Theorem 3.1.

3.1. Outline of the proof of Theorem 3.1

The proof of Theorem 3.1 involves two different parts: (1) simple nonlinear estimates for the Landau collision operator Q and a trapping argument; and (2) stability estimates for the semigroup associated to the linearised operator Λ in the corresponding spaces.

It is worth mentioning that our method is mostly based on these semigroup stability estimates. Furthermore, in order to do that, we develop a method to prove non-uniform (non-exponential) stability estimates of semigroups in large functional spaces, by taking advantage of a weak coercivity estimate in one small space and using an enlargement trick for weakly dissipative operators that we develop in [7].

This enlargement trick we develop in [7] is inspired on the extension theory of [18] (introduced in [22]) and it generalises the theory of [18] to the case in which the operator does not possesses a spectral gap (hence its associated semigroup is not exponentially stable).

Let us now describe our method in more details.

3.1.1. Linear stability

We first investigate the linearised (inhomogeneous) operator Λ defined in (3.2) and prove strong (non uniformly exponential) stability estimates for the associated semigroup $S_\Lambda(t)$ in several large Hilbert spaces. These stability estimates are the crucial part of our method and are obtained in several steps.

Step 1. The linearised version of the H-Theorem implies that the homogeneous linearised collision operator $\mathcal{L}$ (defined in (3.3)) satisfies the following weak coercivity inequality in some "small"

Hilbert space E_0:

$$\forall\, f \in \mathrm{Dom}(\mathcal{L}_{|E_0}), \quad \langle \mathcal{L} f, f\rangle_{E_0} \lesssim -\|\Pi_{\mathcal{L}} f\|^2_{E_{0,*}}, \quad E_{0,*} \not\subset E_0.$$

where $\mathrm{Dom}(\mathcal{L}_{|E_0})$ stands for the domain of $\mathcal{L}$ when acting on the space E_0, $\Pi_{\mathcal{L}}$ denotes the projection onto the orthogonal of $\ker(\mathcal{L})$, and $E_{0,*}$ is another Hilbert space.

Using a hypocoercivity trick, we then obtain an analogous weak coercivity estimate for the inhomogeneous linearised operator $\Lambda = \mathcal{L} - v\cdot\nabla_x$ (defined in (3.2)), still in some "small" Hilbert space E,

$$\forall\, f \in \mathrm{Dom}(\Lambda_{|E}), \quad \langle \Lambda f, f\rangle_E \lesssim -\|\Pi f\|^2_{E_*}, \quad E_* \not\subset E, \tag{3.8}$$

where here $\mathrm{Dom}(\Lambda_{|E})$ stands for the domain of Λ when acting on the space E, Π denotes the projection onto the orthogonal of $\ker(\Lambda)$, and E_* is a second Hilbert space (in the norm of which we express the weak dissipativity property of Λ in E).

Step 2. In several (large) Hilbert spaces X, the operator Λ factorises as $\Lambda = \mathcal{A} + \mathcal{B}$ satisfying the following properties : the operator $\mathcal{A} : X \to X$ is bounded; $\mathcal{B}$ is weakly dissipative in the sense

$$\forall\, f \in \mathrm{Dom}(\Lambda_{|X}), \quad \langle \mathcal{B} f, f\rangle_X \lesssim -\|f\|^2_{X_*}, \quad X_* \not\subset X, \tag{3.9}$$

where again $\mathrm{Dom}(\Lambda_{|X})$ stands for the domain of Λ when acting on the space X and X_* is a second Hilbert space; and some convolution power of the operators $\mathcal{A}S_{\mathcal{B}}$ and $S_{\mathcal{B}}\mathcal{A}$ enjoy suitable regularity properties.

We observe here that one cannot deduce any decay estimate on the associated semigroups ΠS_Λ (resp. $S_{\mathcal{B}}$) directly from inequality (3.8) (resp. inequality (3.9)). This framework of weakly dissipative operators is hence more tricky than the classical dissipative case, in which an analogous estimate is obtained with $X_* = X$ and that already implies an exponential decay estimate for the associated semigroup.

Step 3. Using (3.9) with several choices of spaces X and using an interpolation argument, we first obtain that $S_{\mathcal{B}}$ is strongly (non-uniformly exponentially) stable. More precisely, for several choices of Hilbert spaces $X \subsetneq X_0$, we have first

$$\|S_{\mathcal{B}}(t)\|_{X\to X_0} \le \Theta(t) \to 0, \quad \text{as } t\to\infty, \tag{3.10}$$

for some polynomial or sub-exponential decay function $\Theta = \Theta_{X,X_0}$, as well as the regularisation estimate

$$\|S_{\mathcal{B}}(t)\|_{X'_*\to X_0} \le (t\wedge 1)^{-1/2}\,\Theta_*(t), \tag{3.11}$$

for some polynomial decay function $\Theta_* = \Theta_{X'_*,X_0}$ and where X'_* is the dual of X_* for some suitable duality product.

Step 4. Next, by using an extension trick, we deduce that ΠS_Λ also enjoys the decay and regularisation estimates of $S_{\mathcal{B}}$ presented above. More precisely, recalling the factorisation $\Lambda = \mathcal{A} + \mathcal{B}$ and writing iterated Duhamel formulas with this splitting, we deduce that, for any $\ell, n \in \mathbf{N}$, there holds

$$\begin{aligned} S_\Lambda \Pi = &\sum_{0\le j\le \ell-1} \Pi S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*j)} + \sum_{0\le i\le n-1} (S_{\mathcal{B}}\mathcal{A})^{(*i)} * S_{\mathcal{B}}\Pi * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)} \\ &+ (S_{\mathcal{B}}\mathcal{A})^{(*n)} * S_\Lambda \Pi * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}. \end{aligned}$$

We then use factorisation together with the properties of the operators $\mathcal{A}$ and $\mathcal{B}$ presented above, and the decay of $S_\Lambda\Pi$ in some small space (associated to the weak dissipativity (3.8)), which yields that ΠS_Λ enjoys the same estimates of $S_{\mathcal{B}}$, that is

$$\Pi S_\Lambda \text{ satisfies the decay estimate (3.10)},$$

as well as

$$\Pi S_\Lambda \text{ satisfies the regularisation estimate (3.11)}.$$

Step 5. We then define a new norm, for some convenient choice of $\eta, K > 0$,

$$\forall\, f \in \Pi X, \quad |||f|||^2_X := \eta \|f\|^2_X + \int_0^\infty \|S_\Lambda(\tau) f\|^2_{X_0}\, d\tau, \tag{3.12}$$

which is an equivalent norm in ΠX, and for which Λ satisfies the weak dissipativity estimate

$$\forall\, f \in \mathrm{Dom}(\Lambda_{|X}), \quad \langle\!\langle \Lambda f, f\rangle\!\rangle_X \le -K\|\Pi f\|^2_{X_*}, \tag{3.13}$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle_X$ stands for the duality bracket associated to the $|\!|\!| \cdot |\!|\!|_X$ norm.

3.1.2. Nonlinear estimates

We first prove nonlinear estimates for the quadratic operator in the spaces X, X_* of the form

$$\langle Q(f,f), f \rangle_X \leq C \|f\|_X \, \|f\|_{X_*}^2.$$

Together with the estimates of Λ and ΠS_Λ presented above, we are then able to deduce a similar nonlinear estimate of Q for the new norm $|\!|\!| \cdot |\!|\!|_X$ and duality bracket $\langle\!\langle \cdot, \cdot \rangle\!\rangle_X$:

$$\langle\!\langle Q(f,f), f \rangle\!\rangle_X \leq C |\!|\!| f |\!|\!|_X \, \|f\|_{X_*}^2. \tag{3.14}$$

3.1.3. Nonlinear stability

Combining the weak dissipativity estimate (3.13) with the nonlinear estimate (3.14), we finally establish that for any solution $f = F - \mu$ to the Landau equation (3.1), the following a priori estimate holds

$$\frac{d}{dt} \|\Pi f\|_X^2 \leq \|\Pi f\|_{X_*}^2 (-K + C \|\Pi f\|_X).$$

Our existence, uniqueness and asymptotic stability results are then immediate consequences of that last differential inequality and of the estimates it provides.

3.2. Functional spaces and main estimates

We now present the functional setting we work on and the main weakly dissipative estimates corresponding to (3.9) in the method presented above in Sections 3.1.1, 3.1.2 and 3.1.3.

We denote by $L^2_{x,v} = L^2_{x,v}(\mathbf{T}^3_x \times \mathbf{R}^3_v)$ the standard Lebesgue space on $\mathbf{T}^3_x \times \mathbf{R}^3_v$. For a velocity weight function $m = m(v) : \mathbf{R}^3_v \to \mathbf{R}_+$, we then define the weighted Lebesgue $L^2_x L^2_v(m)$ and Sobolev spaces $H^n_x L^2_v(m)$, associated to the norms

$$\|f\|^2_{L^2_x L^2_v(m)} := \|mf\|^2_{L^2_{x,v}}$$

and

$$\|f\|^2_{H^2_x L^2_v(m)} := \|mf\|^2_{L^2_{x,v}} + \|\nabla_x (mf)\|^2_{L^2_{x,v}} + \|\nabla^2_x (mf)\|^2_{L^2_{x,v}}.$$

We also define the space $\mathcal{H}^1_{x,v}(m)$, for a weight function m satisfying (3.4), as the space associated to the norm defined by

$$\|f\|^2_{\mathcal{H}^1_{x,v}(m)} := \|mf\|^2_{L^2_{x,v}} + \|\nabla_x (mf)\|^2_{L^2_{x,v}} + \|\langle v \rangle^{\frac{\sigma}{4} - \frac{3}{2}} \nabla_v (mf)\|^2_{L^2_{x,v}}, \tag{3.15}$$

where we recall that σ is defined in (3.4). Consider the space $H^1_{v,*}(m)$ associated to the norm

$$\|f\|^2_{H^1_{v,*}(m)} := \|m \langle v \rangle^{(\sigma-3)/2} f\|^2_{L^2_v} + \|\langle v \rangle^{-3/2} \widetilde{\nabla}_v (mf)\|^2_{L^2_v}, \tag{3.16}$$

where $\widetilde{\nabla}_v$ is the anisotropic gradient

$$\widetilde{\nabla}_v f = P_v \nabla_v f + \langle v \rangle (I - P_v) \nabla_v f, \quad P_v \nabla_v f = \left(\frac{v}{|v|} \cdot \nabla_v f \right) \frac{v}{|v|}.$$

When furthermore m is a polynomial weight function, we define the negative Sobolev space $H^{-1}_*(m)$ in duality with $H^1_*(m)$ with respect to the duality product on $L^2(m)$, more precisely

$$\|f\|_{H^{-1}_{v,*}(m)} = \sup_{\|\phi\|_{H^1_{v,*}(m)} \leq 1} \langle mf, m\phi \rangle_{L^2_v}. \tag{3.17}$$

The space $H^2_x(H^1_{v,*}(m))$ is associated to the norm

$$\|f\|^2_{H^2_x(H^1_{v,*}(m))} := \sum_{j=0}^{2} \| \, \|\nabla^j_x f\|_{H^1_{v,*}(m)} \, \|^2_{L^2_x}. \tag{3.18}$$

When furthermore m is a polynomial weight function, we also define the negative weighted Sobolev space $H^2_x(H^{-1}_{v,*}(m))$ in duality with $H^2_x(H^1_{v,*}(m))$ with respect to the $H^2_x L^2_v(m)$ duality product, more precisely

$$\|f\|_{H^2_x(H^{-1}_{v,*}(m))} \quad := \sup_{\|\phi\|_{H^2_x(H^1_{v,*}(m))}\leq 1} \langle mf, m\phi\rangle_{H^2_x L^2_v}.$$

For a weight function m satisfying (3.4), the spaces corresponding to the method presented in Section 3.1 are then

$$X = H^2_x L^2_v(m), \quad X_0 = H^2_x L^2_v,$$
$$X_* = H^2_x(H^1_{v,*}(m)), \quad X'_* = H^2_x(H^{-1}_{v,*}(m)).$$

Let us now present some of the key estimates. First of all, on the space $L^2_v(\mu^{-1/2})$, we classically observe that the homogeneous linearised operator $\mathcal{L}$ is self-adjoint and verifies $\langle \mathcal{L}f, f\rangle_{L^2_v(\mu^{-1/2})} \leq 0$, so that its spectrum satisfies $\Sigma(\mathcal{L}) \subset \mathbf{R}_-$. We also have the following weak coercivity inequality from [11, 3, 19, 21, 24]

$$\langle \mathcal{L}f, f\rangle_{L^2_v(\mu^{-1/2})} \lesssim -\|\Pi_{\mathcal{L}} f\|^2_{H^1_{v,*}(\mu^{-1/2})}, \quad \forall f \in L^2_v(\mu^{-1/2}),$$

where $\Pi_{\mathcal{L}}$ is the projection onto the orthogonal of ker$\mathcal{L}$. From this estimate and a hypocoercivity argument developed in [23], we get an analogous estimate for the inhomogeneous linearised operator Λ in the space $\mathcal{H}^1_{x,v}(\mu^{-1/2})$ associated to an equivalent norm. This gives us the first step presented in Section 3.1.1.

We introduce the factorisation $\Lambda = \mathcal{A} + \mathcal{B}$ with

$$\mathcal{A}f = Q(f,\mu) + M\chi_R f, \quad \mathcal{B}f = Q(\mu, f) - M\chi_R f - v\cdot\nabla_x f,$$

for constants $M, R > 0$ to be chosen large enough and where χ_R is a smooth cutoff function, that is, $\chi_R(\cdot) = \chi(\cdot/R)$, $0 \leq \chi \in C^\infty_c(\mathbf{R}^3)$, $\chi(x) = 1$ if $|x| \leq 1$ and $\chi(x) = 0$ if $|x| > 2$.

The main estimates and properties presented in steps 2 to 5 of Section 3.1.1 are variants and consequences of the following weak dissipativity estimates of $\mathcal{B}$: there exist $M, R > 0$ large enough such that

- $\mathcal{B}$ is weakly dissipative in $H^2_x L^2_v(m)$, in the sense
$$\langle \mathcal{B}f, f\rangle_{H^2_x L^2_v(m)} \lesssim -\|f\|^2_{H^2_x(H^1_{v,*}(m))}.$$
- $\mathcal{B}$ is weakly dissipative in $\mathcal{H}^1_{x,v}(m)$, in the sense
$$\langle \mathcal{B}f, f\rangle_{\widetilde{\mathcal{H}}^1_{x,v}(m)} \lesssim -\|f\|^2_{\widetilde{\mathcal{H}}^1_{x,v}(m\langle v\rangle^{(\sigma-3)/2})}$$
where $\|\cdot\|_{\widetilde{\mathcal{H}}^1_{x,v}(m)}$ is a equivalent norm on $\mathcal{H}^1_{x,v}(m)$ and $\langle\cdot,\cdot\rangle_{\widetilde{\mathcal{H}}^1_{x,v}(m)}$ is its associated scalar product.

From all the estimates obtained in Section 3.1.1, following [19, 8] we can then obtain the corresponding nonlinear estimates (that is, in the same spaces) of Section 3.1.2 and 3.1.3.

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