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Sharp parabolic regularity and geometric flows on singular spaces

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Abstract

This is a brief survey about regularity expansions for solutions of elliptic and parabolic problems on spaces with conic singularities. The results themselves are closely related to classical results about elliptic boundary problems, and analogues of these are expected to hold on quite general stratified spaces with incomplete iterated edge metrics. The emphasis here is on the interpretation and application of these expansions to geometric problems.

1. Introduction

This short paper surveys a few recent results about elliptic and parabolic problems on singular spaces. The focus here is on two nonlinear geometric flows, and the way that what we call sharp regularity for solutions of the linearized problems enters into their analysis.

The study of elliptic and parabolic partial differential equations arising in geometric analysis on stratified spaces has received substantial attention in the past few decades, and certain parts of the subject are reaching some state of maturity. In the simplest geometric settings, isolated conic and simple edge singularities, this linear theory is now quite well understood. Our aim here is to show how some apparently purely technical results concerning 'sharp regularity' for solutions of such linear and nonlinear equations turn out to have interesting geometric meaning. We explain a few such results here. To keep the discussion simple, we focus primarily on spaces with isolated conic singularities. In addition, our examples are in the lowest dimensional cases: the Gauss curvature equation on surfaces with conic singularities and the associated Ricci flow equation, and the flow by curvature of a network of curves in the plane or on a surface. One goal here is to explain why it is fruitful to regard this latter problem as having a conic nature. We conclude by discussing some known and expected extensions of these results to higher dimensions.

2. Surfaces with conic points

Fix a compact surface M and a conformal structure $\mathfrak{c}$ on it. A metric g representing $\mathfrak{c}$ is said to have a conic singularity at a point p if in local holomorphic coordinates compatible with $\mathfrak{c}$ it takes the form
$$g = |z|^{2\beta} e^{2\phi} \overline{g},$$
where $\overline{g}$ is a smooth metric representing $\mathfrak{c}$ and where ϕ is at least bounded (though we require further regularity below). The number β determines the cone angle at p as follows. In the model case $\overline{g} = |dz|^2$ and $g_\beta = |z|^{2\beta}|dz|^2$. In polar coordinates $|dz|^2 = d\rho^2 + \rho^2 d\theta^2$, so changing coordinates by $r = \rho^{1+\beta}/(1+\beta)$ yields that
$$g_\beta = \rho^{2\beta} d\rho^2 + (1+\beta)^2 \rho^{2\beta+2} d\theta^2 = dr^2 + (1+\beta)^2 r^2 d\theta^2.$$

Since $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$, this space is the complete cone $C(S^1_{2\pi(1+\beta)})$ over the circle of circumference $2\pi(1+\beta)$. We always take $\beta > -1$, and shall focus almost entirely on the case where $\beta_j < 0$, i.e., all cone angles are less than 2π.

The first question we consider here involves the relationship between the regularity of the conformal factor ϕ and the Gauss curvature of the metric $g = e^{2\phi}g_\beta$. More generally, if g_1 and $g_2 = e^{2\phi}g_1$ are any two conformally related conic metrics, and ϕ is bounded, then we consider the relationship between the regularity of ϕ and the Gauss curvatures K_1 and K_2 of g_1 and g_2, respectively. To approach this, recall the classical transformation formula

$$\Delta_1 \phi - K_1 + K_2 e^{2\phi} = 0.$$

When g_1 and g_2 are smooth, it is standard that ϕ gains precisely two orders of regularity from the K_j. Thus for example if $K_1, K_2 \in \mathcal{C}^{0,\alpha}$, then $\phi \in \mathcal{C}^{2,\alpha}$. The correct analogue of this statement when g_1 and g_2 are conic lies at the heart of the sharp regularity of the title.

Before stating this result, we review a few facts about the linear problem $\Delta_1 u = f$, when g_1 is conic. These are stated using weighted Sobolev and Hölder spaces,

$$r^\nu H_b^k = \{u = r^\nu v : v \in H_b^k\}$$
$$r^\mu \mathcal{C}_b^{k,\alpha} = \{u = r^\mu v : v \in \mathcal{C}_b^{k,\alpha}\}.$$

In terms of the modified polar coordinates near each cone point, $v \in H_b^k$ if $(r\partial_r)^i \partial_\theta^j v \in L^2$, $i+j \leq k$, while $\mathcal{C}_b^{k,\alpha} = \{v : (r\partial_r)^i \partial_\theta^j v \in \mathcal{C}_b^{0,\alpha} \text{ for all } i+j \leq k\}$, where $v \in \mathcal{C}_b^{0,\alpha}$ if $\sup |v| < \infty$ and

$$\sup_{(r,\theta)\neq(r',\theta')} \frac{|v(r,\theta) - v(r',\theta')|(r+r')^\alpha}{|r-r'|^\alpha + (r+r')^\alpha|\theta - \theta'|^\alpha} < \infty.$$

Notice that the vector fields $r\partial_r$ and ∂_θ are invariant under the homotheties $(r,\theta) \mapsto (\lambda r, \theta)$, as is the Hölder seminorm, so all these space are invariant under these scalings. These spaces are the natural ones with respect to the vector fields $r\partial_r, \partial_\theta$; even the Hölder seminorm represents a fractional derivative with respect to these vector fields.

In two dimensions, near a cone point p with cone angle parameter β,

$$\Delta_1 = e^{-2\phi}\Delta_0, \qquad \Delta_0 = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{(1+\beta)^2 r^2}\partial_\theta^2.$$

Hence the linear problem $\Delta_1 u = f$ is the same as $\Delta_0 u = fe^{2\phi}$.

The following discussion is local, and all of the results below are proved in [22]. The simplest case is when $f \equiv 0$, and it is natural then to search for separable solutions $u = A(r)B(\theta)$. This leads immediately to the family of such decomposable solutions

$$r^{j/(1+\beta)}(a\cos(j\theta) + b\sin(j\theta)), \quad j \in \mathbb{Z}.$$

In the special case $j = 0$, the solutions are 1 and $\log r$. A general solution of this homogeneous equation is an infinite superposition of these. If f is smooth and vanishes to infinite order at $r = 0$, then the correct conclusion is that if u is bounded, then

$$u \sim \sum_{j=0}^\infty r^{j/(1+\beta)}(a_j \cos(j\theta) + b_j \sin(j\theta)),$$

where the sequences a_j and b_j are rapidly decreasing. This is an example of a polyhomogeneous expansion. More generally, a polyhomogeneous expansion is one of the form $u \sim \sum a_{\ell,p} r^{\gamma_\ell}(\log r)^p \phi_\ell(\theta)$ where the γ_ℓ have real parts tending to infinity, there are only finitely many values of $p \in \mathbb{N}$ for each ℓ, and the ϕ_ℓ are $\mathcal{C}^\infty$. These results can be restated as saying that if $\Delta_0 u = f$ and f is smooth and vanishes to all orders, then u is polyhomogeneous. It is not hard to deduce from this that if f is polyhomogeneous, then u us polyhomogeneous.

Now suppose that the conformal factor ϕ is polyhomogeneous at each conic point, and let u be in the maximal domain of Δ_1. This means that $u \in L^2$ and $\Delta_1 u = f \in L^2$. A basic regularity result shows that in this case,

$$u = \tilde{a_0}\log r + a_0 + r^{1/(1+\beta)}(a_{11}\cos\theta + a_{12}\sin\theta) + \tilde{u}, \tag{2.1}$$

where $\tilde{u} \in r^2 H_b^2$ and $a_0, \tilde{a_0}$ and the a_{ij} are constants. When $\beta \in (0, -1/2)$ then $1/(1+\beta) > 2$ so the set of terms which have this exponent should be subsumed into the error term $\tilde{u}$. This is

known as a *partial expansion*. To explain it, note first that local (scale-invariant) elliptic regularity implies that $u \in H_b^2$. However, this only guarantees that $\Delta_1 u \in r^{-2} L^2$, so the extra information that $\Delta_1 u = f \in L^2$ means that any term in the expansion for u vanishing less quickly than r^2 must be annihilated by the operator. The extra initial terms in this partial expansion are precisely of the form so that this happens.

There is a canonical densely defined subspace $\mathcal{D}_{\mathrm{Fr}}$, the so-called Friedrichs domain, which is a self-adjoint extension of the Laplacian (or of $\Delta_1 + V$ for any bounded V, for example). It is characterized by the fact that $u \in \mathcal{D}_{\mathrm{Fr}}$ if $u\nabla u, \Delta_1 u \in L^2$. This condition on the gradient precludes the $\log r$ term in the expansion. In particular, if a solution u to $\Delta_1 u = f$ is bounded, then it must lie in the Friedrichs domain.

There is an analogue of this definition for weighted Hölder spaces. thus if $f \in \mathcal{C}_b^{0,\alpha}$, then local elliptic estimates (taking advantage of the scaling) shows that $u \in \mathcal{C}_b^{2,\alpha}$. However, if this is all we know, then $\Delta_0 u \in r^{-2} \mathcal{C}_b^{0,\alpha}$. As before, this can only happen if there is some cancellation. One proves that u has an expansion exactly as in (2.1), but now with $\tilde{u} \in r^2 \mathcal{C}_b^{2,\alpha}$. The assumption that u is bounded means that $\tilde{a}_0 = 0$. Accordingly, we define the Hölder-Friedrichs domain $\mathcal{D}_{\mathrm{HFr}}$ as the set of $u \in \mathcal{C}_b^{2,\alpha}$ which are bounded and which satisfy $\Delta_0 u \in \mathcal{C}_b^{0,\alpha}$.

We now return to the nonlinear problem. These linear results can be easily adapted to prove the following

Proposition 2.1. *If the two metrics g_1 and g_2 have $\mathcal{C}_b^{2,\alpha}$ coefficients, have the same cone angles, and are conformally related, $g_2 = e^{2\phi} g_1$ where ϕ is bounded, and K_1, $K_2 \in \mathcal{C}^{0,\alpha}$, then*

$$\phi = a_0 + r^{1/(1+\beta)}(a_{11} \cos\theta + a_{12} \sin\theta) + \tilde{\phi}, \qquad \tilde{\phi} \in r^2 \mathcal{C}_b^{2,\alpha}.$$

Now consider the uniformization question in this setting. Fix a compact Riemann surface $(M, \mathfrak{c})$ and suppose that $\mathfrak{p} = \{p_1, \ldots, p_k\}$ is a simple divisor, i.e., a set of distinct points on M, and $\vec{\beta} = \{\beta_1, \ldots, \beta_k\}$ is a k-tuple of cone angle parameters. We ask whether there is a metric g representing $\mathfrak{c}$, with these specified cone angles at the points p_j, and with constant curvature K. This can be recast as a PDE as follows by starting with any conic metric g_1 in this conformal class with these specified cone angles, and which has smooth or polyhomogeneous coefficients in (r, θ). Then $g_2 = e^{2\phi} g_1$, solves this uniformization problem if

$$\Delta_1 \phi - K_1 + K e^{2\phi} = 0.$$

We can of course use this equation as a way to find solutions. There is a constraint given by the analogue of the Gauss-Bonnet formula in this setting, namely that if a solution exists, then

$$\int_M K_{g_0} \, dA_{g_0} = \int_M K_0 \, dA_g = 2\pi \left(\chi(M) + \sum_{j=1}^k \beta_j \right)$$

If all the cone angles are less than 2π, then a complete answer is known: this problem has a solution which is unique up to some obvious symmetries if and only if $\chi(M) \leq 0$ or else $\chi(M) > 0$ and $\vec{\beta}$ must satisfy

$$\beta_j > \sum_{i \neq j} \beta_i \qquad \text{for all } j. \tag{2.2}$$

This is called the *Troyanov* condition; it is the lowest dimensional version of the K-stability condition in Kähler geometry.

Theorem 2.2 (McOwen [27], Troyanov [36])**.** *If all cone angles are less than 2π, then there exists a solution of this problem if and only if $\vec{\beta}$ satisfies the Troyanov condtion. This solution is unique except when the solution metric g is flat, in which case there is a one-parameter family of scalings, or else if $M = S^2$, $k = 2$ and $\beta_1 = \beta_2$, in which case there is a one-parameter family of solutions generated by the family of dilations of S^2 which fix both conic points.*

The existence and uniqueness (or more likely, multiplicity) theory when any of the cone angles are greater than 2π is much less well understood. We refer to the work of Malchiodi and his collaborators [5, 8, 20] and Mondello and Panov [28] for the best current results in these directions.

The deformation theory of conic metrics with constant curvature and all cone angles less than 2π has only recently been carried out [26]. That paper analyzes this deformation theory using

tools of geometric analysis similar to Tromba [35] for smooth metrics without conic points. The key new tools needed in the conic setting are the mapping and regularity properties of solutions of the Laplacian on functions, 1-forms and symmetric two-tensors. Since one is dealing with exactly hyperbolic metrics, the regularity theorem above prove that solutions of the various equations encountered here are polyhomogeneous. Their complete asymptotic expansions involve the basic terms $r^{j/(1+\beta)}(a\cos(j\theta) + b\sin(j\theta))$. The mapping properties of these operators are proved in [22, Section 4]. Refined regularity results about solutions of these various problems follow using the techniques sketched above. To connect this with the general theme of this paper, however, the precise identification of the terms in these expansions with small exponents is required to understand the nonlinear deformations.

We turn now to the associated dynamical problem: the behavior of conic metrics under the Ricci flow. Fix an initial conic metric g_0 and suppose that the 1-parameter family of $g(t)$ satisfies the normalized Ricci flow equation with $g(0) = g_0$. In two dimensions $\mathrm{Ric}(g) = Rg$, where R is the scalar curvature of g, so the flow equation is

$$\frac{\partial g}{\partial t} = \rho g - \mathrm{Ric}(g) = (\rho - R)g;$$

we include the renormalizing constant $\rho = 2\pi\chi(M)/\mathrm{Area}(M, g_0)$ to ensure that the area of $(M, g(t))$ remains constant. This flow preserves the conformal class, so we can rewrite this equation as a scalar quasilinear problem in terms of the function $u(t, \cdot)$, where $g(t) = u(t)g_0$:

$$\frac{\partial u}{\partial t} = \Delta_0 \log u - R_{g_0} + \rho u.$$

There are two separate issues. Unlike when M is smooth, short-time existence is not straightforward here, and even once that has been accomplished, long-time existence and convergence as $t \to \infty$ present further interesting challenges.

The difficulty with short-time existence is that the problem is not well-posed. In fact, there are multiple different solutions emanating from any initial conic metric g_0. One of these is a solution $g(t)$ for which the conic singularity disappears entirely for $t > 0$. This was proved by M. Simon [34] and in somewhat greater generality by Koch and Lamm [18]. Both of these papers study the Ricci flow equation for manifolds with general Lipschitz initial data, of which metrics with a finite number of conic singularities on surfaces are a very special case. Note that in higher dimensions, conic singularities may not be smoothable since spaces with isolated conic singularities need not even be topological manifolds. This result is not so surprising given the regularizing properties of the flow. There is another solution to this flow which is much more surprising: for this solution, produced by Giesen and Topping [12, 13], the incomplete conic metric on $M \setminus \{p_1, \ldots, p_k\}$ becomes instantaneously complete. Finally, a main result of [23] states that there is a unique family of conic metrics $g(t)$ for which the cone angles remain constant. Slightly more generally, there is a solution with any prescribed smooth function $\vec{\beta}(t)$ of cone angle parameters. Another earlier approach to this was given by Yin [37, 38] using an approximation technique. We also refer to the work of Ramos [31]. We advertise the method of [23] however because it addresses the regularity issues directly, and the asymptotics of solutions obtained this way are used in the long-time existence and convergence steps. These asymptotics will also be helpful for studying further problems, for example the evolution of global spectral invariants under the flow.

Consider first the linearized problem

$$\frac{\partial \phi}{\partial t} = L_0 \phi := \Delta_0 \phi - R_0 \phi. \tag{2.3}$$

To preserve the cone angle, we require that ϕ remain bounded near $r = 0$. This determines which solution operator for this parabolic problem we must use. Namely, following the earlier discussion, we must choose the solution such that $\phi(t, \cdot)$ lies in the Hölder-Friedrichs domain for L_0 for each $t > 0$. (This is the solution given by the heat semigroup corresponding to the Friedrichs extension of L_0.)

There are two types of parabolic Hölder spaces here. The first, $\mathcal{C}_{b0}^{k+\alpha,(k+\alpha)/2}$, is based on differentiations with respect to the vector fields $r^2\partial_t, r\partial_r, \partial_\theta$. These spaces are fully dilation invariant with respect to the parabolic rescalings $(t, r, \theta) \mapsto (\lambda^2 t, \lambda r, \theta)$. These do not control regularity in t when $r = 0$, and so do not give information about the terms in the expansion of a solution as $r \to 0$ for

$t > 0$. To focus on this region, we also use spaces $\mathcal{C}_b^{k+\alpha,(k+\alpha)/2}$, which are based on differentiations with respect to $\partial_t, r\partial_r, \partial_\theta$. The clear advantage here is that these control regularity in t even at the conic tip where $r = 0$; the downside is that it is harder to prove estimates for the solution operator on these spaces. In fact, it is necessary to first obtain estimates in the former spaces, which can be done rather simply using scaling arguments and classical local parabolic regularity estimates, and pass from these to estimates on the latter spaces, which is done using commutator and interpolation arguments.

The following is proved in [23].

Proposition 2.3. *If* $f \in \mathcal{C}_b^{k+\alpha,(k+\alpha)/2}$ *then the unique solution* u *to*

$$\frac{\partial u}{\partial t} = L_0 u + f, \qquad u(0, \cdot) = 0$$

lies in $\mathcal{C}_b^{k+2+\alpha,(k+2+\alpha)/2}$ *For all* $t > 0$ *there is a decomposition*

$$u(t, \cdot) = u_0(t) + r^{1/(1+\beta)}(u_{11}(t)\cos\theta + u_{12}(t)\sin\theta) + \tilde{u}$$

where $\tilde{u} \in r^2 \mathcal{C}_b^{k+2+\alpha,(k+2+\alpha/2)}$.

An interpretation of the part of this result concerning expansions is that the Hölder-Friedrichs domain propagates under this linear evolution. It is straightforward to deduce from this that there exists a solution ϕ to the nonlinear Ricci flow equation for some small time interval $0 < t < T$ which is bounded and has a partial expansion near each cone point. If the initial metric g_0 is polyhomogeneous, then $g(t)$ is polyhomogeneous for each fixed $t > 0$. This also implies that if K_0 is bounded, then $K_{g(t)}$ is also bounded when $t > 0$.

We shall not say much about how these results are obtained, referring to [23] for a careful explanation. The main point, however, is that we rely on a detailed geometric description of the Schwartz kernel of the heat operator. This is a distribution on $\mathbb{R}^+ \times M \times M$, and the result which we borrow from geometric microlocal analysis is that the heat kernel lifts to a resolution of this space as a function which is polyhomogeneous at all boundary faces. This structure was proved originally in [29] and used in [16] to obtain certain mapping properties, but ones which do not go as far as the ones stated here.

The next issue is long-time existence of the flow. Hamilton [14] laid out an effective strategy for proving this on surfaces. A key ingredient is the existence of a potential function f for the flow. By definition this is a solution to the equation $\Delta_{g(t)}f = R_{g(t)} - \rho$ such that both f and $|\nabla f|$ are bounded for each t. We have already explained that f must have an expansion as in (2.1), and if we choose the solution in the Hölder-Friedrichs domain, then $\tilde{a_0} = 0$ so f is bounded. This expansion also implies that $|\nabla f|$ is bounded, but interestingly this only holds when all cone angles are less than 2π. It is possible to follow Hamilton's argument through in the conic case for cone angles in this range; long-time existence when some of the cone angles are greater than 2π is not known and might possibly fail.

The final issue is convergence of the solution as $t \to \infty$, which is more difficult. We prove in [23] that if $\vec{\beta}$ does satisfy the Troyanov condition, then $g(t)$ converges exponentially quickly to the uniformizing constant curvature metric. However, if $\vec{\beta}$ does not satisfy the Troyanov condition then one cannot expect such convergence since there is no stationary solution to which $g(t)$ could flow. In this case we prove that there exists a unique point $p \in M = S^2$ such that $u \nearrow \infty$ in any small ball $B_\epsilon(p)$ (measured with respect to the initial metric g_0), and $u \searrow 0$ on $M \setminus B_\epsilon(p)$. We also show the existence of a family F_t of diffeomorphisms of M such that $F_t^* g(t)$ converges to a metric g_∞ which is either smooth and has constant curvature or is a soliton metric with either 1 or 2 conic points. The natural conjecture is that the point of divergence p is one of the conic points (and in fact is the unique point for which (2.2) fails), and furthermore that the diffeomorphism F_t by which one pulls back are conformal. A different approach to convergence using more powerful geometric machinery which includes these stronger results was given by Phong et al. [30].

3. The network flow

We now discuss a rather different looking geometric problem which turns out to have very similar analytic underpinnings. This is the flow by curvature of networks of curves in the plane, or in an

arbitrary surface. This extends the well-known theory, studied in the early days of geometric flows by Gage and Hamilton, Grayson and others, concerning the evolution problem

$$\frac{\partial \gamma}{\partial t} = \kappa\, n, \tag{3.1}$$

where γ is a curve, n its unit normal and κ its curvature. There is a beautiful and complete theory when γ is either closed or else is an arc and satisfies some natural boundary conditions (e.g. the boundary points could be fixed, or the curve could be required to meet a given boundary orthogonally). If the initial curve γ_0 is $\mathcal{C}^2$ and embedded, then $\gamma(t)$ exists on some finite interval $0 \le t < T_*$; no matter the shape of γ_0, there exists an intermediate time $T_0 < T_*$ such that $\gamma(t)$ is convex for $t > T_0$, and becomes increasingly circular and shrinks to a point as $t \nearrow T*$.

The generalization we have in mind is motivated by applications to the evolution of grain boundaries. One considers a network of curves, i.e., a locally finite collection of curve segments $\{\gamma_j\}$, with various subcollections meeting at internal nodes. These curves might also have external boundary points satisfying some auxiliary boundary conditions, but we do not specify these or discuss these further here. One then seeks to flow this entire ensemble according to the evolution law (3.1). This equation makes sense away from the boundary points of each curve, but it is not immediately apparent how to formulate this problem near these points. The first successful attempt was carried out by Brakke [6], who interpreted Γ as a varifold and studied a weak formulation of this curvature flow. This is an efficient and beautiful setup, but as usual with any weak interpretation, leaves open many questions about the geometry of the evolving varifolds. In particular, solutions of this flow are not necessarily unique, and more to the point, one does not obtain easy information about the geometry of these evolving curves and how their intersections vary.

This leads to the goal of interpreting this problem in a strong sense using partial differential equations. The basic formulation is as follows. We first add a tangential term to (3.1) and suppose that each curve γ_j

$$\partial_t \gamma_j = \frac{\partial_s^2 \gamma_j}{|\partial_s \gamma_j|^2}. \tag{3.2}$$

This extra tangential term allows the internal vertices to move. As for the 'internal boundary conditions, if $\gamma_1, \ldots, \gamma_k$ intersect at some node p, corresponding to parameter value $s = 0$ on each curve, then we require that

$$\gamma_1(t,0) = \ldots = \gamma_k(t,0), \qquad \sum_{j=1}^{k} \frac{\partial_s \gamma_j(t,0)}{|\partial_s \gamma_j(t,0)|} = 0. \tag{3.3}$$

The geometric content of (3.3) is that the unit tangents to all curves in the network meeting at p must be equally spaced. It is discussed in [6] that the stable configurations are those for which only three curves meet. Thus we call a node *regular* if it is trivalent and if the curves meeting there make a mutual angle of $2\pi/3$. A network is regular if all of its nodes are regular.

It was proved by Bronsard and Reitich [7] that these nonlinear boundary conditions satisfy the Lopatinski-Shapiro property, and it can then be proved that if an initial network Γ_0 is regular, and if a set of higher order compatibility conditions are satisfied at each vertex, then this system admits at least a short-time solution. The proof invokes standard results for quasilinear parabolic problems in ordinary Hölder spaces. The higher order compatibility conditions are what make it possible to work in these function spaces. A later paper by Mantegazza, Novaga and Tortorelli [21] clarifies and sharpens this analysis, and shows that a short-time solution still exists even without these compatibility conditions, but still requiring that all nodes of Γ_0 are regular.

The full story about the long-time behavior of this flow is not well understood. Complications include that certain curves can pinch off and the evolving network may cease to be regular at these points of time. Results in this direction are obtained [21, 15].

Not only to understand how to continue the flow past these singular times, but also just to have a complete theory, it is necessary to consider the problem of short-time existence when the initial network Γ_0 is not regular, and this is our principal interest here.

The first step toward this is to classify the expanding soliton solutions. By definition, an expanding soliton is a solution of this system on $\mathbb{R}^2$ such that $\Gamma(t)$ is a dilate of the solution at $t = 1$;

the scaling which arises in a natural parametrization is that

$$\gamma_j(t,s) = \sqrt{t}\,\tilde{\gamma}_j(s/\sqrt{t}).$$

The initial network for an expanding soliton is a union of rays emanating from the origin, and the directions of these rays are the same as the asymptotic directions of each of the curves in the soliton network at later times which extend to infinity. If there are only three such rays and they already meet with an angle of $2\pi/3$, then this configuration remains the same for all time. The next case is when Γ_0 consists of three ray meeting at unequal angles. The soliton solutions for such configurations were obtained by Schnürer and Schulze [33]. A later paper by the author and Saez [24] provides a sort of classification of all expanding solitons, which can be described as follows. It is sufficient to consider the network at $t = 1$. Each curve Γ_j in the network $\Gamma(1)$ satisfies a dimensional reduction of the flow equations which is in fact simply the geodesic equations for the metric $g_s = e^{|z|^2}|dz|^2$ on $\mathbb{R}^2$. This is a complete metric with negative curvature, so it is natural to compactify (R^2, g_s) as the interior of the disk B^2. The k rays in the initial configuration correspond to k points $q_1, \ldots, q_k$ on the circle at infinity. The network $\Gamma(1)$ is then a minimizing geodesic Steiner tree with asymptotic boundary at these k points. The local regularity theory for such minimizing trees ensures that each interior node is regular. The existence of at least one such minimizing geodesic Steiner tree for any given collection of points $q_1, \ldots, q_k \in S^1$ is proved in [24] using fairly simple arguments from geometric measure theory. Simple examples show that there may be more than one solution. In other words, for an initial network of k intersecting rays, there is at least one, and usually many different ways that it can 'explode' into a regular network. The single node at the origin bursts into a set of $k-2$ separate nodes for $t > 0$, and the inital configuration of k rays becomes a network of $2k-3$ curves, k of which extend to infinity.

We can now formulate the general short-term existence theorem.

Proposition 3.1. *Let Γ_0 be any initial network of curves such that at any interior vertex p, no two curves meet tangentially. (We prescribe some good boundary conditions at each external vertex.) For each internal vertex, consider the tangent array $\Gamma_0(p)$, which is the set of k rays emanating from 0 which are the tangent-rays for the curves meeting at p. Choose any expanding soliton solution $\Gamma_{ss}(p)$ which desingularizes $\Gamma_0(p)$. Then there exists a unique solution to the network flow associated to this data which exists for some small time interval $0 < t < T_*$. The rays in $\Gamma(t)$ are smooth up to their endpoints for all $t > 0$.*

There is a better regularity statement which requires some additional notation to describe properly. Consider the space $\mathbb{R}_t^+ \times \mathbb{R}^2$, with the initial network Γ_0 lying in $\{0\} \times \mathbb{R}^2$. Now pass to a new space $\mathcal{P}$ by taking the parabolic blowup of each of the points $(p, 0)$, where p is an internal vertex of Γ_0. Namely, for each internal vertex p, translate the coordinates so that p is at the origin. There is a polar coordinate system adapted to the parabolic dilations $(t, x) \mapsto (\lambda^2 t, \lambda x)$, namely $R = \sqrt{t + |x|^2}$, $\omega = (\omega_0, \omega')$, with $\omega_0 = t/R^2$, $\omega' = x/R$. Then $\mathcal{P}$ is obtained by removing each of the points $(0, p)$ and replacing it with the hemisphere S_+^2 parametrized by ω at $R = 0$. The network Γ_0 lifts to a collection of curves on the main boundary component at $t = 0$ of $\mathcal{P}$ which connect these various hemispheres. For each such p, its associated hemisphere $S(p)$ is projectively identified with $\mathbb{R}^2$, and the N_p curves in Γ_0 which meet at p determine N boundary points on the hemisphere, or equivalently, N_p asymptotic directions in this $\mathbb{R}^2$. Any choice of expanding soliton Γ_{ss} for each p gives a system of curves on that hemisphere, and the union of these 'closes up' the lift of Γ_0 to a network which extends over all the boundary components of $\mathcal{P}$.

The corresponding solution of the network flow provided by the theorem determines a set of parallel networks $\{t\} \times \Gamma(t)$ which fit together into a union of smooth sheets. We can now state the precise regularity theorem, which asserts that if the curves in Γ_0 are smooth and polyhomogeneous up to their endpoints, then these sheets extend to $\partial\mathcal{P}$ to be polyhomogeneous surfaces with boundary, where the union of the intersection of these sheets with $S(p)$ is precisely the soliton $\Gamma_{ss}(p)$. This is better visualized with the picture below. The gist of this entire result is that the apparent discontinuity of the evolution of this solution to the network flow at $t = 0$ is resolved by the passage to the space $\mathcal{P}$, and that with this interpretation, solutions appear to be completely smooth.

The proof of these results will appear in a forthcoming paper by Lira, the author and Saez [19]. The main step is to introduce the picture above and then reinterpret it in a way that it may be
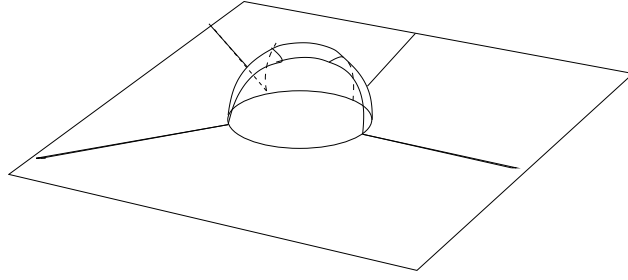
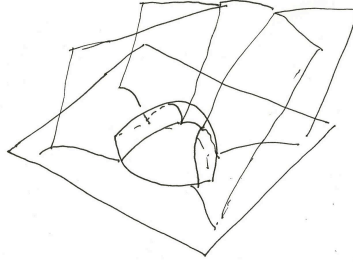Figure 3.1: The blowup and induced network at $t = 0$

Figure 3.2: The intersecting sheets of the flowout of the network

translated into a manifestly conic parabolic problem. As a hint of how to do this, consider the simple case where there is a single interior vertex and four curves intersect there. As pictured in Figure 1 above, the space $\mathcal{P}$ is a half-space with a single point on its boundary blown up to a hemisphere. The initial network Γ_0 is a collection of curves, each of which end at a set of distinct points of this hemisphere. There are several desingularizing solitons in this case: there are two disconnected solutions and two connected ones. Choosing any one of these caps off the lift of Γ_0. The sheets of the flowout are now of two types. There are four sheets, each corresponding to one of the original curves and meeting the original boundary of $\mathcal{P}$ at that curve but also intersecting $S(p)$, and one new sheet which only meets $\partial\mathcal{P}$ in that hemisphere. We consider this as a collection
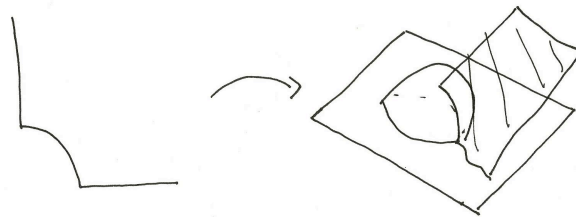

Figure 3.3: The space Q and the mapping into $\mathcal{P}$

of mappings: $U = (u_1, u_2, u_3, u_4, u_s)$. For $j = 1, 2, 3, 4$, u_j is initially a map from $\mathbb{R}_t^+ \times [0,1]_s$ into $\mathbb{R}_t^+ \times \mathbb{R}_x^2$, but in fact lifts to a polyhomogeneous map of the space Q obtained from $\mathbb{R}_t^+ \times [0,1]_s$ by blowing up $(0,0)$ parabolically, and has image in $\mathcal{P}$, which is of course a similar blowup of $\mathbb{R}_t^+ \times \mathbb{R}_x^2$. The final map u_s is a map from $\mathbb{R}_t^+ \times [0,1]_\sigma$ into $\mathcal{P}$, where $u_s(0, \sigma)$ parametrizes the curve in the soliton Γss(p) which connects the interior vertices and does not touch the boundary of the hemisphere. Thus altogether,

$$U : Q^4 \times \mathbb{R}^+ \times [0,1] \longrightarrow \mathcal{P}.$$

The original linking boundary conditions extend naturally to boundary conditions on the vertical boundaries of this blown up domain space. The main observation is that when interpreted this way, the underlying PDE is another example of a conic problem. The estimates from [23] may then be invoked almost verbatim to prove the theorem.

We note that there is an earlier proof of short-time existence for this general network flow, along with some long-time existence results, due to Ilmanen, Neves and Schulze [15]. Their approach uses a method of approximation, and while very satisfactory from many points of view, does not contain the sharp regularity statement at the vertices.

4. Extensions

We conclude this survey by mentioning a few natural generalizations of the problems considered here along with some results and problems in these directions.

The natural class of spaces generalizing those with isolated conic singularities are the stratified spaces with iterated edge metrics. There are now a number of limited results about linear elliptic theory in this general setting, see [1, 2]. On the other hand, this theory for spaces with simple edge singularities is now quite mature, see [22, 25].

The most direct generalization of the problem about constant curvature metrics with conic singularities concerns the existence and nature of Kähler-Einstein metrics on a Kähler manifold with edge singularities along a divisor. These metrics were considered initially by Tian in the early '90's, and more recently introduced by Donaldson as a crucial component of the resolution of the log Fano conjecture. The paper [17] contains a proof of the existence of these 'Kähler-Einstein edge metrics' with cone angles less than 2π, along with a complete regularity theory very similar to the one outlined above. That paper contains many further references to other approaches and results appearing in the now extensive literature on this problem. We also mention the work of Chen and Wang [9], who extended the approach of Donaldson [10] to a parabolic theory which attacks the dynamical study of this problem.

One may also consider the Ricci flow in higher dimensions on manifolds with conic and edge singularities. We mention the work of Bahuaud, Dryden and Vertman [3, 4] as initial steps. However, this problem has turned out to present some formidable technical obstacles and further progress has been difficult.

As for generalizations of the network flow, the most obvious extension to higher dimensions is to consider 'fans' of surfaces which meet along arcs and at vertices and to attempt to flow these by mean curvature. Freire [11] provides the analogue of [21] in this setting. He establishes a short-time existence result when the initial fan is 'regular' (this allows triples of surfaces to meet along a curve with angle $2\pi/3$. A more complicated configuration is also possible, where six surfaces meet at a vertex in an arrangement like the cone over the barycenter of a regular simplex. The analogue of the classification result of [24] for expanding fan solitons remains unknown. There are plausible conjectures about what the general short-term existence result should look like, but this remains an open problem.

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