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Renato Lucà

Abstract

We investigate the size of the regular set of weak solutions of the Navier–Stokes equation which are close, in an appropriate sense, to strong solutions. More precisely, if w is a strong solution with initial datum w_0, we focus on weak solutions evolving by initial data u_0 such that the difference $u_0 - w_0$ is small in the weighted $[L^2(\mathbb{R}^3)]^3$ space with weight $|x|^{-1}$. This is different by any smallness assumption in translation invariant critical Banach spaces. We also prove similar results in the small data setting.

1. Introduction and main results

We consider the Navier–Stokes problem

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \Delta u &= -\nabla P \\ \nabla\cdot u &= 0 \\ u(0,x) &= u_0(x) \quad (\nabla\cdot u_0 = 0). \end{cases} \tag{1.1}$$

in $(0,\infty)\times\mathbb{R}^3$. This describes the motion a viscous incompressible fluid without external forces. The velocity field has been denoted by u and the pressure by P.

For simplicity we use similar notations for the norm of scalar, vector or tensor quantities. For instance we write:

$$\|P\|_{L^2} := (\int P^2 dx)^{\frac{1}{2}}, \qquad \|u\|^2_{L^2} := \sum_j \|u_j\|^2_{L^2}, \qquad \|\nabla u\|^2_{L^2} := \sum_{j,k} \|\partial_k u_j\|^2_{L^2}.$$

We also write $L^2(\mathbb{R}^3)$ instead of $[L^2(\mathbb{R}^3)]^3$, or $C^\infty(\mathbb{R}^3)$ instead of $[C^\infty(\mathbb{R}^3)]^3$ *etc.*

In the small data framework, the equation (1.1) can be viewed as a perturbation of the heat equation. This is more clear when we consider its integral formulation, namely

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)(s)\, ds \qquad \text{in} \quad (0,\infty)\times\mathbb{R}^3 \tag{1.2}$$

where $\mathbb{P}$ is the Leray projection

$$\mathbb{P}f := f - \nabla\Delta^{-1}(\nabla\cdot f),$$

that project the vector field f onto the divergence free subspace.

The Picard iteration scheme for Problem (1.2) is

$$u_1 := e^{t\Delta}u_0, \qquad u_n := e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u_{n-1}\otimes u_{n-1})(s)\, ds.$$

Once the velocity is known, the pressure can be calculated by

$$P = -\Delta^{-1}\nabla\otimes\nabla\cdot(u\otimes u).$$

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An abstract fixed point theorem for (1.2) is the following one.

Proposition 1.1 ([26]). *Let $X \subset \bigcap_{s<\infty} L^2_t L^2_{uloc,x}((0,s)\times\mathbb{R}^3)$*[1] *be a Banach space such that the bilinear form*

$$B(u,v) := \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes v)(s)\ ds \tag{1.3}$$

is bounded from $X\times X$ to X:

$$\|B(u,v)\|_X \le C_X\|u\|_X\|v\|_X.$$

Moreover, let $X_0 \subset \mathscr{S}'(\mathbb{R}^3)$ be a normed space such that $e^{t\Delta}: X_0 \to X$ is bounded:

$$\|e^{t\Delta}f\|_X \le A_{X_0,X}\|f\|_{X_0}.$$

Then for every data u_0 such that $\|u_0\|_{X_0} < 1/4C_XA_{X_0,X}$ the sequence u_n is Cauchy in X and converges to a solution u of the integral equation (1.2)*. The solution satisfies*

$$\|u\|_X \le 2A_{X_0,X}\|u_0\|_{X_0}.$$

Following [26] we say that X is an *admissible (path) space*, while X_0 is an *adapted space*. Many adapted spaces have been studied: $\dot{H}^{1/2}$ [12], L^3 [19], the Morrey space $\dot{M}^3_2$ [17, 36, 20, 10, 23], the Besov spaces $\dot{B}^{-1+3/q}_{q,\infty}$ [4, 30], *etc.* This approach culminated in the work of Koch and Tataru [22] in which the authors consider small initial data in BMO^{-1}. This is the most general result in literature.

Regarding large data, global weak solutions of (1.1) have been proved to exist by Leray [27] for any $u_0 \in L^2$ (namely for finite kinetic energy). The Leray's proof is based on a compactness argument and the uniqueness and persistence of regularity of the Leray's weak solutions is a long standing open problem.

Stronger results are available once we restrict to u_0 with some specific geometric properties. For instance u_0 axisymmetric [38, 25, 5, 15, 14], helical [29] or two dimensional. The last case, in particular, is (essentially) completely understood.

Other interesting classes of large initial data, without symmetry, have been studied in [11, 13, 18, 7, 6].

Once these special solutions are known, it is natural considering their small perturbation. For instance if we take a large axisymmetric w_0 without swirl, for which the problem is known to be well-posed [38], do we still have well-posedness for u_0 such that $u_0 - w_0$ is small in an appropriate functional space ?

This 'perturbative' approach has been introduced by Ponce, Racke, Sideris and Titi in [31]. They worked with small H^1 perturbations. Like in the small data case, the class of functional spaces in which perturbative solutions have been constructed has been extended to several functional spaces. For instance L^3 [21], Besov spaces [16] and BMO^{-1} [1].

A key feature of both small data and perturbative setting is to consider functional spaces that are scaling and translation invariant. More precisely, since problem 1.1 is invariant under the family of symmetries

$$u \mapsto \lambda u(\lambda^2 t, \lambda(x-\bar{x})), \qquad \lambda\in(0,\infty),\quad \bar{x}\in\mathbb{R}^3, \tag{1.4}$$

it is natural to look at initial data belonging to Banach spaces invariant under

$$u_0(x) \mapsto \lambda u_0(\lambda(x-\bar{x})), \qquad \lambda\in(0,\infty),\quad \bar{x}\in\mathbb{R}^3.$$

It is interesting that partial results have been obtained by working with norms which are only scaling, but not translation, invariant. Namely invariant under

$$u_0(x-\bar{x}) \mapsto \lambda u_0(\lambda(x-\bar{x})), \qquad \lambda\in(0,\infty),$$

where now $\bar{x}\in\mathbb{R}^3$ is a fixed vector. To our knowledge this point of view has been first exploited in [3] in which, among the other things, the authors prove a very general class of weak solutions

[1] The space L^2_{uloc} consists of the functions that are uniformly locally square-integrable (see [26] Definition 11.3). The operator (1.3) is well-defined on $\bigcap_{s<\infty} L^2_t L^2_{uloc,x}((0,s)\times\mathbb{R}^3)\times\bigcap_{s<\infty} L^2_t L^2_{uloc,x}((0,s)\times\mathbb{R}^3)$. We refer to [26], Chapter 11, for more details.

exhibiting a nice behavior around the point $\bar{x} \in \mathbb{R}^3$, for all times, provided that the weighted L^2 norm

$$\Big(\int_{\mathbb{R}^3} |x-\bar{x}|^{-1}|u_0(x)|^2 dx\Big)^{\frac{1}{2}} = \||x-\bar{x}|^{-1/2}u_0\|_{L^2(\mathbb{R}^3)}, \tag{1.5}$$

of the initial data u_0 is small enough. The precise statement is given below.

The aim of this note is to give various extensions and improvements of this result, in both the small data and perturbative setting.

Let recall a classical notion of regularity for Problem 1.1.

Definition 1.2. A point $(t_0, x_0) \in (0,\infty) \times \mathbb{R}^3$ is *regular* for a solution $u(t,x)$ of (1.1) if u is essentially bounded on a neighborhood of (t_0, x_0). In particular this implies [34] that $u(t,x)$ is smooth (in space) in a neighborhood of (t_0, x_0). A subset of $(0,\infty) \times \mathbb{R}^3$ is *regular* if all its points are regular.

We use the notation

$$\Pi_{\alpha,\bar{x}} := \left\{(t,x) \in (0,\infty) \times \mathbb{R}^3 \;:\; t > \frac{|x-\bar{x}|^2}{\alpha}\right\}$$

for the interior of the paraboloid of aperture α in the upper half space $(0,\infty) \times \mathbb{R}^3$, with vertex at $(0,\bar{x})$. If $\bar{x} = 0$ we simply write Π_α in place of $\Pi_{\alpha,0}$. Note that $\Pi_{\alpha,\bar{x}} \supset \Pi_{\beta,\bar{x}}$ if $\alpha > \beta$.

The following result (Theorem D in [3]) holds for *suitable* weak solutions, which we are going to define in Section 2. Let mention that the weak solutions constructed by the Leray's approximation procedure are, for instance, suitable (Theorem 2.3 in [33]).

Theorem 1.3 (Caffarelli–Kohn–Nirenberg, [3])**.** *There exists a constant $\varepsilon_0 > 0$ such that the following holds. If*

$$\||x-\bar{x}|^{-1/2}u_0\|^2_{L^2(\mathbb{R}^3)} =: \varepsilon < \varepsilon_0 \tag{1.6}$$

then the set

$$\Pi_{\varepsilon_0-\varepsilon,\bar{x}} = \left\{(t,x) \;:\; t > \frac{|x-\bar{x}|^2}{\varepsilon_0-\varepsilon}\right\}$$

is regular for any suitable weak solution u of problem (1.1) *with divergence free initial datum $u_0 \in L^2(\mathbb{R}^3)$.*

Thus, if the weighted L^2 norm of the datum is small enough, then the solution is smooth in the interior of a paraboloid above the point $(0,\bar{x})$, where $\bar{x}$ is the center of the weight.

The interest of this result is that the condition (1.6) does not force u_0 to be small at the points x far enough from $\bar{x}$. This makes it different by any possible translation invariant smallness assumption on u_0. We clarify this fact in the following remark.

Remark 1.1. There exist initial data such that the norms $\||x-\bar{x}|^{-1/2}u_0\|_{L^2(\mathbb{R}^3)}$ are arbitrarily small while the norms $\|u_0\|_{BMO^{-1}(\mathbb{R}^3)}$ are arbitrarily large. Assume indeed, for simplicity, $\bar{x} = 0$ and let $\phi \in C_c^\infty(\mathbb{R}^3)$ be a divergence free vector field. Write $\phi_K(x) := \phi(x - K\xi)$ for the translate of ϕ by the vector ξK, with $\xi \in \mathbb{R}^3$, $|\xi| = 1$ and $K > 1$. It is immediate to check that[2]

$$\||x|^{-1/2}\phi_K\|_{L^2(\mathbb{R}^3)} \simeq K^{-1/2}.$$

Thus, by the translation invariance of BMO^{-1}, as $K \to +\infty$

$$\||x|^{-1/2}\phi_K\|_{L^2(\mathbb{R}^3)} \to 0 \quad \text{while} \quad \|\phi_K\|_{BMO^{-1}(\mathbb{R}^3)} = const. \tag{1.7}$$

The size of the regular set clearly depends on the size of u_0. Let notice, in particular, that $\Pi_{\varepsilon_0-\varepsilon,\bar{x}}$ converges to a maximal regular set $\Pi_{\varepsilon_0,\bar{x}}$ as $\||x-\bar{x}|^{-1/2}u_0\|_{L^2(\mathbb{R}^3)} =: \varepsilon \to 0$; namely as the initial data converge to the zero solution. This behavior can be improved by showing that the regular set actually invades the whole half space $\{t > 0\}$ in the limit $\varepsilon \to 0$.

Theorem 1.4. [8] *There exists a constant $\delta_0 > 0$ such that the following holds. If $M \geq 1$, the set*

$$\Pi_{M\delta_0,\bar{x}} = \left\{(t,x) \;:\; t > \frac{|x-\bar{x}|^2}{M\delta_0}\right\}$$

[2] As usual we write $A \lesssim B$ if $A \leq CB$ for a certain constant $C > 0$ and $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.

is regular for any suitable weak solution u of problem 1.1 with divergence free initial datum $u_0 \in L^2(\mathbb{R}^3)$, provided that

$$\||x-\bar{x}|^{-1/2}u_0\|_{L^2(\mathbb{R}^3)} \leq \delta_0 e^{-M^2/\delta_0}.$$

Thus, taking $M \to +\infty$, we see that as weighted L^2 norm of the data goes to zero, the regular set invades the whole half space $\{t>0\}$, as claimed above.

Definition 1.5. A couple (r,q) is *admissible* if $2 \leq r < \infty$ and $2/r + 3/q = 1$.

Definition 1.6. A weak solution $w \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$ of problem 1.1 with divergence free initial datum $w_0 \in L^2(\mathbb{R}^3)$ is a $\mathcal{K}$*-reference* solution if

$$\int_0^\infty \left(\int_{\mathbb{R}^3} |w(t,x)|^q dx\right)^{\frac{r}{q}} dt =: \|w\|^r_{L^r_t L^q_x} =: \mathcal{K} < \infty \tag{1.8}$$

for an admissible couple (r,q) and

$$\limsup_{r\to 0} \frac{1}{r} \int\int_{Q^*_r(t,x)} |\nabla w|^2 < \varepsilon^* \tag{1.9}$$

for all $(t,x) \in (0,\infty)\times\mathbb{R}^3$, where ε^* is the absolute constant in (2.9) and $Q^*_r(t,x)$ is the parabolic cylinder defined in (2.8).

We also say that u is a strong solution if it is a $\mathcal{K}$-reference solution for some $\mathcal{K}>0$. Thus $\mathcal{K}$ can be considered as a measure of the size of a strong solution.

Regarding the notations, we write L^q_x when the norm is taken over all the space variables $x \in \mathbb{R}^3$ and L^r_t when the time integration is over $t>0$. We write $\|f\|_{XY} := \left\|\|f\|_Y\right\|_X$ for nested norms and denote XY the completion of the Schwartz class with respect to these norms.

We can now state the main result of this note.

Theorem 1.7. [9] *Let $\bar{x} \in \mathbb{R}^3$ and let w be a $\mathcal{K}$-reference solution of problem 1.1 with (divergence free) initial datum $w_0 \in L^2(\mathbb{R}^3)$. There exists a constant $\delta_1 > 0$ such that the following holds. The set*

$$\Pi_{\delta_1,\bar{x}} := \left\{(t,x) \ : \ t > \frac{|x-\bar{x}|^2}{\delta_1}\right\} \tag{1.10}$$

is regular for every suitable weak solution $u \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$ of 1.1 with (divergence free) initial datum $u_0 \in L^2(\mathbb{R}^3)$ satisfying

$$\||x-\bar{x}|^{-1/2}(u_0 - w_0)\|_{L^2_x} \leq \delta_1 e^{-\mathcal{K}/\delta_1}. \tag{1.11}$$

As already pointed out, the existence of suitable weak solutions $u \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$ has been proved in [33, Theorem 2.3] for any $u_0 \in L^2(\mathbb{R}^3)$.

Interesting examples of $\mathcal{K}$-reference solutions with large w_0 can be found in [38, 6]. The author would like to thank Jean-Yves Chemin for the second reference.

Thus, weak solutions evolving by small weighted L^2 perturbations of the initial data of strong solutions are still regular around the center of the weight.

As observed in Remark 1.1 there exists perturbations which are (at $t=0$) small in the sense of (1.11) but such that the norms $\|u_0 - w_0\|_{BMO^{-1}(\mathbb{R}^3)}$ are arbitrarily large.

We use the stability Theorem 1.7 to construct weak solutions which are smooth around $\bar{x}$.

We focus on perturbations of (possibly large) axisymmetric vector fields with zero swirl. Let $\Theta \in \mathbb{T}$, $\mathbf{r} \in (0,+\infty)$ be polar coordinates in $\mathbb{R}^2$:

$$x_1 =: \mathbf{r}\cos\Theta, \qquad x_2 =: \mathbf{r}\sin\Theta. \tag{1.12}$$

We say that a vector field f is axisymmetric (with respect to the x_3-axis)[3] if its expression in cylindrical polar coordinates $(\Theta, \mathbf{r}, x_3)$ is independent on the variable Θ, namely

$$f = f_{e_\mathbf{r}}(\mathbf{r},x_3)e_\mathbf{r} + f_{e_\Theta}(\mathbf{r},x_3)e_\Theta + f_{e_{x_3}}(\mathbf{r},x_3)e_{x_3}.$$

The swirl of f is f_{e_Θ}.

[3]For simplicity we consider x_3 as symmetry axis. Since problem 1.1 is invariant under rotations, we can of course choose any other direction.

Proposition 1.8. [9] *Let $\bar{x} \in \mathbb{R}^3$ and let $w_0 \in H^4(\mathbb{R}^3) \cap L^2(|x-\bar{x}|^{-1}dx)$ be a zero swirl axisymmetric divergence free vector field. There exists a constant $\delta_2 = \delta_2(w_0)$ such that the following holds. Any suitable weak solution u of problem (1.1) with (divergence free) initial datum $u_0 \in L^2(\mathbb{R}^3)$ satisfying*

$$\| |x-\bar{x}|^{-1/2}(u_0 - w_0)\|_{L^2(\mathbb{R}^3)} \leq \delta_3, \tag{1.13}$$

is regular in the interior of a paraboloid with vertex at $(0, \bar{x})$.

Regarding the small data setting, we give a result which covers the gap between Theorem 1.3 and the L^3 Kato's theorem.

We consider initial data such that the critical weighted L^p norm

$$\| |x-\bar{x}|^{\alpha} u_0\|_{L^p(\mathbb{R}^3)}, \quad 2 < p < 3, \quad \alpha = 1 - \frac{3}{p} \tag{1.14}$$

is very small, and we prove a local regularity result which improves as p increases. We recover full regularity in the limit $p \to 3^-$ (namely $\alpha \to 0^+$).

Let introduce

$$\theta_1(p) := \left(\frac{p-2}{3-p}\right)^{1-p/3}, \qquad \theta_2(p) := \left(\frac{p-2}{3-p}\right)^{1-p/2}. \tag{1.15}$$

It is straightforward to check that

$$\lim_{p \to 2^+} \theta_1(p) = 0, \quad \lim_{p \to 3^-} \theta_1(p) = 1, \tag{1.16}$$

while θ_2 behaves in the opposite way

$$\lim_{p \to 2^+} \theta_2(p) = 1, \quad \lim_{p \to 3^-} \theta_2(p) = 0. \tag{1.17}$$

Theorem 1.9. [9] *Let $\bar{x} \in \mathbb{R}^3$, $2 < p < 3$, $\alpha = 1 - 3/p$. Let $u_0 \in L^2(\mathbb{R}^3)$ be a divergence free vector field and let u be a suitable weak solution of problem 1.1 with initial datum u_0. There exists a constant $\delta_4 > 0$ such that the following holds. For every $M \geq 1$, if*

$$\theta_1 \| |x-\bar{x}|^{\alpha} u_0\|^{p/3}_{L^p(\mathbb{R}^3)} \leq \delta_4, \quad \theta_2 \| |x-\bar{x}|^{\alpha} u_0\|^{p/2}_{L^p(\mathbb{R}^3)} \leq \delta_4 e^{-M^2/\delta_4}, \tag{1.18}$$

then the set $\Pi_{M\delta,\bar{x}}$ is regular for u.

We again remark that there exist data u_0 which are arbitrarily large in $BMO^{-1}(\mathbb{R}^3)$ but such that the (1.18) is satisfied.

The result can be interpreted in the following way. We have observed that $\theta_2(p) \to 0$ as $p \to 3^-$, so we can choose $p = p_M$ as a function of M in such a way that

$$e^{M^2/\delta_4} \cdot \theta_2(p_M) \to 0 \quad \text{as} \quad M \to \infty.$$

Since we are taking $p_M \to 3^-$, we also have $\theta_1(p_M) \to 1$, so that the theorem implies, for all sufficiently large M:

$$\| |x-\bar{x}|^{-1/2} u_0\|_{L^{p_M}_x} \leq \delta_4/2, \quad \Rightarrow \quad \Pi_{M\delta_4,\bar{x}} \text{ is a regular set for } u.$$

In other words, if we take $M \to \infty$ and the norm $\| |x-\bar{x}|^{-1/2} u_0\|_{L^{p_M}}$ is less than $\delta_4/2$, the regular set invades the whole half space $\{t > 0\}$. We also refer to Theorem 1.5 in [8] for a similar result.

Here we only give the proof of Theorem 1.4. The proof of Theorem 1.9 is somewhat similar, while different ideas are necessary to treat the perturbative case, namely Theorem 1.7. For this we refer to a forthcoming paper [9] written in collaboration with Piero D'Ancona. Let remark that Proposition 1.8 is a simple consequence of Theorem 1.7.

2. Preliminaries

Definition 2.1. Let $u_0 \in L^2(\mathbb{R}^3)$. Following [3, 26, 28] we say that u is a *suitable* weak solution of problem 1.1 if:

1. there exists $P \in L^{3/2}_{loc}((0,\infty) \times \mathbb{R}^3)$ such that (u, P) satisfies (1.1) in the sense of distributions;
2. $u(t) \to u_0$ weakly in L^2 as $t \to 0$;

3. for some constants E_0, E_1
$$\int_{\mathbb{R}^3} |u(t,x)|^2\, dx \le E_0,$$
for all $t > 0$ and
$$\int_0^\infty \int_{\mathbb{R}^3} |\nabla u(t,x)|^2\, dtdx \le E_1;$$

4. for all non negative $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^3)$ and for all $t > 0$

$$\int_{\mathbb{R}^3} |u|^2\phi(t) + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi \\ \le \int_{\mathbb{R}^3} |u_0|^2\phi(0) + \int_0^t \int_{\mathbb{R}^3} |u|^2(\phi_t + \Delta\phi) + \int_0^t \int_{\mathbb{R}^3} (|u|^2 + 2P)u\cdot\nabla\phi. \quad (2.1)$$

Suitable weak solutions are known to exist for all L^2 initial data (see Theorem 2.3 in [33] or the Appendix in [3]) and are L^2-weakly continuous as functions of time (see [37], pp. 281–282), namely
$$\int_{\mathbb{R}^3} u(t,x)w(x)\, dx \to \int_{\mathbb{R}^3} u(t',x)w(x)\, dx$$
for all $w \in L^2(\mathbb{R}^3)$ as $t \to t'$ $(t, t' \in [0,+\infty))$; thus it makes sense to impose the initial condition (2). Actually, by taking advance of the energy inequality, also strong convergence to the initial data can be proved: $\lim_{t\to 0^+} \|u(t) - u_0\|_{L^2(\mathbb{R}^3)}$.

Since we are only considering the whole $\mathbb{R}^3$ case, we have the well known representation formula for the pressure
$$P = \Delta^{-1}\nabla\otimes\nabla(u\otimes u) = R\otimes R\cdot(u\otimes u),$$
where $R := (R_1, R_2, R_3)$ and R_j is the Riesz transform oriented in the direction of the j-th coordinate. By this and by (3) easily follows that P belongs to $L^{5/3}((0,\infty)\times\mathbb{R}^3)$ (see [3] for details).

We also need a stronger version of inequality (2.1), namely for almost every $t_0 \in (0,\infty)$ and for any $t > t_0$

$$\int_{\mathbb{R}^3} |u|^2\phi(t) + 2\int_{t_0}^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi \\ \le \int_{\mathbb{R}^3} |u(t_0)|^2\phi(t_0) + \int_{t_0}^t \int_{\mathbb{R}^3} |u|^2(\phi_t + \Delta\phi) + \int_{t_0}^t \int_{\mathbb{R}^3} (|u|^2 + 2P)u\cdot\nabla\phi, \quad (2.2)$$

provided that $\phi \in C_c^\infty([t_0,\infty)\times\mathbb{R}^3)$ is non negative. This property is actually satisfied by suitable weak solutions as defined above.

Lemma 2.2 (strong generalized energy inequality). *Let u be a suitable weak solution of problem 1.1. The inequality* (2.2) *holds for for almost every $t_0 \in (0,+\infty)$, for all $t > t_0$ and for any non negative test function $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^3)$*

Proof. Let $\varepsilon > 0$, $t_0 > 0$ and ϕ a non negative test function. We consider the auxiliary test functions $\phi_\varepsilon(t,x) := \eta_\varepsilon(t)\phi(t,x)$ where $\eta_\varepsilon(t)$ is an ε-mollification of the step function with jump in t_0; namely $\eta_\varepsilon(t) := \chi_{[t_0,+\infty]} * \rho_\varepsilon$, being $\chi_{[t_0,+\infty]}$ the indicator function of $[0,+\infty]$ and $\rho_\varepsilon := \varepsilon^{-1}\rho(\varepsilon^{-1}t)$ with ρ a smooth non negative function supported in $[-\varepsilon,+\varepsilon]$ and such that $\int_{\mathbb{R}}\rho = 1$. We have clearly
$$\partial_t\phi_\varepsilon(t,x) = \rho_\varepsilon(t-t_0)\phi(t,x) + \eta_\varepsilon\partial_t\phi(t,x) \quad (2.3)$$
and, as $\varepsilon \to 0$
$$\eta_\varepsilon \to \chi_{[t_0,+\infty]}, \quad \phi_\varepsilon \to \chi_{[t_0,+\infty]}\phi, \quad \nabla\phi_\varepsilon \to \chi_{[t_0,+\infty]}\nabla\phi, \quad \Delta\phi_\varepsilon \to \chi_{[t_0,+\infty]}\Delta\phi. \quad (2.4)$$
Let now $t > t_0$, our aim is to apply the inequality (2.1) with ϕ_ε and passing to the limit $\varepsilon \to 0$. In order to do this we recall that $P \in L_t^{5/3}L_x^{5/3}$, $u \in L_t^\infty L_x^2$ and $\nabla u \in L_t^2 L_x^2$, so that by

Sobolev embedding and interpolation we aslo have $u \in L^4_t L^3_x \cap L^{20/3}_t L^{5/2}_x$. This allows to apply the dominated convergence theorem, so that by (2.3, 2.4) we obtain

$$\int_{\mathbb{R}^3} |u|^2\phi(t) + 2\int_{t_0}^t\int_{\mathbb{R}^3} |\nabla u|^2\phi \;\le\; \lim_{\varepsilon\to 0}\int_0^t \rho_\varepsilon(s-t_0)\left(\int_{\mathbb{R}^3}|u|^2\phi\right)(s)ds \\ + \int_{t_0}^t\int_{\mathbb{R}^3}|u|^2(\phi_t+\Delta\phi) + \int_{t_0}^t\int_{\mathbb{R}^3}(|u|^2+2P)u\cdot\nabla\phi. \quad (2.5)$$

Thus the lemma is proved once we show that for almost every $t_0 \in (0,+\infty)$

$$\lim_{\varepsilon\to 0}\int_0^t \rho_\varepsilon(s-t_0)\left(\int_{\mathbb{R}^3}|u|^2\phi\right)(s)ds = \int_{\mathbb{R}^3}|u(t_0)|^2\phi(t_0), \quad (2.6)$$

which is actually true for any Lebesgue point of the function $t \to \int_{\mathbb{R}^3}|u|^2\phi(t)$ and so almost everywhere, being this function bounded by the third assumption in Definition 2.1. □

A more direct approach, also sufficient for our purposes, would be to assume the property (2.2) in Definition 2.1 and then to show directly that the weak solutions constructed in [33] satisfies this property, which is indeed the case.

Corollary 2.3. *Let u be a suitable weak solution of Problem 1.1 and ϕ_n with $n \in \mathbb{N}$ be a sequence of non negative test functions. Then there is a measurable subset $\mathcal{A} \subset (0,\infty)$ with*[4] $(0,\infty)/\mathcal{A}| = 0$ *such that*

$$\int_{\mathbb{R}^3}|u|^2\phi_n(t) + 2\int_{t_0}^t\int_{\mathbb{R}^3}|\nabla u|^2\phi_n \\ \le \int_{\mathbb{R}^3}|u(t_0)|^2\phi_n(t_0) + \int_{t_0}^t\int_{\mathbb{R}^3}|u|^2(\partial_t\phi_n+\Delta\phi_n) + \int_{t_0}^t\int_{\mathbb{R}^3}(|u|^2+2P)u\cdot\nabla\phi_n, \quad (2.7)$$

holds for any ϕ_n and for any $t_0 \in \mathcal{A}$ and $t > t_0$.

Next we define the *parabolic cylinder* of radius r and top point (t,x) as

$$Q_r(t,x) := \left\{(s,y): \quad |x-y|<r, \quad t-r^2<s<t\right\}$$

while the *shifted parabolic cylinder* is

$$Q^*_r(t,x) := Q_r(t+r^2/8,x) \equiv \left\{(s,y): \quad |x-y|<r, \quad t-7r^2/8<s<t+r^2/8\right\} \quad (2.8)$$

The crucial regularity result in [3] ensures that:

Lemma 2.4 (Caffarelli–Kohn–Nirenberg)**.** *There exists an absolute constant ε^* such that if u is a suitable weak solution of* (1.1) *and*

$$\limsup_{r\to 0}\frac{1}{r}\int\int_{Q^*_r(t,x)}|\nabla u|^2 \le \varepsilon^*, \quad (2.9)$$

then (t,x) is a regular point.

We shall make frequent use of the following interpolation inequality from [2].

Lemma 2.5. *Assume that*

1. $r \ge 0$, $0 < a \le 1$, $\gamma < 3/r$, $\alpha < 3/2$, $\beta < 3/2$;
2. $-\gamma + 3/r = a(-\alpha+1/2) + (1-a)(-\beta+3/2)$;
3. $a\alpha + (1-a)\beta \le \gamma$;
4. *when* $-\gamma+3/r = -\alpha+1/2$, *assume also that* $\gamma \le a(\alpha+1)+(1-a)\beta$.

Then

$$\|\sigma^\gamma_\nu u\|_{L^r(\mathbb{R}^3)} \le C\|\sigma^\alpha_\nu\nabla u\|^a_{L^2(\mathbb{R}^3)}\|\sigma^\beta_\nu u\|^{1-a}_{L^2(\mathbb{R}^3)}, \quad (2.10)$$

where $\sigma_\nu := (\nu+|x|^2)^{-1/2}$, $\nu \ge 0$, with a constant C independent of ν.

[4] $|\cdot|$ denotes the Lebesgue measure.

3. Proof of Theorem 1.4

Let $\xi \in \mathbb{R}^3 \setminus \{0\}$, $T \geq 1$ and consider the segment

$$L(T,\xi) := \{(s,\xi s) \ : \ s \in (0,T)\}.$$

We will investigate for which (T,ξ) the set $L(T,\xi)$ is a regular. To this purpose we introduce the change of variables

$$(t,y) = (t, x - \xi t), \qquad u_\xi(t,y) = u(t,x), \tag{3.1}$$

which takes (1.1) into the system

$$\begin{cases} \partial_t u_\xi + ((u_\xi - \xi)\cdot\nabla)u_\xi + \nabla P_\xi - \Delta u_\xi &= 0 \\ \nabla \cdot v_\xi &= 0 \\ u_\xi(0) &= u_0 \\ P_\xi &= R \otimes R \ (u_\xi \otimes u_\xi) \end{cases} \tag{3.2}$$

and maps the segment $L(T,\xi)$ in $(0,T)\times\{0\}$ (a vertical segment above the origin of the space-time). We fix an arbitrary $M \geq 1$ and define the set

$$S(M,T,\xi) := \left\{ s \in (0,T] \ : \ \int_s^{s+T/M} \int_{\mathbb{R}^3} |y|^{-1} |\nabla u_\xi(\tau,y)|^2 \ d\tau dy > M \right\} \tag{3.3}$$

and the number $\bar{s} \geq 0$

$$\bar{s} := \begin{cases} \inf\{s \in S(M,T,\xi)\} & \text{if} \quad S(M,T,\xi) \neq \emptyset \\ T & \text{otherwise.} \end{cases} \tag{3.4}$$

From the definition of $\bar{s}$ one has immediately

$$\int_0^{\bar{s}} \int_{\mathbb{R}^3} |y|^{-1} |\nabla u_\xi(\tau,y)|^2 \ d\tau dy \leq M(M+1) \leq 2M^2. \tag{3.5}$$

We next distinguish two cases.

3.1. First case: $\bar{s} = T$

In this case the entire segment $L(T,\xi)$ is a regular set. To prove this, we first note that by (3.5), once we come back to the old variables, we get

$$\int_0^T \int_{\mathbb{R}^3} \frac{|\nabla u(\tau,x)|^2}{|u - \xi\tau|} \ d\tau dx < +\infty. \tag{3.6}$$

Let $0 < s < T$ and let $r > 0$ be so small that $0 < s - 7r^2/8 < s + r^2/8 < T$ and $|\xi| r \leq 1$. For each $(\tau,x) \in Q_r^*(s,\xi s)$ we have

$$|x - \xi\tau| \leq |x - \xi s| + |\xi||s - \tau| \leq r + r^2|\xi| \leq 2r$$

which implies

$$\frac{1}{r} \int\int_{Q_r^*(s,\xi s)} |\nabla u(\tau,x)|^2 \ d\tau dx \leq 2 \int_{s-\frac{7}{8}r^2}^{s+\frac{1}{8}r^2} \int_{\mathbb{R}^3} \frac{|\nabla u(\tau,x)|^2}{|x - \xi\tau|} \ d\tau dx.$$

Because of this and the (3.6) we immediately see that the regularity condition (2.9) is satisfied at all $(s,\xi s) \in L(T,\xi)$, *i.e.* $L(T,\xi)$ is a regular set as claimed.

3.2. Second case: $0 \leq \bar{s} < T$

Since u_ξ is a suitable weak solution of problem 3.2, the following generalized energy inequality is valid: for all $t \geq 0$ and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^3)$:

$$\begin{aligned} &\int_{\mathbb{R}^3} \phi(t,x)|u_\xi|^2 dx + 2\int_0^t \int_{\mathbb{R}^3} \phi |\nabla u_\xi|^2 \\ &\quad \leq \int_{\mathbb{R}^3} \phi(0,x)|u_0|^2 dx + \int_0^t \int_{\mathbb{R}^3} |u_\xi|^2 (\phi_t - \xi\cdot\nabla\phi + \Delta\phi) + \int_0^t \int_{\mathbb{R}^3} (|u_\xi|^2 + 2P_\xi) u_\xi \cdot \nabla\phi. \end{aligned} \tag{3.7}$$

This is indeed the inequality (2.1) after the change of variables (3.1). By a standard approximation procedure (see the proof of Lemma 8.3 in [3]) the estimate is valid for any test function of the form

$$\phi(t,y) := \psi(t)\phi_1(y)$$

with $\phi_1 \in C_c^\infty(\mathbb{R}^3)$, $\phi_1 \geq 0$, and

$$\psi : (0,\infty) \to \mathbb{R} \quad \text{absolutely continuous with} \quad \dot{\psi} \in L^1(0,\infty).$$

We shall choose here

$$\psi(t) \equiv 1, \qquad \phi_1 = \sigma_\nu(y)\chi(\delta|y|),$$

where $\nu, \delta > 0$,

$$\sigma_\nu(y) = (\nu + |y|^2)^{-\frac{1}{2}},$$

and $\chi : [0,+\infty) \to [0,+\infty)$ is a smooth nonincreasing function such that

$$\chi = 1 \text{ on } [0,1], \qquad \chi = 0 \text{ on } [2,+\infty].$$

Passing to the limit $\delta \to 0$ we obtain

$$\left[\int_{\mathbb{R}^3} \sigma_\nu |u_\xi|^2\right]_0^t + 2\int_0^t \int_{\mathbb{R}^3} \sigma_\nu |\nabla u_\xi|^2 \leq \int_0^t \int_{\mathbb{R}^3} |u_\xi|^2(-\xi\cdot\nabla\sigma_\nu + \Delta\sigma_\nu) + \int_0^t \int_{\mathbb{R}^3} (|u_\xi|^2 + 2P_\xi)u_\xi \cdot \nabla\sigma_\nu. \tag{3.8}$$

Notice that this is allowed by the integrability properties of the functions involved via dominated convergence; see Section 2. Since

$$|\nabla\sigma_\nu| \leq (\nu + |y|^2)^{-1} = \sigma_\nu^2, \qquad \Delta\sigma_\nu < 0, \tag{3.9}$$

we deduce the estimate

$$\left[\int_{\mathbb{R}^3} \sigma_\nu |u_\xi|^2\right]_0^t + 2\int_0^t \int_{\mathbb{R}^3} \sigma_\nu |\nabla u_\xi|^2 \leq |\xi| \int_0^t \int_{\mathbb{R}^3} \sigma_\nu^2 |u_\xi|^2 + \int_0^t \int_{\mathbb{R}^3} \sigma_\nu^2 (|u_\xi|^3 + 2|P_\xi||u_\xi|). \tag{3.10}$$

We now use the (3.10) to obtain a Gronwall-type inequality for the quantities

$$a_\nu(t) = \|\sigma_\nu^{1/2} u_\xi(t,x)\|_{L_x^2}^2, \qquad B_\nu(t) = \int_0^t \|\sigma_\nu^{1/2}\nabla u_\xi(s,x)\|_{L_x^2}^2 ds.$$

We first estimate the term which contains P_ξ. Let write

$$I := 2\int_{\mathbb{R}^3} \sigma_\nu^2 |u_\xi||P_\xi| = 2\int_{\mathbb{R}^3} \sigma_\nu^2 |u_\xi||R\otimes R\ (u_\xi \otimes u_\xi)|. \tag{3.11}$$

We use the the weighted L^p inequality for the Riesz transform (see [35]), uniform in $\nu \geq 0$

$$\|\sigma_\nu^m R\phi\|_{L^s} \leq Z\|\sigma_\nu^m \phi\|_{L^s}, \qquad 1 < s < \infty, \qquad m \in \left(-\frac{3(s-1)}{s}, \frac{3}{s}\right). \tag{3.12}$$

Here and in the following, as usual, Z denotes several universal constants, possibly different from line to line. We have

$$\begin{aligned} I &\leq 2\|\sigma_\nu R\otimes R\ (u_\xi\otimes u_\xi)\|_{L^2}\|\sigma_\nu u_\xi\|_{L^2} \leq Z\|\sigma_\nu |u_\xi|^2\|_{L^2}\|\sigma_\nu u_\xi\|_{L^2} \\ &\leq Z\|\sigma_\nu^{1/2} u_\xi\|_{L^4}^2 \|\sigma_\nu u_\xi\|_{L^2} \end{aligned}$$

and by the Caffarelli–Kohn–Nirenberg inequality (2.10) we obtain

$$\begin{aligned} I \;&\leq\; Z\|\sigma_\nu^{1/2}\nabla u_\xi\|_{L^2}^{3/2}\|\sigma_\nu^{1/2}u_\xi\|_{L^2}^{1/2} \cdot \|\sigma_\nu^{1/2}\nabla u_\xi\|_{L^2}^{1/2}\|\sigma_\nu^{1/2}u_\xi\|_{L^2}^{1/2} \\ &=\; Z\dot{B}_\nu a_\nu^{1/2} \;\leq\; \frac{\dot{B}_\nu}{3} + Z\dot{B}_\nu a_\nu. \end{aligned} \tag{3.13}$$

Consider now the other terms in (3.10). Proceeding as above, we have by CKN (2.10)

$$|\xi|\int_{\mathbb{R}^3} \sigma_\nu^2 |u_\xi|^2 \leq Z|\xi|\|\sigma_\nu^{1/2}\nabla u_\xi\|_{L^2}\|\sigma_\nu^{1/2}u_\xi\|_{L^2} = Z|\xi|(\dot{B}_\nu a_\nu)^{1/2} \leq \frac{\dot{B}_\nu}{3} + Z|\xi|^2 a_\nu; \tag{3.14}$$

and again by CKN (2.10)

$$\int_{\mathbb{R}^3} \sigma_\nu^2 |u_\xi|^3 = \|\sigma_\nu^{2/3} u_\xi\|_{L^3}^3 \le Z \|\sigma_\nu^{1/2} \nabla u_\xi\|_{L^2}^2 \|\sigma_\nu^{1/2} u_\xi\|_{L^2} = Z \dot{B}_\nu a_\nu^{1/2} \le \frac{\dot{B}_\nu}{3} + Z \dot{B}_\nu a_\nu. \tag{3.15}$$

Now recalling (3.10), summing all the inequalities and absorbing a term $\int_0^t \dot{B}_\nu(s) ds = B_\nu(t)$ from the left hand side, we obtain

$$a_\nu(t) + B_\nu(t) \le a_\nu(0) + Z \int_0^t \left(|\xi|^2 + \dot{B}_\nu(s)\right) a(s)\ ds,$$

and passing to the limit $\nu \to 0$, we arrive at the estimate

$$a(t) + B(t) \le a(0) + Z \int_0^t \left(|\xi|^2 + \dot{B}(s)\right) a(s)\ ds,$$

for some universal constant Z, where

$$a(t) = \int_{\mathbb{R}^3} |y|^{-1} |u_\xi(t,y)|^2 dy, \qquad B(t) = \int_0^t \int_{\mathbb{R}^3} |y|^{-1} |\nabla u_\xi(s,y)|^2 ds dy.$$

By a standard application of Gronwall's lemma we get for $0 \le t \le \bar{s}$

$$a(t) \le e^{ZA} a(0), \qquad A = B(\bar{s}) + \bar{s}|\xi|^2.$$

By (3.5) we have $A \le 2M^2 + \bar{s}|\xi|^2$. Thus once we restrict to the vectors ξ such that[5]

$$|\xi|^2 \bar{s} \le M^2 \tag{3.16}$$

the estimate becomes

$$a(t) \le e^{ZM^2} \epsilon^2 \quad \text{for} \quad 0 \le t \le \bar{s}, \tag{3.17}$$

where we have set $\epsilon^2 := a(0)$ and Z is a larger constant.

Let $\bar{s}_n < \bar{s}$ be a sequence converging to $\bar{s}$ and such that the strong version (in the sense of Lemma 2.2 with $t_0 = \bar{s}_n$) of the generalized energy inequality (3.7) applies for the sequence of test functions we are going to define. The existence of the sequence $\bar{s}_n$ is ensured by Corollary 2.3. We now repeat the previous argument, starting from the point $(\bar{s}_n, \bar{s}_n \xi)$, so we write the energy inequality (3.7) on the time interval $\bar{s}_n \le s \le t$ with $t \le \bar{s} + T$, choosing as test function $\phi(t,y) := \psi_\nu(t) \sigma_\nu(y) \chi(\delta|y|)$ where χ and σ_ν are as before[6], while

$$\psi_\nu(t) := e^{-kB_{\bar{s}_n,\nu}(t)}, \qquad B_{\bar{s}_n,\nu}(t) := \int_{\bar{s}_n}^t \int_{\mathbb{R}^3} \sigma_\nu |\nabla u_\xi|^2$$

with k a positive constant to be specified. Note that $B_{\bar{s}_n,\nu}$ is bounded if $\nu > 0$ by the properties of v. In this way we obtain, letting $\delta \to 0$,

$$\begin{aligned}[\int_{\mathbb{R}^3} \psi_\nu \sigma_\nu |u_\xi|^2]_{\bar{s}_n}^t &+ 2\int_{\bar{s}_n}^t \int_{\mathbb{R}^3} \psi_\nu \sigma_\nu |\nabla u_\xi|^2 \\ &\le \int_{\bar{s}_n}^t \int_{\mathbb{R}^3} \psi_\nu |u_\xi|^2 (-k\dot{B}_{\bar{s}_n,\nu} \sigma_\nu - \xi \cdot \nabla \sigma_\nu + \Delta \sigma_\nu) + \int_{\bar{s}_n}^t \int_{\mathbb{R}^3} \psi_\nu (|u_\xi|^2 + 2P_\xi u_\xi) \cdot \nabla \sigma_\nu\end{aligned}$$

for $\bar{s}_n \le t \le \bar{s} + T$, and this implies, recalling (3.9),

$$\begin{aligned}[\int_{\mathbb{R}^3} \psi_\nu \sigma_\nu |u_\xi|^2]_{\bar{s}_n}^t &+ 2\int_{\bar{s}_n}^t \int_{\mathbb{R}^3} \psi_\nu \sigma_\nu |\nabla u_\xi|^2 \\ &\le \int_{\bar{s}_n}^t \int_{\mathbb{R}^3} \psi_\nu |u_\xi|^2 (|\xi| \sigma_\nu^2 - k\dot{B}_{\bar{s}_n,\nu} \sigma_\nu) + \int_{\bar{s}_n}^t \psi_\nu \int_{\mathbb{R}^3} \sigma_\nu^2 (|u_\xi|^3 + 2|P_\xi||u_\xi|).\end{aligned} \tag{3.18}$$

Now the goal is to prove an integral inequality involving the quantities

$$a_\nu(t) = \int_{\mathbb{R}^3} \sigma_\nu |v_\xi(t)|^2, \qquad B_{\bar{s}_n,\nu}(t) = \int_{\bar{s}_n}^t \int_{\mathbb{R}^3} \sigma_\nu |\nabla v_\xi|^2.$$

---

[5]Remember that $\bar{s}$ is a function of ξ.

[6]To be precise we should consider vanishing sequences ν_l, δ_m instead of ν, δ in order to use Corollary 2.3. For simplicity we omit this detail and we continue to write simply ν, δ.

We estimate the terms at the right hand side of (3.18). We write again

$$I := 2\int_{\mathbb{R}^3}\sigma_\nu^2|P_\xi||u_\xi| = 2\int_{\mathbb{R}^3}\sigma_\nu^2|v_\xi||R\otimes R\ (v_\xi\otimes v_\xi)|.$$

With computations similar to those of the first step, using the (weighted) boundedness of the Riesz transform and the CKN inequality, we obtain

$$I \le \frac{\dot B_{\bar s_n,\nu}}{2} + Z\dot B_{\bar s_n,\nu}a_\nu. \tag{3.19}$$

Next we have

$$\begin{aligned}|\xi|\int_{\mathbb{R}^3}\sigma_\nu^2|u_\xi|^2 \;&=\; |\xi|\|\sigma_\nu u_\xi\|_{L^2}^2 \;\le\; Z|\xi|\|\sigma_\nu^{1/2}\nabla u_\xi\|_{L^2}\\ &=\; Z|\xi|(\dot B_{\bar s_n,\nu}a_\nu)^{1/2} \;\le\; |\xi|^2 + Z\dot B_{\bar s_n,\nu}a_\nu;\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3}\sigma_\nu^2|u_\xi|^3 \;&=\; \|\sigma_\nu^{2/3}u_\xi\|_{L^3}^3 \;\le\; Z\|\sigma_\nu^{1/2}\nabla u_\xi\|_{L^2}^2\|\sigma_\nu^{1/2}u_\xi\|_{L^2}\\ &=\; Z\dot B_{\bar s_n,\nu}a_\nu^{1/2} \;\le\; \frac{\dot B_{\bar s_n,\nu}}{2} + Z\dot B_{\bar s_n,\nu}a_\nu.\end{aligned} \tag{3.21}$$

We now plug the previous inequalities in (3.18) and we obtain

$$\begin{aligned}&a_\nu(t)\psi_\nu(t) - a_\nu(\bar s_n) + 2\int_{\bar s_n}^t \dot B_{\bar s_n,\nu}(s)\psi_\nu(s)ds\\ &\qquad\le \int_{\bar s_n}^t \psi_\nu(s)[\dot B_{\bar s_n,\nu} + 3Z\dot B_{\bar s_n,\nu}a_\nu + |\xi|^2 - k\dot B_{\bar s_n,\nu}a_\nu](s)ds.\end{aligned}$$

We subtract the first term at the right hand side from the left hand side; then we choose $k = 3Z$ and note that

$$\int_{\bar s_n}^t \dot B_{\bar s_n,\nu}\psi_\nu \equiv -\frac{1}{3Z}\int_{\bar s_n}^t\dot\psi_\nu = \frac{\psi_\nu(\bar s_n)-\psi_\nu(t)}{3Z} = \frac{1-\psi_\nu(t)}{3Z}$$

so that, for $\bar s_n \le t \le \bar s + T$, we obtain

$$a_\nu(t)\psi_\nu(t) - a_\nu(\bar s_n) + \frac{1-\psi_\nu(t)}{3Z} \le |\xi|^2\int_{\bar s_n}^t\psi_\nu(s)ds. \tag{3.22}$$

Consider now the increasing functions, for $t \ge \bar s$,

$$B_{\bar s}(t) := \int_{\bar s}^t\int_{\mathbb{R}^3}|y|^{-1}|\nabla v_\xi(s,y)|^2dyds, \quad B_{\bar s,\nu}(t) := \int_{\bar s}^t\int_{\mathbb{R}^3}\sigma_\nu|\nabla v_\xi(s,y)|^2dyds, \tag{3.23}$$

the first one may become infinite at some point $t > \bar s_n$. By the definition of $\bar s$, we know that $B_{\bar s}(t) \ge M$ for $t \ge \bar s + T/M$; since $B_{\bar s,\nu} \to B_{\bar s}$ pointwise as $\nu \to 0$, we have also

$$B_{\bar s,\nu}(s) \ge \frac{M}{2} \quad \text{for} \quad s \ge \bar s + \frac{T}{M} \quad \text{and } \nu \text{ small enough.}$$

Using this estimate for $s \ge \bar s + T/M$ and the obvious one $B_{\bar s,\nu} \ge 0$ for $s \le \bar s + T/M$, we have easily

$$\begin{aligned}\int_{\bar s_n}^{\bar s+T}\psi_\nu(s)\ ds \;&=\; \int_{\bar s_n}^{\bar s+T}e^{-3ZB_{\bar s_n,\nu}(s)}\ ds \;\le\; \int_{\bar s_n}^{\bar s+T}e^{-3ZB_{\bar s,\nu}(s)}\ ds\\ &\le\; \bar s - \bar s_n + \frac{T}{M} + e^{-\frac{3}{2}ZM}\left(T - \frac{T}{M}\right) \;\le\; \bar s - \bar s_n + \frac{3T}{2M}\end{aligned} \tag{3.24}$$

(here we use $Z \ge 1$). Since $T \ge 1$ and $\bar s_n \to \bar s$ as $n \to +\infty$, we can choose $n = n(M)$ large enough in such a way that $\bar s - \bar s_n \le \frac{T}{2M}$, so that (3.24) gives

$$\int_{\bar s_n}^{\bar s+T}\psi_\nu(s)\ ds \le \frac{2T}{M}. \tag{3.25}$$

We now use the estimate $a_\nu(\bar s_n) \le a(\bar s_n) \le e^{ZM^2}\epsilon^2$, proved in (3.17), and note that we can assume

$$\epsilon \le 1 \quad \Rightarrow \quad a(\bar s_n) \le e^{ZM^2}\epsilon. \tag{3.26}$$

Thus inequality (3.22) implies

$$(a_\nu(t) - \frac{1}{3Z})\psi_\nu(t) + \frac{1}{3Z} - e^{ZM^2}\epsilon - 2|\xi|^2\frac{T}{M} \le 0$$

or equivalently

$$a_\nu(t) + (\frac{1}{3Z} - e^{ZM^2}\epsilon - 2|\xi|^2\frac{T}{M})e^{3ZB_{\bar{s}_n,\nu}(t)} \le \frac{1}{3Z}. \tag{3.27}$$

We now assume ϵ is so small that

$$e^{ZM^2}\epsilon \le \frac{1}{9Z}, \tag{3.28}$$

(this also ensures (3.26)), so that (3.27) implies

$$a_\nu(t) + (\frac{2}{9Z} - 2|\xi|^2\frac{T}{M})e^{3ZB_{\bar{s}_n,\nu}(t)} \le \frac{1}{3Z}. \tag{3.29}$$

Assume in addition that ξ satisfies

$$(\frac{2}{9Z} - 2|\xi|^2\frac{T}{M}) > 0 \quad i.e. \quad |\xi|^2 T < \frac{M}{9Z}. \tag{3.30}$$

Note that this condition is stronger than the first condition (3.16) on ξ, *i.e.* $|\xi|^2\bar{s} \le M^2$, since $M, Z \ge 1$ and $\bar{s} \le T$. Then, if we let $\nu \to 0$, we have[7]

$$a_\nu(t) \to a(t) := \int_{\mathbb{R}^3} |y|^{-1}|v_\xi(t,y)|^2 dy,$$

$$B_{\bar{s}_n,\nu}(t) \to B_{\bar{s}_n}(t) := \int_{\bar{s}_n}^t \int_{\mathbb{R}^3} |y|^{-1}|\nabla v_\xi(s,y)|^2 dyds$$

and (3.29) implies, for all $\bar{s}_n \le t \le \bar{s} + T$

$$a(t) + (\frac{2}{9Z} - 2|\xi|^2\frac{T}{M})e^{3ZB_{\bar{s}_n}(t)} \le \frac{1}{3Z}. \tag{3.31}$$

In particular we see that $a(t)$ and $B_{\bar{s}_n}(t)$ (and so $B_{\bar{s}}(t)$) are finite for $\bar{s}_n \le t \le \bar{s} + T$. Since by the definition of $\bar{s}$ we already know that $B(\bar{s}) \le 2M^2 < +\infty$, we conclude that

$$B(s) < +\infty \quad \text{for all} \quad 0 \le s \le \bar{s} + T.$$

In particular we have

$$B(T) = \int_0^T \int |y|^{-1}|\nabla v_\xi(s,y)|^2 dyds = \int_0^T \int |x - s\xi|^{-1}|\nabla v(s,x)|^2 dyds < +\infty \tag{3.32}$$

and then the same argument used to conclude the proof in the first case ($\bar{s} = T$) gives, also in the second case ($\bar{s} < T$), that $L(T,\xi)$ is a regular set, provided (3.28, 3.30) are satisfied.

3.3. Conclusion of the proof

Summing up, we have proved that there exists a universal constant Z such that for any $M \ge 1$, $T \ge 1$ and $\xi \in \mathbb{R}^3 \setminus \{0\}$ the following holds: if $\epsilon = \||x|^{-1/2}u_0\|_{L^2(\mathbb{R}^3)}$ is small enough to satisfy (3.28), and T, ξ are such that (3.30) holds, then the segment $L(T,\xi)$ is a regular set for the weak solution u.

Now define

$$\delta_0 = \frac{1}{9Z}.$$

The (3.28) is implied by

$$\theta_2\epsilon \le \delta_0 e^{-ZM^2/\delta_0} \tag{3.33}$$

while (3.30) is implied by

$$|\xi|^2 T < M\delta_0 \qquad \Longleftrightarrow \qquad T > \frac{|T\xi|^2}{M\delta_0}$$

or equivalently

$$(T, T\xi) \in \Pi_{M\delta_0}, \qquad \Pi_{M\delta_0} := \{(t,x) \in (0,\infty) \times \mathbb{R}^3 \colon t > \frac{|x|^2}{M\delta_0}\}. \tag{3.34}$$

[7]Notice that it may be $a(t) = \infty$, $B_{\bar{s}_n}(t) = \infty$. Our aim is to use the estimate (3.31) to prevent this.

In other words, if ϵ satisfies (3.33) and $(T, T\xi)$ belongs to the paraboloid $\Pi_{M\delta_0}$, then $L(T, \xi)$ is a regular set. Since $\Pi_{M\delta_0}$ is the union of such segments for arbitrary $T \geq 1$ and $\xi \in \mathbb{R}^3 \setminus \{0\}$, we conclude that $\Pi_{M\delta_0}$ is a regular set for the solution u, provided (3.33) holds.

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