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Abstract

We discuss global well-posedness for the Kadomtsev-Petviashvili II in two and three space dimensions with small data. The crucial points are new bilinear estimates and the definition of the function spaces. As by-product we obtain that all solutions to small initial data scatter as $t \to \pm\infty$.

1. Introduction and main results

In this survey we study the Cauchy problem for the 2 and 3-dimensional Kadomtsev-Petviashvili II (KP-II) equation

$$\begin{cases} \partial_x \left(\partial_t u + \partial_x^3 u + \partial_x(u^2)\right) + \triangle_y u = 0 & (t,x,y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \\ u(0,x,y) = u_0(x,y) & (x,y) \in \mathbb{R} \times \mathbb{R}^d \end{cases} \tag{1.1}$$

where $d = 1$ is the two dimensional and $d = 2$ the three dimensional KP-II equation.

The Kadomtsev-Petviashvili (KP) equations describe nonlinear wave interactions of almost parallel waves. They come with at least four different flavors: The KP-II equation for which the line soliton is supposed to be stable, the KP-I equation with localized solitons, and the modified KP-I and KP-II equations with cubic nonlinearities.

The KP-II equation is invariant under

i) Translations in x, y and t.

ii) Scaling: $\lambda^2 u(\lambda x, \lambda^2 y, \lambda^3 t)$ is a solution if u satisfies the KP-II equation (1.1).

iii) Galilean transform: Let $c \in \mathbb{R}^2$. Then $u(t, x - c \cdot y - |c|^2 t, y + 2ct)$ is a solution if u satisfies (1.1). On the Fourier side the transform is $\hat{u}(\tau - |c|^2\xi - 2c\cdot\eta, \xi, \eta + c\xi)$ where τ is the Fourier variable of t, $\xi \in \mathbb{R}$ is the Fourier variable of x and η the one of y.

iv) Isometries of the y plane.

v) Simultaneous reflections of x, t and u.

The Galilean invariance is often a consequence of the rotational symmetry of full systems for which certain solutions are asymptotically described by a KP equation. The interest in the KP equations comes from the expectation that they describe waves in a certain asymptotic regime for a large class of problems, for which one does not even have to formulate a full model, similar to the role of the nonlinear Schrödinger equation in nonlinear optics.

Keywords: Kadomtsev-Petviashvili, Galilean transform, Bilinear estimate.

We search for spaces of initial data and solutions which reflect the symmetries. The norms

$$\|D_x^{-1/2}u_0\|_{L^2} = \||\xi|^{-1/2}\hat{u}_0\|_{L^2}$$

for $d = 1$ and

$$\|D_x^{1/2}u_0\|_{L^2} = \||\xi|^{1/2}\hat{u}_0\|_{L^2}$$

are invariant under scaling and the shear of the Galilean transform.

We base our construction of the solution space on the space V^2 of functions of bounded 2 variation V^2 adapted to the dimensional KP-II equation. We refer to [4] for their properties.

The x variable plays a prominent role and we use a Littlewood-Paley decomposition in the x variables: for $\lambda \in 2^{\mathbb{Z}}$ we define u_λ by its Fourier transform with respect to x,

$$\hat{u}_\lambda = \chi_{\lambda \le |\xi| < 2\lambda}\hat{u}.$$

We search the solution to the two dimensional problem in the function space X defined by the norm

$$\|u\|_X = \left(\sum_\lambda (\lambda^{-1/2}\|u_\lambda\|_{V^2})^2\right)^{\frac{1}{2}}.$$

We postpone the dicussion of the spaces V^2. Then the following theorem holds.

Theorem 1.1 (Hadac, Herr, Koch '09)**.** *There exists $\varepsilon > 0$ such that if $\|u_0\|_{\dot{H}^{-1/2,0}} < \varepsilon$ then there exists a unique solution u in a function space X which satisfies $\|u\|_{C(\mathbb{R};\dot{H}^{\frac{1}{2},0})} \le c\|u\|_X \le c\|u_0\|_{\dot{H}^{-\frac{1}{2},0}}$. The solution scatters.*

In [1], Bourgain settled the global well-posedness of the two dimensional version of (1.1) in $L^2(\mathbb{R}^2)$. The assertion was then extended by Takaoka and Tzvetkov [14] (see also Isaza and Mejía [7]) from $L^2(\mathbb{R}^2)$ to H^{s_1,s_2} with $s_1 > -\frac{1}{3}$, $s_2 \ge 0$. In [15], Takaoka obtained local well-posedness for $s_1 > -\frac{1}{2}$, $s_2 = 0$ under an additional assumption on the low frequencies which was later removed by Hadac in [3]. Hadac, Herr and the first author [4] studied the two dimensional KP-II equation in the critical case $s_1 = -\frac{1}{2}, s_2 = 0$. They obtained global well-posedness and scattering result in the homogeneous Sobolev space $\dot{H}^{-1/2,0}(\mathbb{R}^2)$ with small initial data. A local well posedness in $H^{-1/2,0}(\mathbb{R}^2)$ were also obtained in [4]. Some recent results on the two dimensional KP-II equation can be found in [9].

Much less is known for KP II in three dimensional spaces. Tzvetkov [16] obtained the local well-posedness in $H^s(\mathbb{R}^3)$ with the additional condition $\partial_x^{-1}u \in H^s(\mathbb{R}^3)$ for $s > \frac{3}{2}$. Here $H^s(\mathbb{R}^3)$ denotes the isotropic Sobolev space. Isaza, López and Mejía [6] constructed unique local solutions in Sobolev space $Y_{s,r}(\mathbb{R}^3)$ defined by

$$Y_{s,r}(\mathbb{R}^3) := \{f \in \mathcal{S}'(\mathbb{R}^3); \|f\|_{Y_{s,r}(\mathbb{R}^3)} := \| <\xi>^s <\zeta>^r \hat{f}(\zeta)\|_{L^2_\zeta} < \infty\}$$

for $s > 1, r > 0$. Hadac [2] in his Ph.D thesis extended the local well-posedness result to almost all the subcritical cases. He obtained local well posed for (1.1) in $Y_{s,r}(\mathbb{R}^3)$ for $s > \frac{1}{2}, r > 0$. To our best knowledge our result is the first result for initial data in a scaling invariant spaces, and the first scattering result for the three dimensional problem. Also the bilinear estimates accounting for dispersion in y seem to be new.

We will sketch a proof for a toy problem in the next section. The proof immediately carries over Theorem 1.1. The three dimensional KP-II equation requires some delicate improvements which we will discuss together with the statement in the last section.

I report on work with M. Hadac and Sebastian Herr [5] on the two dimensional case and with junfeng Li on the three dimensional case.

2. A toy problem: A nonresonant derivative NLS

Consider in $\mathbb{R} \times \mathbb{R}^2 \ni (t,x)$

$$i\partial_t u + \Delta u = \partial_{x_1}\bar{u}^2$$

with initial condition $u(0,x) = u_0(x)$.

Theorem 2.1. *There exists $\varepsilon > 0$ such that for all u_0 with $\|u_0\|_{L^2} < \varepsilon$ there exists a unique global in time solution u. It scatters at $t \to \pm\infty$: The limit*

$$\lim_{t\to\infty} e^{-it\Delta} u(t)$$

exists in L^2.

Let $\lambda \in 2^{\mathbb{Z}}$ and $\hat{u}_\lambda = \chi_{\lambda \le |\xi| < 2\lambda} \hat{u}$. Let

$$\|u\|_X = \left(\sum_{\lambda \in 2^{\mathbb{Z}}} \|u_\lambda\|_{V^2}^2 \right)^{1/2}.$$

Then

$$v(t) = \begin{cases} e^{it\Delta} u_0 & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

satisfies

$$\|v\|_X \le \sqrt{2} \|u_0\|_{L^2}$$

and

$$\left\| \int_0^t S(t-s) f(s) ds \right\|_{V^2} \le 2 \sup_{\|v\|_{V^2} \le 1} \left| \int f \bar{v} dx dt \right|.$$

We claim that always

$$\left| \int_{\mathbb{R}\times\mathbb{R}^2} \bar{u} \bar{v} \partial_{x_1} \bar{w} dx dt \right| \le c \|u\|_X \|v\|_X \|w\|_X. \tag{2.1}$$

Then, by duality

$$\left\| \int_0^t e^{i(t-s)\Delta} \partial_{x_1} \bar{u} \bar{v} ds \right\|_X \le c \|u\|_X \|v\|_X$$

and the theorem follows by standard arguments.

To prove (2.1) we expand the integrand and the functions

$$u = \sum_{\lambda \in 2^{\mathbb{Z}}} u_\lambda$$

where

$$\hat{u}_\lambda = \chi_{\lambda \le |\xi| < 2\lambda} \hat{u}.$$

The dyadic estimate

$$\sum_{\mu \le \lambda} \left| \int \bar{u}_\mu \bar{v}_\lambda \bar{w}_\lambda dx dt \right| \le c \lambda^{-1} \left(\sum_{\mu \le \lambda} \|u_\mu\|_{V^2}^2 \right)^{1/2} \|v_\lambda\|_{V^2} \|v_\lambda\|_{V^2}. \tag{2.2}$$

implies convergence of the fixed point argument. We postpone of (2.2) and show that it implies (2.1). We expand (with sums over $2^{\mathbb{Z}}$)

$$\int \bar{u} \bar{v} \partial_{x_1} \bar{w} dx dt \le \sum_{\lambda_1, \lambda_2, \lambda_2} \left| \int \bar{u}_{\lambda_1} \bar{v}_{\lambda_2} \partial_{x_1} \bar{w}_{\lambda_3} . dx dt \right|.$$

Since the integral of the product is the evaluation of the Fourier transform of the triple convolution at 0, there is only a contribution if there are frequencies adding up to zero in the support, i.e.

$$\xi_1 + \xi_2 + \xi_3 = 0, \lambda_j \le |\xi_j| \le 2\lambda_j.$$

Then necessarily the two larger numbers of λ_j are of similar size. To simplify the notation we assume that they are equal and we denote them by λ and the smaller number by μ.

Moreover

$$\|\partial_{x_1} w_{\lambda_3}\|_{V^2} \le 2\lambda_3 \|w_{\lambda_3}\|_{V^2}$$

and we may replace the derivative with a multiplication by λ_3.

We bound using (2.2)

$$\sum_\lambda \sum_{\mu\le\lambda} \left|\lambda \int \bar u_\mu \bar v_\lambda \bar w_\lambda dxdt\right| \le c \sum_\lambda \left(\sum_{\mu\le\lambda} \|u_\mu\|_{V^2}^2\right)^{1/2} \|u_\lambda\|_{V^2}\|w_\lambda\|_{V^2}$$
$$\le c\|u\|_X\|v\|_X\|w\|_X$$

and

$$\sum_\lambda \sum_{\mu\le\lambda} \mu \left|\int \bar u_\lambda \bar v_\lambda \bar w_\mu dxdt\right| \le c \sum_\lambda \sum_{\mu\le\lambda} \frac{\mu}{\lambda} \|u_\lambda\|_{V^2}\|v_\lambda\|_{V^2}\|w_\mu\|_{V^2}.$$
$$\le c\|u\|_X\|v\|_X\|w\|_X.$$

It remains to prove inequality (2.2). Its prove is based on Strichartz estimates, bilinear estimates, high modulation (i.e. non resonance) considerations and the adapted function spaces U^p and V^p.

The linear Schrödinger equation

$$i\partial_t u + \Delta u = 0$$

has a fundamental solution

$$g_t(x) = ((4\pi i t)^{1/2})^{-n} e^{\frac{|x|^2}{4it}}$$

with Fourier transform

$$\hat g_t(x) = e^{it|\xi|^2}$$

hence

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \qquad \|u(t)\|_{L^\infty} \le |4\pi t|^{-n/2}\|u_0\|_{L^1}$$

It defines a unitary group $S(t)$ (Fourier transform). Solutions are called *free waves.* They satisfy linear Strichartz and bilinear L^2 estimates. Bourgain has shown how to build function spaces which allow to use the Strichartz estimates and the bilinear estimates in the iteration. More precisely

$$\|u\|_{X^{s,b}} = \|S(-t)u(t)\|_{H^b(\mathbb{R};H^{s,0}(\mathbb{R}^{1+d}))} = \left\||\xi|^s|\tau - |\xi|^2|^b \hat u\right\|_{L^2}.$$

This pull back allows a reformulation of the equation. With $v(t) = S(-t)u(t)$ it becomes

$$v_t = S(-t)\partial_x(S(t)v(t))^2$$

This formulation is often useful for the study of scattering. It has often good regularity properties and it allows a transparent study of resonances. For critical problems one would want to use $X^{s,1/2}$ and $X^{s,-1/2}$ - which usually does work due to failing imbeddings like $H^{1/2} \not\subset C(\mathbb{R})$, $L^1 \not\subset H^{-1/2}$.

There is a replacement (Koch and Tataru [10], Hadac & Herr & Koch [5], Koch, [11]): The spaces U^p and V^p. We refer to [5] for the definition and explain here some properties. They 'interpolate' between the extremal Besov spaces:

$$B^{\frac12}_{2,1} \subset U^2 \subset V^2_{rc} \subset B^{\frac12}_{2,\infty}$$

Functions in V^2_{rc} are bounded and norms can be bounded via duality

$$\|u\|_{U^2(\mathbb{R})} \le \sup\left\{\int vu'dt : \|v\|_{V^2_{rc}} \le 1\right\}.$$

The integral on the right hand is a suggestive notation which takes some care to make sense of, see [5]. There are natural embeddings for $p < q$

$$\dot B^{\frac1p}_{p,1} \subset U^p \subset V^p_{rc} \subset \dot B^{\frac1p}_{p,\infty}. \tag{2.3}$$

The spaces of bounded p variation have been used by Wiener, Lepingle [12] and Bourgain and others in harmonic analysis and by T. Lyons [13] in the rough path theory of stochastic ODE's.

We follow the recipe of Bourgain to adapt the function spaces to an unitary operator and define

$$\|u\|_{U^p_S} = \|S(-t)u\|_{U^p}$$

and similarly we deal with V^p. We will omit the index $_S$.

Step 2. We want to bound the left hand side of (2.2), in particular (neglecting powers of 2π)

$$\left|\int \bar u_\mu \bar v_\lambda \bar w_\lambda dxdt\right| = |\hat u_\mu * \hat v_\lambda * \hat w_\lambda(0)|.$$

The integral is zero unless there are points in the support which add up to 0. If $\tau_1 = |\xi_1|^2$ and $\tau_2 = |\xi_2|^2$ and $\tau_3 = -\tau_1 - \tau_2$ and $\xi_3 = -\xi_1 - \xi_2$ then

$$\tau_3 - |\xi_3|^2 = -|\xi_1|^2 - |\xi_2|^2 - |\xi_1 + \xi_2|^2$$

Thus, with $\mu \le \lambda$, in

$$\int \bar u_\mu \bar v_\lambda \bar w_\lambda dx\, dt$$

at least one of the terms has high modulation - i.e. vertical distance $\lambda^2/3$ to the characteristic set, otherwise the integral is zero.

We denote this term by ^{h} and we have to bound using the Strichartz estimate

$$\|u\|_{L^4} \le c\|u\|_{U^4_S}$$

which is an immediate consequence of the Strichartz estimate for free waves:

$$\begin{aligned}\left|\int \bar u^h_\mu \bar v_\lambda \bar w_\lambda dxdt\right| &\le \|u^h_\mu\|_{L^2}\|(v_\lambda w_\lambda)_\mu\|_{L^2} \\ &\le \lambda^{-1}\|u_\mu\|_{V^2}\|v_\lambda\|_{L^4}\|w_\lambda\|_{L^4} \\ &\le \lambda^{-1}\|u_\mu\|_{V^2}\|v_\lambda\|_{U^4}\|w_\lambda\|_{U^4}\end{aligned}$$

This completes the estimate in this case since

$$\|v_\lambda\|_{U^4} \le c\|v_\lambda\|_{V^2}$$

and

$$\left(\sum_{\mu\le\lambda} \|(v_\lambda w_\lambda)_\mu\|^2_{L^2}\right)^{1/2} \le \|v_\lambda w_\lambda\|_{L^2}.$$

If the high modulation falls on a high frequency term we will use a bilinear estimate:

$$\|u_\lambda v_\mu\|_{L^2} \le c(\mu/\lambda)^{\frac12}\|u_\lambda\|_{U^2_S}\|v_\lambda\|_{U^2_S}$$

which again is a consequence of the analogous estimate for free waves to which we will return below. The bilinear estimate can be interpolated with the Strichartz estimate to yield

$$\|u_\lambda v_\mu\|_{L^2} \le c(\mu/\lambda)^{\frac12}\ln^2(1+\lambda/\mu)\|u_\lambda\|_{V^2_S}\|v_\lambda\|_{V^2_S}.$$

It is now an easy exercise to deduce the bound for

$$\int u_\mu v_\lambda w^h_\lambda dtdxdy.$$

The Strichartz estimates for free waves are a consequence of the stationary phase, Stein's complex interpolation and the weak Young inequality with refinements at the endpoint, see [8].

The Strichartz estimates for free waves

$$\|u\|_{L^p_t L^q} \le c\|u(0)\|_{L^2}$$

imply

$$\|u\|_{L^p_t L^q} \le c\|u\|_{U^p}$$

where in the case of KP-I and KP-II in 2d $\frac1p + \frac1q = \frac12$, $2 \le q < \infty$.

The bilinear estimate has a more geometric flavor. The Fourier transform of a solution to the linear equation with initial data u_0 is

$$2\pi \hat u_0(\xi)\delta_\phi$$

where $\phi(\tau,\xi) = \tau - |\xi|^2$ for the Schrödinger equaation. The distance to Σ measures the deviation from being a solution. We consider first free solutions resp. distributions supported on a surface:

$$\|\hat u_0\delta_\phi * \hat v_0\delta_\phi\|_{L^2} \le C\|u_0\|_{L^2}\|v_0\|_{L^2}$$

where (dyadic localization)

$$C^2 = \sup_{(\tau_i = |\xi_i|^2)} \int \delta_{(\phi(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1),\phi(\tau-\tau_2,\xi-\xi_2,\eta-\eta_2))} dx.$$

This is a $d-1$ dimensional integral. It can easily expressed as an integral with respect to the Hausdorff measure. Then

$$\|S(t)u_0 S(t)v_0\|_{L^2} \le C\|u_0\|_{L^2}\|v_0\|_{L^2}.$$

This calculus of Delta functions used here can be justified by replacing the Dirac functions by and approximation, and then applying the coarea formula to separate the effect of the regularization and the surface integrals. The proof boils down to an application of the Cauchy Schwartz inequality. See [11] for more details.

The bilinear estimates for free solutions

$$\|S(t)u_{0,\mu}S(t)v_{0,\lambda}\|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \le c(\mu/\lambda)^{1/2}\|u_{0,\mu}\|_{L^2(\mathbb{R}^2)}\|u_{0,\lambda}\|_{L^2(\mathbb{R}^2)}$$

imply

$$\|u_\mu v_\lambda\|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \le c(\mu/\lambda)^{1/2}\|u_\mu\|_{U^2}\|u_\lambda\|_{U^2}.$$

This completes the proof, up to replacing some U^2 by V^2 which can be done in this situation.

3. KP-II in 2d

The argument sketched above applies with small changes to the two dimensional KP-II equation. Due to the Galilean symmetry we apply the Littlewood-Paley decomposition only in the ξ frequencies. The U^p and V^p spaces are defined with respect to the KP-II evolution.

4. KP-II in 3d

The Strichartz estimates are

Lemma 4.1. *Suppose that* $2 \le p \le \infty$ *and*

$$\frac{2}{p} + \frac{3}{q} = \frac{3}{2}. \tag{4.1}$$

Then the following estimate holds for all $u_0 \in \mathcal{S}$

$$\|u\|_{L^p_t L^q_x} \lesssim \||D_x|^{\frac{1}{3p}} u_0\|_{L^2}.$$

If $2 \le q < \infty$

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} \tag{4.2}$$

then

$$\|u\|_{L^p_t L^q_x} \lesssim \||D_x|^{\frac{2}{p}} u_0\|_{L^2}.$$

The proof follows the same lines as in the two dimensional setting. The endpoint $p=2$ and $q=6$ follows from [8].

There is an important special case of (4.2):

$$\|u\|_{L^4(\mathbb{R}^4)} \le c\||D_x|^{\frac{1}{2}} u_0\|_{L^2(\mathbb{R}^3)}. \tag{4.3}$$

The proof of the main theorem relies crucially on the following bilinear refinements. We denote by $u_{<\mu}$ the Fourier projection to all ξ frequencies less in absolute value than μ, by $u_{>\lambda}$ the Fourier projection to ξ frequencies with absolute value $>\lambda$ and by $u_{\mu,\Gamma}$ the Fourier projection to

$$\Big\{(\xi,\eta) : \mu < |\xi| \le 2\mu, \frac{\eta}{\xi} \in \Gamma\Big\}.$$

Let $|\Gamma|$ denote the Lebesgue measure of Γ. With this notation the following variant or sharpening of the bilinear estimate is true.

Theorem 4.1. *Let $0 < \mu, \lambda$. Then*

$$\|u_{<\mu} v_{>\lambda}\|_{L^2} \le c\mu \|u_0\|_{L^2} \|v_0\|_{L^2}, \tag{4.4}$$

and, if $\mu \le \lambda$, if $\Gamma \subset \mathbb{R}^2$ is measurable, and if either

- $\mu \le \lambda/8$ *or*
- $\lambda/8 < \mu \le \lambda$ *and $\Gamma \subset B_\lambda(0)$ and the support of the Fourier transform of v_λ is disjoint from $\mathbb{R} \times \mathbb{R} \times B_{10\lambda^2}(0)$*

then

$$\Big\| \int_{\mathbb{R}\times\mathbb{R}^2} \Big(\lambda + \Big|\frac{\eta_1}{\xi_1} - \frac{\eta-\eta_1}{\xi-\xi_1}\Big|\Big) \hat{u}_{\mu,\Gamma}(t,\xi_1,\eta_1) \hat{v}_\lambda(t,\xi-\xi_1,\eta-\eta_1) d\xi_1 d\eta_1 \Big\|_{L^2}$$
$$\lesssim \mu |\Gamma|^{\frac12} \|u_{0,\mu,\Gamma}\|_{L^2} \|v_{0,\lambda}\|_{L^2}. \tag{4.5}$$

Let

$$\|u\|_X = \sup_\lambda \Big(\lambda^{1/2} \|u_\lambda\|_{U^2} + \lambda^{-1} \|u_\lambda\|_{X^{0,1}} \Big)$$

Then

$$\|u_\lambda u_\lambda\|_{L^2} \le c\lambda \|u_\mu\|_{U^2} \|u_\lambda\|_{U^2}$$

implies

$$\mu^{-1} \Big\| \Big(\int_0^t S(t)\partial_x (u_\lambda u_\lambda) ds \Big)_\mu \|_{X^{0,1}} \le c\lambda \|u_\lambda\|_{V^2} \|u_\lambda \Big\|_{V^2}$$

and by duality

$$\begin{aligned} \lambda^{\frac12} \| \int_0^t S(t-s)\partial_x u_\mu^h u_\lambda dx dt \|_{V^2} &= \lambda^{\frac32} \sup_{\|w_\lambda\|_{U^2} \le q} \int u_\mu^h u_\lambda w_\lambda dx dt dy \\ &\le \lambda^{\frac32} (\mu\lambda^2)^{-1} \|u_\mu\|_{X^{0,1}} \lambda \|u_\lambda\|_{V^2} \\ &\le (\lambda^{\frac12} \|u_\lambda\|_{V^2})(\mu^{-1} \|u_\mu\|_{X^{0,1}} \end{aligned}$$

It is immediately obvious that these estimates are tight.There are three areas we need to improve the argument.

i) Replace V^2 by U^2.

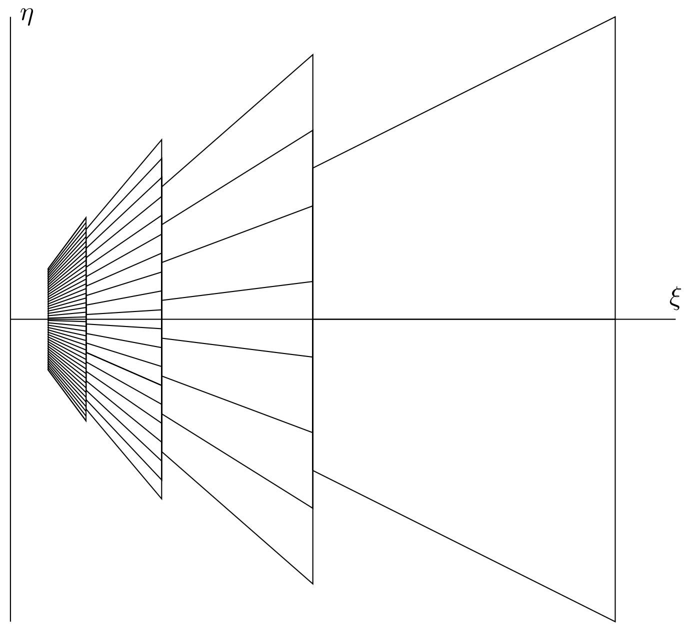
ii) Deal with the summation over the Littlewood-Paley pieces

iii) The solution to the linear problem $\chi_{t\ne 0} S(t) u_0$ is not in $X^{0,1}$ unless it is trivial since the characteristic function is not in H^1_{loc}.

We recall the resonance relation

$$(\xi_1+\xi_2)^3 - \frac{|\eta_1+\eta_2|^2}{\xi_1+\xi_2} - (\xi_1^3 - \frac{|\eta_1|^2}{\xi_1}) - (\xi_2^3 - \frac{|\eta_2|^2}{\xi_2}) = \xi_1\xi_2(\xi_1+\xi_2) \left(3 + \frac{\Big|\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}\Big|^2}{|\xi_1+\xi_2|^2} \right)$$

In the previous argument we only used the first summand of the second factor on the right hand side. The second term expresses some gain when the slope of the lines through (ξ_j, η_j) differs. There is a similar gain in the second part of the bilinear estimate. This motivates are finer partition of the frequency space. The only way - up to equivalent choices - to partition three dimensional space is shown in the graphic



By scaling we may focus on $\lambda = 1$ and $\mu \le \lambda$. Both the bilinear estimate and the resonance relation indicate that we may limit considerations to frequencies $\{(\xi,\eta) : |\eta| \le |\xi| \le 1\}$. The problem is that for μ small there are about μ^2 pieces interacting with the same set with $\frac{1}{2} \le |\xi| \le 1$. The previous bilinear estimate shows that we obtain tight estimates if we use function spaces with l^2 summation with respect to η.

For fixed λ, we partition the set $\{(\xi,\eta) \in \mathbb{R} \times \mathbb{R}^2 : \lambda \le |\xi| < 2\lambda\}$ into sets $\Gamma_{\lambda,k}$ for $k \in \lambda \cdot \mathbb{Z}^2$ defined by

$$\Gamma_{\lambda,k} = \left\{ (\xi,\eta) : \lambda \le |\xi| < 2\lambda, \left| \frac{\eta}{\xi} - k \right|_\infty \le \frac{\lambda}{2} \right\}$$

where

$$|a|_\infty = \max\{|a_1|, |a_2|\}.$$

For any $1 \le q < \infty$, $1 \le p < \infty$, we say a distribution f is in $l^q l^p L^2$ if is in the closure of C_0^∞ with respect to the norm

$$\|f\|_{l^q l^p L^2} := \left\{ \sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^{\frac{q}{2}} \left(\sum_{k \in \lambda \cdot \mathbb{Z}^2} \|f_{\Gamma_{\lambda,k}}\|_{L^2}^p \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} < \infty.$$

The case $p, q = \infty$ follows an obvious modification.

We need also the homogeneous Fourier restriction space $\dot{X}^{0,b}$ for $|b| \le 1$ which is defined by

$$\|u_1\|_{\dot{X}^{0,b}} = \| |\partial_t - \partial_x^3 + \partial_x^{-1} \Delta_y|^b u_1 \|_{L^2} := \| |\tau - \omega(\xi,\eta)|^b \hat{u}_1 \|_{L^2} < +\infty$$

for distributions supported in $[0,\infty) \times \mathbb{R} \times \mathbb{R}^2$.

Here $\omega(\xi,\eta) = \xi^3 - \frac{|\eta|^2}{\xi}$ is the dispersion function associated to KP-II equation. We define

$$\|u\|_{l^q \dot{X}^{0,b}} = \|\lambda^2 u_\lambda(\lambda x, \lambda^2 y, \lambda^3 t)\|_{l^q_\lambda \dot{X}^{0,b}} = \left(\sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^{(2-3b)q} \|u_\lambda\|_{\dot{X}^{0,b}}^q \right)^{\frac{1}{q}}.$$

Where the l^p denotes the p summation over the same partition as above. Finally we define the function space for the fixed point map by

$$\|u\|_X = \|u\|_{l^\infty l^p V^2_{KP}} + \|u\|_{l^q \dot{X}^{0,b}} < \infty.$$

We obtain

Theorem 4.2 (Koch, Li 15)**.** *There exist* $\frac{4}{3} < p < 2$, $b < 1$ *and* $\varepsilon > 0$ *so that for*

$$\|u_0\|_{l^\infty l^p L^2} < \varepsilon$$

there is a unique global solution $u = S(t)u_0 + w$ *with*

$$\|w\|_X \leq c\|u_0\|^2_{l^\infty l^p L^2}.$$

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