

# Sharp $L^p$ Carleman estimates and unique continuation

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## Abstract

We will present a unique continuation result for solutions of second order differential equations of real principal type  $P(x, D)u + V(x)u = 0$  with critical potential  $V$  in  $L^{n/2}$  (where  $n$  is the number of variables) across non-characteristic pseudo-convex hypersurfaces. To obtain unique continuation we prove  $L^p$  Carleman estimates, this is achieved by constructing a parametrix for the operator conjugated by the Carleman exponential weight and investigating its  $L^p - L^{p'}$  boundedness properties.

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## 1. Introduction

The aim of these notes is to present a unique continuation result for solutions of second order differential equations with a potential with critical regularity in the

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Lebesgues spaces. Let  $P(x, D)$  be a second order differential operator of real principal type with  $C^\infty$  coefficients and  $u$  be the solution of the differential equation

$$P(x, D)u + V(x)u = 0, \quad x \in \Omega \subset \mathbb{R}^n \quad (1)$$

where  $V \in L_{loc}^{n/2}$  and let  $S$  be a  $C^\infty$  hypersurface in  $\mathbb{R}^n$  locally defined by

$$S \equiv \psi(x) = \psi(x_0) \quad (2)$$

in a neighbourhood  $\Omega$  of  $x_0$ . We consider the two sides of this orientated hypersurface

$$S_\pm = \{x \in \Omega : \psi(x) \gtrless \psi(x_0)\}$$

and say that a solution  $u$  of (1) has unique continuation across the hypersurface  $S$  if when  $u$  vanishes on the side  $S_-$  then it vanishes on a whole neighbourhood of  $x_0$ . The  $L^{n/2}$  regularity for the potential is the critical one, there are known counter-examples showing that uniqueness may fail for potentials in  $L^p$ ,  $p < n/2$ .

The usual tool for proving unique continuation results is the Carleman estimates; these are weighted exponential inequalities, which in the  $L^p$  framework take the form

$$\|e^{-\sigma\phi}v\|_{L^p} \leq C\|e^{-\sigma\phi}P(x, D)v\|_{L^{p'}} \quad (3)$$

for  $C^\infty$  compact supported functions  $v$ , when  $\sigma > 0$  is large enough. The phase  $\phi$  of the exponential weight is chosen such that if  $p_\sigma(x, \xi) = p(x, \xi - i\sigma\phi'(x))$  stands for the principal symbol of the conjugated operator  $e^{-\sigma\phi}P(x, D)e^{\sigma\phi}$  then

$$p_\sigma = 0 \Rightarrow \frac{1}{\sigma}\{\operatorname{Re} p_\sigma, \operatorname{Im} p_\sigma\} > 0. \quad (4)$$

Related to the  $L^p$  Carleman estimates is the question of proving Strichartz type estimates ( $L^p - L^{p'}$  estimates) for pseudo-differential operators with complex symbols; in particular understanding the underlying geometry in the case of symplectic type pseudo-differential operator (for which the Poisson bracket of the real and imaginary parts of the symbol is positive on the characteristic set) is of substantial interest. We refer the reader to [4] and [5] for the investigation of such estimates in two different geometric cases.

The unique continuation problem for equations with potential in the Lebesgues spaces has been widely studied. Jerison and Kenig initiated the theory of  $L^p$  Carleman estimates for the Laplace operator and proved unique continuation results for elliptic constant coefficient operators in [9]. This was latter generalised to elliptic variable coefficient operators by Sogge [16]. There were further improvements by Wolff [21] for elliptic operators with less regular coefficients and by Koch and Tataru [11] who considered the problem with gradient terms.

For second order hyperbolic operators, unique continuation across space-like hypersurfaces was obtained by Sogge in [17]. For the case of time-like hypersurfaces, the first results were obtained by Kenig, Ruiz and Sogge in [13] who proved unique continuation across any hyperplane when  $P$  is a constant coefficient operator (with symbol a non-degenerate quadratic form). Then Tataru, studying function spaces adapted to the differential operator  $P$  in [19], proved unique continuation for second

operators across pseudo-convex hypersurfaces in the subcritical case ( $V \in L_{loc}^p$ ,  $p > n/2$ ). To be mentioned are also the results of Escauriaza and Vega [6] on the heat equation.

To this one should add a forthcoming paper [12] of Koch and Tataru investigating estimates for operators with complex symbols; their results are complementary to some obtained by the author (see [4] and [5]) and allow them to obtain unique continuation results similar to those described in these notes.

## 2. Statement of the main result

We will need some additional assumptions on the hypersurface  $S$  to obtain unique continuation. We will suppose that  $S$  is non-characteristic and pseudo-convex. Let us recall the definition of (strict<sup>1</sup>) pseudo-convexity:

**Definition 2.1.** *The hypersurface  $S \equiv \{x \in \Omega : \phi(x) = \phi(x_0)\}$  is said to be (strictly) pseudo-convex at  $x_0 \in \Omega$  with respect to the real principal type differential operator  $P$  of order 2 (with principal symbol  $p$ ) whenever*

$$p(x_0, \xi) = H_p \phi(x_0, \xi) = 0 \Rightarrow H_p^2 \phi(x_0, \xi) < 0, \quad \forall \xi \in \mathbb{R}^n.$$

**Remark 2.2.** The geometric interpretation is the following: the transverse characteristics cross the hypersurface whereas the tangential ones remain on one side.

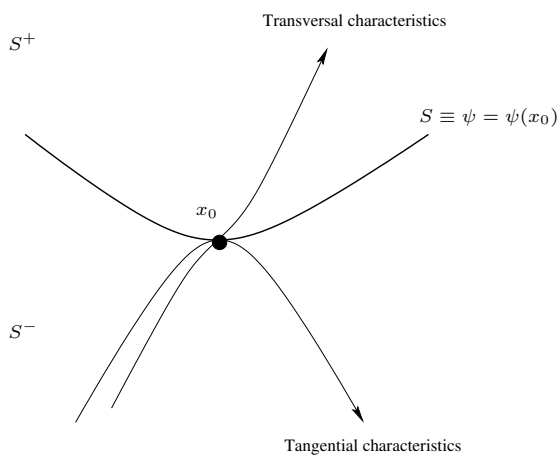


Figure 1: Pseudo-convexity.

**Remark 2.3.** In the case of the wave operator, a space-like hypersurface is always pseudo-convex.

Let us now state the main theorem.

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<sup>1</sup>For second order real principal type operators, the notions of pseudo-convexity and strict pseudo-convexity coincide, hence we shall give here the definition of strict pseudo-convexity which is simpler.

**Theorem 2.4.** *Let  $P(x, D)$  be a differential operator of order 2 of real principal type defined on an open set  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$  and  $S$  be a hypersurface defined by (2) non-characteristic and pseudo-convex with respect to  $P$  at  $x_0$ . If  $u \in H^1$  is solution of the differential equation (1) on  $\Omega$  with  $V \in L_{loc}^{n/2}$  and  $u|_{S_-} = 0$  then  $x_0 \notin \text{supp } u$ .*

Typically, we would like to consider the wave operator with a time-like hypersurface satisfying some convexity assumption.

### 3. The Carleman method

The Carleman method consists in establishing adapted weighted  $L^p$  inequalities with an exponential weight  $e^{-\sigma\phi}$ ,  $\sigma$  being a large enough parameter and  $\phi$  a  $C^\infty$  function, the level set of which is in a suitably convexified situation (see figure 2) and to deduce unique continuation therefrom. This is the object of the following proposition.

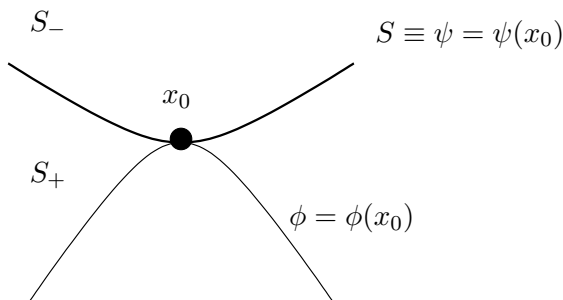


Figure 2: Relative situations of the level sets of  $\phi$  and  $\psi$ .

**Proposition 3.1.** *Let  $P(x, D)$  be a differential operator of order 2, defined on an open set  $\Omega \subset \mathbb{R}^n$ ,  $n > 2$ . Suppose that there exists  $M \geq 1$ , a neighbourhood  $\Omega_0$  of  $x_0 \in \Omega$  and a function  $\phi \in C^\infty$  verifying  $\{x \in \Omega_0 : x \neq x_0, \phi(x) \leq \phi(x_0)\} \subset S_-$  such that the Carleman estimate*

$$\|e^{-\sigma\phi}v\|_{L^{\frac{2n}{n-2}}} \leq C\|e^{-\sigma\phi}P(x, D)u\|_{L^{\frac{2n}{n+2}}} \quad (5)$$

*holds for all  $v \in C_0^\infty(\Omega_0)$  and  $\sigma \geq M$ . Then if  $u \in H^1(\Omega)$  is solution of the equation (1) on  $\Omega$  where  $V \in L_{loc}^{n/2}$  and  $u$  vanishes on  $S_-$ , then there exists a neighbourhood of  $x_0$  on which  $u$  vanishes.*

Let us briefly recall why the Carleman estimate implies unique continuation. Suppose  $x_0 = 0$  and  $\psi(0) = 0$  and let  $\chi \in C_0^\infty(\Omega_0)$  equal 1 on a compact neighbourhood  $\Omega_1$  of 0. Applying the Carleman inequality to the function  $v = \chi u$  with compact support gives

$$\|e^{-\sigma\phi}v\|_{L^{\frac{2n}{n-2}}} \leq C\|V\|_{L^{\frac{n}{2}}(\text{supp } \chi)}\|e^{-\sigma\phi}v\|_{L^{\frac{2n}{n-2}}} + C\|e^{-\sigma\phi}[P(x, D), \chi]u\|_{L^{\frac{2n}{n-2}}}.$$

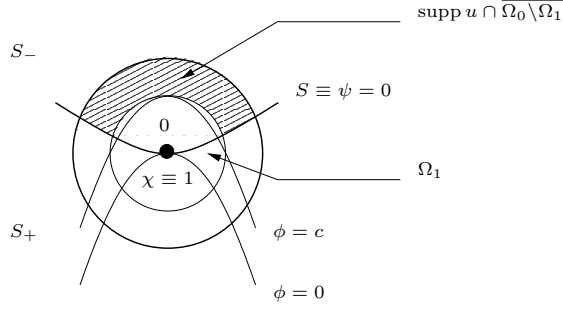


Figure 3: The Carleman method.

Then we choose  $\chi$  with sufficiently small support so that the first right handside term may be absorbed in the left handside term, thus

$$\|e^{-\sigma\phi}\chi u\|_{L^{\frac{2n}{n-2}}} \leq 2C\|e^{-\sigma\phi}[P(x, D), \chi]u\|_{L^{\frac{2n}{n+2}}}.$$

But  $[P(x, D), \chi]u$  is supported in  $\text{supp } u \cap \overline{\Omega_0 \setminus \Omega_1} \subset \{\phi > 0\}$  where one has  $\phi > c > 0$  (see figure 3), which implies

$$\|e^{-\sigma\phi}\chi u\|_{L^{\frac{2n}{n-2}}} \leq 2Ce^{-\sigma c}\|[P(x, D), \chi]u\|_{L^{\frac{2n}{n+2}}} \leq C'e^{-\sigma c}\|u\|_{H^1}.$$

and hence that  $\|e^{-\sigma(\phi-c)}\chi u\|_{L^{\frac{2n}{n-2}}}$  is bounded, which is impossible unless  $u = 0$  when  $\phi < c$ . This completes the proof of unique continuation.

The Carleman estimates are merely  $L^p - L^{p'}$  estimates for the conjugated operator  $P_\sigma = P(x, D - i\sigma\phi'(x))$ , whose principal symbol  $p(x, \xi - \sigma\phi'(x))$  is always *complex*. Besides, one should have in mind that because of the essential property (4) it is important to keep track of the *positivity* of the parameter  $\sigma$ . Note that an important ingredient in the proof of  $L^p - L^{p'}$  estimates is the curvature of the characteristic set; in our case, this is contained in the fact that the principal symbol of  $P$  is a non-degenerate quadratic form (hence it vanishes on a codimension 1 manifold with  $(n - 2)$  non-vanishing principal curvatures<sup>2</sup>).

We shall concentrate on the proof of the Carleman estimates. The proof is organised along the following principal points: microlocalisation, reduction of the operator to a normal form, construction of a microlocal parametrix, investigation of the  $L^p - L^{p'}$  boundedness properties of the parametrix, estimation of the remainders. We will give some ideas of the proof.

We suppose that  $x_0 = 0$ . After a change of coordinates, we may suppose that the principal symbol of the operator takes the form

$$p(x, \xi) = \xi_1^2 + r(x, \xi') \quad (6)$$

where  $r$  is a non-degenerate quadratic form in  $\xi' \in \mathbb{R}^{n-1}$ , that the hyperplane  $x_1 = 0$  is pseudo-convex with respect to  $P$  and that the hypersurface  $S$  is convex with

<sup>2</sup>Because of the homogeneity, this is the maximal curvature we may have on the zero set of the principal symbol.

respect to  $x_1 = 0$  (see figure 2). We choose  $\phi(x) = x_1 - Ax_1^2/2$  for the exponential weight. The pseudo-convexity at 0 reads

$$r(0, \xi') = 0 \Rightarrow \frac{\partial r}{\partial x_1}(0, \xi') > 0$$

and allows us to choose  $A > 0$  such that the property (4) is true.

The principal symbol of the conjugated operator is then:

$$p_\sigma(x, \xi) = \xi_1^2 + r(x, \xi') - \sigma^2(1 - Ax_1)^2 - 2i\sigma(1 - Ax_1)\xi_1. \quad (7)$$

We are interested in constructing a parametrix microlocally in a conic neighbourhood of the characteristic set

$$\begin{aligned} \Sigma &= \{(\xi, \sigma) \in \mathbb{R}^n \setminus \{0\} \times \mathbb{R}^+ : p(0, \xi - i\sigma\phi'(0)) = 0\} \\ &= \{\xi_1^2 + r(0, \xi') = \sigma^2\} \cap \{\xi_1\sigma = 0\}. \end{aligned}$$

Three zones may be distinguished :

1. in the neighbourhood of  $\Sigma^+ = \{r(0, \xi') = \sigma^2\} \cap \{\xi_1 = 0\} \cap \{|\xi'| \lesssim \sigma\}$ , since  $r > 0$ , the situation is (microlocally) similar to the one which arises in the case of unique continuation for elliptic operators (see [16]),
2. in the neighbourhood of  $\Sigma^- = \{r(0, \xi') = -\xi_1^2\} \cap \{\sigma = 0\} \cap \{|\xi'| \lesssim |\xi_1|\}$ , since  $r < 0$ , the situation is (microlocally) similar to the one which arises in the case of unique continuation for hyperbolic operators (see [17]) across a space-like hypersurface,
3. in the neighbourhood of  $\Sigma_0 = \{r(0, \xi') = 0\} \cap \{\sigma = 0\} \cap \{\xi_1 = 0\}$  the situation is unknown.

In what follows we will concentrate on the third case.

On a conic neighbourhood of  $(0, \xi'_0, 0)$  with  $r(0, \xi'_0) = 0$ , assuming for instance that  $\partial r / \partial \xi_n(0, \xi'_0) \neq 0$ , using the Malgrange-Weierstrass theorem, the symbol  $p_\sigma$  reads

$$\xi_n + \alpha(x, \xi_1, \xi'', \sigma) + i\sigma\beta(x, \xi_1, \xi'', \sigma)$$

(with  $\xi = (\xi_1, \xi'', \xi_n)$ ) modulo an elliptic factor homogeneous of order 1. Using the Littlewood-Paley theory, we may also restrict our construction to a zone where  $\sigma + |\xi| \sim |\xi'| \sim \lambda$ .

We will take  $t = x_n$  as the evolution variable and change a little notations by taking  $(t, x) = (t, x_1, \dots, x_{n-1})$  as what used to be the  $x$  variables. Conjugating the operator  $P_\sigma$  by a Fourier integral operator  $U_1(t)$  with real phase reduces the problem to the construction of a parametrix for

$$D_t + i\sigma f(t, x, D_x, \sigma).$$

But since the property (4) is invariant by symplectic change of coordinates, we have  $\partial f / \partial t > 0$ , thus the latter operator takes the form

$$D_t + i\sigma(t + \omega(x, D_x, \sigma))\Lambda(t, x, D_x, \sigma)$$

where  $\Lambda > 0$ , this may also be reduced after conjugation by another Fourier integral operator with real phase  $U_2$  (independent of the time) to

$$D_t + i\sigma(t + \lambda^{-1}D_{x_1})\Lambda(t, x, D_x, \sigma) \quad (8)$$

with  $\Lambda > 0$ .

If we sum up, after conjugation by a Fourier integral operator  $U(t) = U_1(t)U_2$  with real phase, the problem of finding a parametrix to the operator  $P_\sigma$  reduces to the construction of a parametrix for the model operator (8). This is fortunate since the fact that the change of sign of the imaginary part of the symbol of (8) occurs on a fairly simple set will allow us to use the method of Treves [20] to construct such a parametrix. The phase of the Fourier integral operator  $U(t)$  satisfies some non-degeneracy property which will be useful in the investigation of the  $L^p - L^{p'}$  boundedness properties of the parametrix. This property follows basically from the fact that  $p(x, \xi)$  is a non-degenerate quadratic form (hence vanishes on a codimension one manifold with  $n - 2$  non-vanishing principal curvatures), and may be tracked along the different reductions of the symbol.

#### 4. A parametrix construction

To find a left parametrix for the reduced operator (8) we seek a right parametrix for  $D_t - i\sigma(t + \lambda^{-1}D_{x_1})\Lambda$  and then take the adjoint.

Let us first study the model case  $D_t - i\sigma(t + \lambda^{-1}D_{x_1})$ . In this case, the construction is fairly easy since we can take the Fourier transform with respect to  $x_1$  and solve an ordinary differential equation according to the sign of the quantity  $t + \lambda^{-1}\eta_1$ . Namely we have<sup>3</sup>

$$Eu(t, x) = e^{-\sigma(t-s)^2/2} \iiint \omega e^{i(x-y)\cdot\eta} e^{-\sigma(t-s)(s+\lambda^{-1}\eta_1)} a(\eta, \sigma, \lambda) u(s, y) dy d\eta ds \quad (9)$$

$$\text{where } \omega(s, t, \eta_1, \lambda) = H(t-s) - \check{H}(\lambda s + \eta_1)$$

and where  $a$  is just a localising function in the set  $\sigma + |\xi| \sim \lambda$  and in the neighbourhood of a point of  $\Sigma_0$ . Observe that the integral lies precisely on a set where the real exponentials are decreasing.

In the general case, we seek the parametrix under the form

$$Eu(t, x) = \iiint \omega e^{iw(s,t,x,\eta,\sigma,\lambda) - iy\cdot\eta} a(s, t, x, \eta, \sigma, \lambda) u(s, y) dy d\eta ds \quad (10)$$

where  $\text{Im } w \geq 0$  when  $\omega \neq 0$  and  $\sigma \geq 0$ , and where  $w$  is a solution (at least in an approximate sense) of the *complex* eiconal equation

$$\begin{aligned} \partial_t w &= i\sigma(t + \partial_{x_1} w) \tilde{\Lambda}(t, x, \partial_x w) \\ w|_{t=s} &= x \cdot \eta \end{aligned}$$

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<sup>3</sup> $H$  is the Heaviside function, and  $\check{H}(t) = H(-t)$ .

where  $\tilde{\Lambda}$  is an almost analytic extension of  $\Lambda$ , and where  $a$  is solution of a complex transport equation. This requires a careful application of the construction of Treves [20] for solving complex eiconal equation.

We find that the phase looks very much like the one in the model case

$$w(s, t) = x \cdot \eta + i\sigma(t - s)(s + \lambda^{-1}\eta_1)\Lambda(s, x, \eta) + \sigma(t - s)^2 v(s, t)$$

where  $v(s, t)$  is a complex function with positive imaginary part. Besides it is fairly simple to obtain that the operator

$$W(t, s)u = \iiint \omega e^{iw(s,t,x,\eta,\sigma,\lambda) - iy \cdot \eta} a(s, t, x, \eta, \sigma, \lambda) u(s, y) dy d\eta \quad (11)$$

is bounded on  $L^2$ .

## 5. The $L^p - L^{p'}$ estimates

It is time now to study the  $L^p - L^{p'}$  boundedness of the complete microlocal parametrix  $U(t)E^*U^*(t)$  of the operator  $P_\sigma$ . Recall that  $U(t)$  is the Fourier integral operator with real phase reducing the conjugated operator  $P_\sigma$  to the normal form (8).

**Proposition 5.1.** *We have the following estimate:*

$$\|U(t)E^*U^*(t)u\|_{L^p} \leq C\lambda\|u\|_{L^{p'}} \quad (12)$$

for all functions  $u \in C_0^\infty(\mathbb{R}^n)$ .

The operator  $U(t)E^*U^*(t)$  takes the form

$$\int U(t)W^*(t, s)U^*(s) ds$$

where  $W(s, t)$  is given by (11), hence it suffices to prove the estimate

$$\|U(t)W^*(t, s)U^*(t)f\|_{L^p(\mathbb{R}_x^d)} \leq C\lambda|t - s|^{-(1-\frac{2}{n})}\|f\|_{L^{p'}(\mathbb{R}_x^d)} \quad (13)$$

which gives

$$\|U(t)EU^*(t)u\|_{L^p(\mathbb{R}_x^d)} \leq C\lambda \int |t - s|^{-(1-\frac{2}{n})}\|u(s)\|_{L^p(\mathbb{R}_x^d)} ds$$

and implies the estimate (12) thanks to the Hardy-Littlewood-Sobolev inequality since

$$1 - \frac{1}{p} = 1 - \frac{1}{p'} + \frac{2}{n}.$$

The estimate (13) is obtained by interpolation between the  $L^2$  estimate

$$\|U(t)W^*(t, s)U^*(t)f\|_{L^2(\mathbb{R}_x^d)} \leq \|f\|_{L^2(\mathbb{R}_x^d)}$$



which as said previously is fairly simple to obtain, and the  $L^\infty - L^1$  estimate

$$\|U(t)W^*(t, s)U^*(t)f\|_{L^\infty(\mathbb{R}_x^d)} \leq \lambda^{n/2}|t-s|^{-\frac{n-2}{2}}\|f\|_{L^1(\mathbb{R}_x^d)}. \quad (14)$$

To give an idea of the proof of (14) we will restrict ourselves to the model case where  $U(t) = e^{it|D_x|}$  — which basically contains all the non-degeneracy properties of the phase which will be needed — and where  $W(s, t)$  is given by (11) in the model case (9).

The estimate (14) comes from a  $L^\infty$  bound on the kernel of  $U(t)W^*(t, s)U^*(s)$  which in our model case takes the form

$$e^{-\sigma(t-s)^2/2} \int [H(s-t) - \check{H}(\lambda t + \eta_1)] e^{i(x-y)\cdot\eta} \times e^{i(t-s)|\eta|} e^{-\sigma(s-t)(t+\lambda^{-1}\eta_1)} a(\eta, \sigma, \lambda) d\eta \quad (15)$$

where the amplitude  $a$  is supported in a neighbourhood of a point of  $\Sigma_0$  and in a set where  $\sigma + |\xi| \sim \lambda$ . After a change of variable, we have to prove the following bound

$$\left| \int [H(s-t) - \check{H}(t + \eta_1)] e^{i\lambda(x-y)\cdot\eta} e^{i\lambda(t-s)|\eta|} e^{-\sigma(s-t)(t+\eta_1)} a(\lambda\eta, \sigma, \lambda) d\eta \right| \leq C(\lambda|t-s|)^{-\frac{n-2}{2}} \quad (16)$$

where  $a$  is now compact supported with support independent from  $\lambda$  and its derivatives are bounded independently from  $\lambda$ .

We restrict ourselves to the case  $s \geq t$ . Two cases occur: either  $|x' - y'| \gtrsim |t - s|$  and we obtain the bound by integration by parts in  $\eta'$ , or  $|x' - y'| \lesssim |t - s|$  and for the sake of simplicity we may as well suppose that  $x' = y'$ . Suppose that the amplitude is supported in a neighbourhood of  $\eta = (0, \dots, 1)$ , then the Hessian of  $|\eta|$  with respect to  $(\eta_2, \dots, \eta_{n-2})$  is non-degenerate, we can take  $\lambda|t - s|$  as a parameter and apply the stationary phase in the integral with respect to  $(\eta_2, \dots, \eta_{n-2})$ . Hence it remains to bound the one dimensional integral

$$\left| \int_{-t}^{\infty} e^{i\lambda(x_1 - y_1)\cdot\eta_1} e^{i\lambda(t-s)\sqrt{\eta_1^2 + \eta_{n-1}^2}} e^{-\sigma(s-t)(t+\eta_1)} a(\eta_1, \eta_{n-1}) d\eta_1 \right| \leq C(\lambda|t-s|)^{-\frac{1}{2}}.$$

When  $|x_1 - y_1| \gtrsim |t - s|$  this is just an integration by part (considering the real exponential as an amplitude), hence for the sake of simplicity we may as well suppose  $x_1 = y_1$ . Finally we have to prove

$$\left| \int_{-t}^{\infty} e^{i\lambda(t-s)\sqrt{\eta_1^2 + \eta_{n-1}^2}} e^{-\sigma(s-t)(t+\eta_1)} a(\eta_1, \eta_{n-1}) d\eta_1 \right| \leq C(\lambda|t-s|)^{-\frac{1}{2}}$$

but to do this we only have to apply Van der Corput's lemma since the phase is non-degenerate when  $(\eta_1, \eta_{n-1}) = (0, 1)$  putting the real exponential in the amplitude, since in Van der Corput's lemma we only need the amplitude to be bounded and the  $L^1$  norm of the derivative of the amplitude to be bounded by constants independent of  $\lambda$ . This is of course the case of  $e^{-\sigma(s-t)(t+\eta_1)} a$ . This achieves the proof in the model case.

## 6. Comments

The general basic ideas of the proof are similar to those we have described in the model case presented here. Technical difficulties arise from the fact that the parametrix constructed in section 4 is not a Fourier integral operator with complex phase. Therefore we can not use the symbolic calculus to compute the composition with the Fourier integral operator  $U(t)$ . Nevertheless repeated use of the stationary phase provides an adequate substitute to the calculus.

One point that we have not so much insisted on is that along the different reductions from the general form of  $P_\sigma$  given by the Malgrange-Weierstrass theorem to the normal form (8), we may obtain a description of the phase of the Fourier integral operator  $U(t)$  allowing us to apply the same general principles as in the model case.

The last remark concern the remainders: when microlocalising, constructing a parametrix, etc., remainders appear. But the Carleman estimate (5) may not be *a posteriori* pertubated by lower order terms. Hence we have to be particularly careful in the proof when dealing with the remainders. The solution is to use estimates involving  $L^2$  for the remainders. Namely we need the additional estimates

$$\begin{aligned}\|U(t)E^*u\|_{L^p} &\leq C\lambda^{1/2}\|u\|_{L^2} \\ \|E^*U^*(t)u\|_{L^2} &\leq C\lambda^{1/2}\|u\|_{L^{p'}}\end{aligned}$$

the proof of which is very similar in spirit to the proof of the  $L^p - L^{p'}$  estimates for the principal term. Indeed using the  $TT^*$  argument these reduce to proving  $L^p - L^{p'}$  estimates.

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