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Existence globale et diffusion pour l'équation de Schrödinger nonlinéaire répulsive cubique sur $\mathbf{R}^3$ en dessous l'espace d'énergie

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Abstract

Nous profilons une demonstration de l'existence globale et diffusion pour l'équation de Schrödinger nonlinéaire répulsive cubique avec données à $H^s(\mathbf{R}^3)$ pour $s > \frac{4}{5}$. La raisonnement utilise une estimation nouvelle de type de Morawetz. Nous détaillerons la demonstration ailleurs.

1. Introduction

Consider the following initial value problem for a cubic defocussing nonlinear Schrödinger equation,

$$i\partial_t \phi(x,t) + \Delta\phi(x,t) = |\phi(x,t)|^2\phi(x,t) \quad x \in \mathbf{R}^3, t \geq 0 \tag{1}$$
$$\phi(x,0) = \phi_0(x) \ \in H^s(\mathbf{R}^3). \tag{2}$$

Here $H^s(\mathbf{R}^3)$ denotes the usual inhomogeneous Sobolev space. Our goal is to loosen the regularity requirements [4, 15] on the initial data which ensure global-in-time solutions. In addition, we aim to loosen the symmetry assumptions on the data which were previously used [4] to prove scattering for rough solutions.

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It is known [7] that (1)-(2) is well-posed locally in time in $H^s(\mathbf{R}^3)$ when[1] $s > \frac{1}{2}$. In addition, these local solutions enjoy L^2 conservation,

$$\|\phi(\cdot,t)\|_{L^2(\mathbf{R}^3)} = \|\phi_0(\cdot)\|_{L^2(\mathbf{R}^3)}, \tag{3}$$

and the $H^1(\mathbf{R}^3)$ solutions have the following conserved energy,

$$E(\phi)(t) \equiv \int_{\mathbf{R}^3} \frac{1}{2}|\nabla_x \phi(x,t)|^2 + \frac{1}{4}|\phi(x,t)|^4 \, dx = E(\phi)(0). \tag{4}$$

Together, energy conservation and the local-in-time theory immediately yield global-in-time well-posedness of (1)-(2) from data in $H^s(\mathbf{R}^3)$ when $s \geq 1$. It is conjectured that (1)-(2) is in fact globally well-posed in time from all data included in the local theory. The obvious impediment to claiming global-in-time solutions in H^s, with $\frac{1}{2} < s < 1$, is the lack of any applicable conservation law.

The first argument extending the lifespan of rough solutions to (1)-(2) in a range $s_0 < s < 1$ was given in [4] (see also [5]). In what might be called a "Fourier truncation" approach, Bourgain observed that from the point of view of regularity, the high frequency component of the solution ϕ is well-approximated by the corresponding linear evolution of the data's high frequency component. More specifically: one makes a first approximation to the solution for a small time step by evolving the high modes linearly, and the low modes according to the nonlinear flow (for which one has energy conservation). The correction term one must add to match this approximation with the actual solution is shown to have finite energy. This correction is added to the low modes as data for the nonlinear evolution during the next time step, where the high modes are again evolved linearly. For $s > \frac{11}{13}$, one can repeat this procedure to an arbitrarily large time provided the distinction between "high" and "low" frequencies is made at $|\xi| = N$ for a sufficiently large number N.

The argument in [2] has been applied to other subcritical initial value problems with sufficient smoothing in their principal parts. (See e.g. [5], [8], [16], [22], [28], and [29]). It is important to note that the Fourier truncation method demonstrates more than just rough data global existence. Indeed, write $S^{NL}(t)$ for the nonlinear flow[2] of (1)-(2), and let $S^L(t)$ denote the corresponding *linear* flow. The Fourier truncation method shows then that for $s > \frac{11}{13}$ and for all $t \in [0, \infty)$,

$$S^{NL}(t)\phi_0 - S^L(t)\phi_0 \in H^1(\mathbf{R}^3). \tag{5}$$

Besides being part of the conclusion, the smoothing property (5) seems to be a crucial constituent of the Fourier truncation argument itself.

In this paper we will use a modification of the above arguments, originally put forward to analyze equations where the smoothing property (5) is not available because it is either false (e.g. wave maps [20][3]) or simply not known (e.g. Maxwell-Klein-Gordon equations [19], for which we suspect (5) is false). In this "almost conservation law" approach, one controls the growth in time of a rough solution by

[1]In addition, there are local in time solutions from $H^{\frac{1}{2}}$ data. However, it is not yet known whether the time interval of existence for such solutions depends only on the data's Sobolev norm.

[2]That is, $S^{NL}(t)(\phi_0)(x) = \phi(x,t)$, where ϕ, ϕ_0 is as in (1)-(2).

[3]See the appendix of [19] for the failure of (5) for Wave Maps.

monitoring the energy of a certain smoothed out version of the solution. It can be shown that the energy of the smoothed solution is *almost conserved* as time passes, and controls the solution's sub-energy Sobolev norm.

The almost conservation approach to global rough solutions has proven to be quite robust [20], [19], [10], [13], [15] and has been improved significantly by adding additional correction terms to the original almost conserved energy functional. As a result, one obtains even stronger bounds on the growth of the solution's rough norm, and at least in some cases sharp global well-posedness results [14], [11], [12].

The above work, along with the theorem outlined below, is motivated by a number of considerations. We mention here just a few examples. First and most obviously, we aim to better understand the global in time evolution properties of rough solutions to these nonlinear equations. Second, our results for rough solutions yield polynomial in time bounds for the growth of the below-energy Sobolev norms of *smooth* solutions. Such bounds give, for example, a qualitative understanding of how the energy in a smooth solution moves from high frequencies to low frequencies[4]. Third, we hope that the techniques developed for these subcritical, rough solution problems can be used to address open problems for relatively smooth solutions. As an example of this, our arguments below give a new proof of the finite energy scattering result of [17]. The bounds we obtain on the global Schrödinger admissible space-time norms of the solution depend polynomially on the energy of the initial data, whereas previous bounds were exponential. (See remark in [4], page 276, and (9) below.) There are of course much more interesting examples where low-regularity techniques have helped to solve open problems for smooth solutions, e.g. [3, 30].

Our main result is the following:

Theorem 1.1. *The initial value problem* (1)-(2) *is globally-well-posed from data* $\phi_0 \in H^s(\mathbf{R}^3)$ *when* $s > \frac{4}{5}$*. In addition, there is scattering for these solutions: given data* $\phi_0 \in H^s(\mathbf{R}^3)$*,* $s > \frac{4}{5}$*, there is a unique function* $\phi_0^L \in H^s(\mathbf{R}^3)$ *so that* [5]

$$\lim_{t\to\infty} \|\phi(t) - S^L(t)\phi_0^L\|_{H^s(\mathbf{R}^3)} = 0 \tag{6}$$

where $S^L(t)\phi_0^L$ *is as defined above, the evolution of the data* ϕ_0^L *according to the linear Schrödinger equation.*

By "globally-well-posed", we mean that given data $\phi_0 \in H^s(\mathbf{R}^n)$ as above, and any time $T > 0$, there is a unique solution to (1)-(2)

$$\phi(x,t) \in C([0,T]; H^s(\mathbf{R}^n)) \tag{7}$$

which depends continuously in (7) upon $\phi_0 \in H^s(\mathbf{R}^n)$.

Scattering in the space $H^1(\mathbf{R}^3)$ was shown in [17]. Theorem 1.1 extends to some degree the work in [4, 5] where global well-posedness was shown when $s > \frac{11}{13}$ in

[4]If one has a smooth solution with large but finite energy, the below-energy Sobolev norms can start relatively small and grow large when the low frequency pieces of the solution grow in (for example) L^2, while the high frequency pieces decrease in L^2. A polynomial bound on the rough norm's growth says that this "movement of energy from high to low frequencies" cannot occur to an enormous extent.

[5]More precisely, we are asserting here asymptotic completeness.

the case of general $H^s(\mathbf{R}^3)$ data ϕ_0. In the case of radially symmetric data, [4, 5] establish global well-posedness and scattering[6] for $\phi_0 \in H^s(\mathbf{R}^3)$, $s > \frac{5}{7}$. In a different sense, the result here is weaker than the results of [4, 5] as we obtain no information whatsoever along the lines of (5). Theorem 1.1 also extends the result of [15], where we showed global existence for $s > \frac{11}{13}$, with no scattering statement. The extension of global well-posedness to the cases $s > \frac{4}{5}$, as well as the proof of scattering in a nonradial context, depend very heavily on a new Morawetz-type inequality for (1) whose proof we sketch very briefly below. Details of all arguments outlined in this lecture will appear elsewhere.

We do not expect our results here to be sharp. For example, we hope to extend Theorem 1.1 to allow lower values of s, using the additional correction terms mentioned above (see [14], [11], [12]).

2. Sketch of Proof

In sketching the proof of Theorem 1.1, we emphasize the following three main ingredients, the first two of which are relatively new.

Ingredient 1: A New Morawetz Estimate

Suppose that we take initial data $\phi_0 \in C_0^\infty(\mathbf{R}^3)$, so that global existence for (1) follows easily from the local theory and energy conservation. In studying the asymptotic behavior of this global solution, past work (e.g. [23, 17, 5, 4]) has demonstrated the usefulness of the following Morawetz-type estimate (see [23], or [24], [25] for the motivating estimates in the context of nonlinear Klein-Gordon equations),

$$\int_0^\infty \int_{\mathbf{R}^3} \frac{|\phi|^4}{|x|} dx dt \leq 2\|\phi_0\|^2_{H^1(\mathbf{R}^3)} + \|\phi_0\|^4_{L^4(\mathbf{R}^3)} \lesssim E(\phi_0) + \|\phi_0\|^2_{L^2(\mathbf{R}^3)}. \tag{8}$$

We will use heavily instead the following related global L^4 space-time bound for solutions of (1)-(2),

$$\int_0^T \int_{\mathbf{R}^3} |\phi(x,t)|^4 dx dt \lesssim \|\phi(0)\|^2_{L^2(\mathbf{R}^3)} \cdot \sup_{0 \leq t \leq T} \|\phi(t)\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^3)}. \tag{9}$$

Ingredient 2: Almost Conservation Law

Keeping in mind that the energy (4) of our solutions might be infinite, our aim will be to control the growth in time of $E(I\phi)(t)$, where $I\phi$ is a certain smoothed version of ϕ. The operator I depends on a parameter $N >> 1$ to be chosen later, and the level of regularity $s < 1$ at which we are working[7].

Definition 2.1. Given $s < 1$ and $N >> 1$, define the following Fourier multiplier operator I,

$$\widehat{If}(\xi) \equiv m_N(\xi)\hat{f}(\xi), \tag{10}$$

[6] The phrase here again refers to asymptotic completeness, (6).

[7] We abuse notation and suppress this dependence, writing simply I instead of $I_{s,N}$.

where the multiplier $m_N(\xi)$ is smooth, radially symmetric, nonincreasing in $|\xi|$ and

$$m_N(\xi) = \begin{cases} 1 & |\xi| \leq N \\ \left(\frac{N}{|\xi|}\right)^{1-s} & |\xi| \geq 2N. \end{cases} \tag{11}$$

The following two inequalities follow quickly from the definition of I, the L^2 conservation (3), and by considering separately those frequencies $|\xi| \leq N$ and $|\xi| \geq N$.

$$E(I\phi)(t) \lesssim \left(N^{1-s}\|\phi(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)}\right)^2 + \|\phi(t,\cdot)\|^4_{L^4(\mathbb{R}^n)}, \tag{12}$$

$$\|\phi(\cdot,t)\|^2_{H^s(\mathbb{R}^n)} \lesssim E(I\phi)(t) + \|\phi_0\|^2_{L^2(\mathbb{R}^n)}. \tag{13}$$

In studying the possible growth of our solution in time, we will not estimate the norm $\|\phi(t)\|_{H^s(\mathbf{R}^3)}$ directly. Instead, we will use (13). Of course, since (1) is a nonlinear equation, it's certainly not true that $I\phi(x,t)$ solves (1). In particular, one doesn't expect $E(I\phi)(t)$ to be constant. One of the main ingredients of Theorem 1.1 is proving that this quantity is uniformly bounded in time. This upper bound, or more precisely some local in time result which contributes to the proof of such a bound, is what we mean by an *almost conservation law*. Global well-posedness follows from (13), a uniform bound on $E(I\phi)(t)$ in terms of $\|\phi_0\|_{H^s(\mathbf{R}^3)}$, and the local in time theory.

Combining these first two ingredients with a scaling argument, we will be able to prove the following Proposition giving uniform bounds in terms of the rough norm of the initial data.

Proposition 2.2. *Suppose $\phi(x,t)$ is a global in time solution to* (1)*-*(2) *from data $\phi_0 \in C_0^\infty(\mathbf{R}^3)$. Then so long as $s > \frac{4}{5}$, we have*

$$\|\phi\|_{L^4([0,\infty]\times\mathbf{R}^3)} \lesssim C(\|\phi_0\|_{H^s(\mathbf{R}^3)}) \tag{14}$$

$$\sup_{0\leq t<\infty} \|\phi(t)\|_{H^s(\mathbf{R}^3)} \leq C(\|\phi_0\|_{H^s(\mathbf{R}^3)}). \tag{15}$$

Ingredient 3: Previous Work on Local Well-Posedness and Scattering

Assuming Proposition 2.2 for the moment, we turn to the proof of Theorem 1.1. We have explained above why the global well-posedness statement in Theorem 1.1 follows from (15). It remains only to sketch the argument for asymptotic completeness (6) using the following well-known arguments. (See e.g. [23, 17, 4, 7].)

We'll bootstrap a family of norms using the following Strichartz estimates for the linear Schrödinger equation. (See e.g. [26, 27, 18, 31, 21]). The estimates involve the following definition: a pair of Lebesgue space exponents are called *Schrödinger admissible* for $\mathbf{R}^{3+1}$ when $q, r \geq 2$, and

$$\frac{1}{q} + \frac{3}{2r} = \frac{3}{4}. \tag{16}$$

Proposition 2.3 (Strichartz estimates in 3 space dimensions). *Suppose that* (q,r) *and* $(\tilde{q},\tilde{r})$ *are any two Schrödinger admissible pairs as in* (16)*. Suppose too that* $\phi(x,t)$ *is a (weak) solution to the problem*

$$(i\partial_t + \Delta)\phi(x,t) = F(x,t) \quad (x,t) \in \mathbf{R}^3 \times [0,T] \tag{17}$$
$$\phi(x,0) = \phi_0(x) \tag{18}$$

for some data u_0 *and* $T > 0$*. Then we have the estimate*

$$\|\phi\|_{L^q_t L^r_x([0,T]\times\mathbf{R}^3)} \lesssim \|\phi_0\|_{L^2(\mathbf{R}^3)} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x([0,T]\times\mathbf{R}^3)}. \tag{19}$$

We aim for a uniform bound of the form,

$$Z(t) \equiv \sup_{q,r \text{ admissible}} \|\langle\nabla\rangle^s \phi\|_{L^q_t L^r_x([0,t]\times\mathbf{R}^3)} \tag{20}$$
$$\lesssim C(\|\phi_0\|_{H^s(\mathbf{R}^3)}). \tag{21}$$

By (14), we can decompose the time interval $[0,\infty)$ into a finite number of disjoint intervals $J_1, J_2, \dots J_K$ where for $i = 1, \dots K$ we have

$$\|\phi\|_{L^4_{x,t}(J_i\times\mathbf{R}^3)} \leq \epsilon \tag{22}$$

for a constant $\epsilon(\|\phi_0\|_{H^s(\mathbf{R}^3)})$ to be chosen momentarily.

We apply $\langle\nabla\rangle^s$ to both sides of (1). Choosing $\tilde{q}, \tilde{r} = \frac{10}{7}$, (19) and a fractional Leibnitz rule give us that for all $t \in J_1$,

$$Z(t) \lesssim \|\langle\nabla\rangle^s \phi_0\|_{L^2(\mathbf{R}^3)} + \|(\langle\nabla\rangle^s\phi)\phi\phi\|_{L^{10/7}_{t,x}([0,t]\times\mathbf{R}^3)}. \tag{23}$$

We apply Hölder's inequality to the last term on the right, taking the factors here in $L^{\frac{10}{3}}, L^5$, and L^5, respectively. The factor ending up in $L^{\frac{10}{3}}$ is bounded by $Z(t)$. The remaining $L^5_{x,t}$ factors are bounded using interpolation involving the small $L^4_{x,t}(J_1 \times \mathbf{R}^3)$ norm and the quantity $Z(t)$. To be more precise: interpolate between $\|\phi\|_{L^4_{x,t}}$ and $\|\phi\|_{L^6_{x,t}}$. The latter norm is bounded by $Z(t)$ using Sobolev embedding:

$$\|\phi\|_{L^6_{x,t}} \lesssim \|\langle\nabla\rangle^{\frac{2}{3}}\phi\|_{L^6_t L^{18/7}_x} \leq Z(t).$$

We conclude

$$Z(t) \lesssim \|\phi_0\|_{H^s(\mathbf{R}^3)} + \epsilon^{\delta_1} Z(t)^{(1+\delta_2)}. \tag{24}$$

for some constants $\delta_1, \delta_2 > 0$. For sufficiently small choice of ϵ, the bound (24) yields (21) for all $t \in J_1$, as desired. Since we are assuming the bound (15), we may repeat this argument to handle the remaining intervals J_i.

The scattering claim in Theorem 1.1 follows from (21) and the following standard arguments. Given $\phi_0 \in H^s(\mathbf{R}^3)$, we look for a ϕ_0^L satisfying (6). Set,

$$\phi_0^L \equiv \phi_0 - i\int_0^\infty S^L(t-\tau)\left(|\phi|^2\phi\right) d\tau \tag{25}$$

which will make sense once we show the integral on the right hand side converges in $H^s(\mathbf{R}^3)$. Equivalently, we want

$$\lim_{t\to\infty} \left\| \int_t^\infty \langle\nabla\rangle^s S^L(-\tau) \left(|\phi|^2\phi\right) d\tau \right\|_{L^2(\mathbf{R}^3)} = 0. \tag{26}$$

With this,

$$\lim_{t\to\infty} \left\|S^L(t)\phi_0^L - \phi(t)\right\|_{H^s(\mathbf{R}^3)} = \lim_{t\to\infty} \left\|\langle\nabla\rangle^s S^L(t) \int_t^\infty S^L(-\tau) \left(|\phi|^2\phi\right) d\tau\right\|_{L^2(\mathbf{R}^3)} = 0$$

since we are assuming (26). To prove (26), test the time integral on the left against an arbitrary $L^2(\mathbf{R}^3)$ function $F(x)$. Using the fractional Leibnitz rule,

$$\begin{aligned}\left\langle F(x)\,,\ \int_t^\infty \langle\nabla\rangle^s S^L(-\tau) \left(|\phi|^2\phi\right) d\tau \right\rangle_{L^2(\mathbf{R}^3)} &\approx \left\langle S^L(\tau)F(x)\,,\ (\nabla^s\phi)\phi\phi\right\rangle_{L^2_{x,t}([t,\infty)\times\mathbf{R}^3)} \\ &\leq \|S^L(\tau)F(x)\|_{L^{10/3}_{x,t}} \|\nabla^s\phi\|_{L^{10/3}_{x,t}} \|\phi\|^2_{L^5_{x,t}([t,\infty)\times\mathbf{R}^3)} \to 0,\end{aligned}$$

where in the last step we've used (21) and the $L^5_{x,t}$ argument before (24) sketched above.

3. Proofs for the ingredients

Three topics remain to be discussed: the basic steps behind the Morawetz inequality (9); a sketch of the almost conservation law; and some indication of how these two estimates yield the global bounds in Proposition 2.2.

In discussing (9), we assume ϕ solves (1)-(2) with $\phi_0 \in C_0^\infty(\mathbf{R}^3)$. Define then

$$M_0(t) \equiv \operatorname{Im} \int_{\mathbf{R}^3} \overline{\phi(x,t)}\partial_r\phi(x,t)dx \tag{27}$$

with $\partial_r \equiv \frac{x}{|x|} \cdot \nabla$. It can be shown relatively easily using Plancherel's theorem that

$$|M_0(t)| \lesssim \|\phi(t)\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^3)} \tag{28}$$

while the work in [23] yields

$$\partial_t M_0(t) = 4\pi^2|\phi(t,0)|^2 + \int \frac{2}{r}|\nabla\!\!\!/_0\phi(t,x)|^2\, dx + \int \frac{1}{r}|\phi(t,x)|^4\, dx \tag{29}$$

where $\nabla\!\!\!/_0$ is the angular component of the derivative:

$$\nabla\!\!\!/_0\phi := \nabla\phi - \frac{x}{|x|}\left(\frac{x}{|x|}\cdot\nabla\phi\right).$$

Integrating in time, (28) and (29) together give us the standard Morawetz estimates [23],

$$\int_0^T |\phi(t,0)|^2\, dt \lesssim \sup_{0\le t\le T}\|\phi(t)\|^2_{\dot H^{1/2}_x} \tag{30}$$

$$\int_0^T \int \frac{|\nabla\!\!\!/\,\phi(t,x)|^2}{|x|}\, dtdx \lesssim \sup_{0\le t\le T}\|\phi(t)\|^2_{\dot H^{1/2}_x} \tag{31}$$

$$\int_0^T \int \frac{|\phi(t,x)|^4}{|x|}\, dtdx \lesssim \sup_{0\le t\le T}\|\phi(t)\|^2_{\dot H^{1/2}_x}. \tag{32}$$

As remarked before, (32) is the bound that has been used the most. Note that this bound arises from the presence of the nonlinear term in the equation, or equivalently from the third term on the right side of (29). In contrast, the estimate we use (9) arises eventually from the linear part of the equation, more specifically from the presence of the first term on the right side of (29).

Of course, we can translate M_0 in space, creating for every point $y \in \mathbf{R}^3$ a Morawetz action

$$M_y(t) := \mathrm{Im} \int_{\mathbf{R}^3} \overline{\phi}(t,x) \left[\frac{x-y}{|x-y|}\cdot\nabla\right] \phi(t,x)\, dx,$$

and by translating (29) we obtain

$$\partial_t M_y(t) = 4\pi^2|\phi(t,y)|^2 + \int \frac{2}{|x-y|}|\nabla\!\!\!/_y\phi(t,x)|^2\, dx + \int \frac{1}{|x-y|}|\phi(t,x)|^4\, dx \tag{33}$$

where

$$\nabla\!\!\!/_y\phi(x) := \nabla\phi(x) - \frac{x-y}{|x-y|}\left(\frac{x-y}{|x-y|}\cdot\nabla\phi(x)\right)$$

is the portion of the x-gradient which is orthogonal to $x-y$. Of course one still has

$$|M_y(t)| \lesssim \|\phi\|^2_{\dot H^{1/2}_x}.$$

This can be used to prevent repeated concentration at a single stationary point y over long times, but it is difficult to preclude the solution's concentration near a moving point using these sorts of estimates. Hence, it is difficult to prove scattering using the above arguments. (In particular, scattering for solutions in $H^s, s<1$ was known only in the radial case.)

The way we take around this is to construct a *Morawetz interaction potential*

$$M(t) \equiv \int |\phi(y)|^2 M_y(t)\, dy.$$

In other words we average all the Morawetz actions together, weighted by the probability density of ϕ itself. Clearly we have

$$|M(t)| \lesssim \|\phi\|^2_{L^2_x}\|\phi\|^2_{\dot H^{1/2}_x}. \tag{34}$$

Notice that $M(t)$ is a double integral in both x and y. A slightly more involved argument than that behind (29) yields

$$\partial_t M(t) \geq \quad 4\pi^2 \int |\phi(y)|^4 \, dy + \int \int |\phi(y)|^2 |\phi(x)|^4 \, \frac{dxdy}{|x-y|}.$$

In particular $M(t)$ is monotone increasing. Combining this with (34) and taking advantage of L^2 norm conservation we obtain in particular the global $L^4_{x,t}$ estimate

$$\int_0^T \int |\phi(t,x)|^4 \, dxdt \lesssim \|\phi(0)\|^2_{L^2_x} \sup_{0 \leq t \leq T} \|\phi(t)\|^2_{\dot{H}^{1/2}_x}. \tag{35}$$

The details of this argument will appear elsewhere.

We turn to a more precise statement of Ingredient 2, the almost conservation law, and give some indication of its proof.

Proposition 3.1 (Almost Conservation Law). *Assume we have $s > \frac{4}{7}$, $N \gg 1$, initial data $\phi_0 \in C_0^\infty(\mathbf{R}^3)$, and a solution of (1)-(2) on a time interval $[0,T]$ for which*

$$\|\phi\|_{L^4_{x,t}([0,T]\times\mathbf{R}^3)} \lesssim \epsilon. \tag{36}$$

Assume in addition that $E(I\phi_0) \lesssim 1$.

We conclude that for all $t \in [0,T]$,

$$E(I\phi)(t) = E(I\phi)(0) + O(N^{-1+}). \tag{37}$$

Equation (37) asserts that $I\phi$, though not a solution of the nonlinear problem (1), enjoys something akin to energy conservation. If one could replace the increment N^{-1+} in $E(I\phi)$ on the right side of (37) with $N^{-\alpha}$ for some $\alpha > 0$, one could repeat the argument we give below to prove global well-posedness of (1)-(2) for all $s > \frac{3+\alpha}{3+2\alpha}$. In particular, if $E(I\phi)(t)$ is conserved (i.e. $\alpha = \infty$), one could show that (1)-(2) is globally well-posed when $s > \frac{1}{2}$. Recall that the scale-invariant Sobolev space is $\dot{H}^{\frac{1}{2}}(\mathbf{R}^3)$.

The proof of Proposition 3.1 proceeds by pretending that $I\phi$ is a solution of (1) and using the usual proof of energy conservation. We look at the resulting space-time integral in Fourier space, where we estimate various frequency interactions separately. In doing so, we'll need control of local-in-time norms similar to those that are usually handled by the local existence theorem. Here the norms will include the operator I, and so you might call the following bootstrap argument a modified local theory. The argument is a straightforward analog of the one we gave for (20).

We aim for,

$$Z_I(t) \equiv \sup_{q,r \text{ admissible}} \|\nabla I\phi\|_{L^q_t L^r_x([0,t]\times\mathbf{R}^3)} \tag{38}$$

$$\lesssim C(\|\phi_0\|_{H^s(\mathbf{R}^3)}). \tag{39}$$

Apply $I\nabla$ to both sides of (1). Choosing $\tilde{q}, \tilde{r} = \frac{10}{7}$, (19) and a Leibnitz rule give us that for all $0 \le t \le T$,

$$Z_I(t) \lesssim \|\nabla I\phi_0\|_{L^2(\mathbf{R}^3)} + \|(\nabla I\phi)\phi\phi\|_{L^{\frac{10}{7}}_{x,t}([0,t]\times\mathbf{R}^3)}.$$

We apply Hölder's inequality to the last term on the right, taking the factors here in $L^{\frac{10}{3}}, L^5$, and L^5, respectively. The factor ending up in $L^{\frac{10}{3}}$ is bounded by $Z_I(t)$. Roughly speaking, the remaining $L^5_{x,t}$ factors are bounded using interpolation involving the small $L^4_{x,t}([0,T]\times\mathbf{R}^3)$ norm and the quantity $Z(t)$. (To be more precise: one decomposes ϕ dyadically. For dyadic pieces ϕ_j supported on $|\xi| \le N$, we interpolate between the norms $\|\phi_j\|_{L^4_{x,t}}$ and $\|\phi_j\|_{L^{10}_{x,t}}$. The latter norm here is bounded by $Z_I(t)$ using homogeneous Sobolev embedding and the fact that I is the identity on $|\xi| \le N$. High frequency components ϕ_j are estimated using the norms $\|I\phi_j\|_{L^{10}_{x,t}}$ and $\|\nabla I\phi_j\|_{L^{\frac{10}{3}}_{x,t}}$ which are both bounded by $Z_I(t)$.) For sufficiently large parameter N, these observations give

$$Z_I(t) \lesssim 1 + \epsilon^{\delta_1} Z_I(t)^{(1+\delta_2)}, \tag{40}$$

for some constants $\delta_1, \delta_2 > 0$. For sufficiently small choice of ϵ, the bound (40) yields (39) for all $0 \le t \le T$, as desired.

We're now ready to prove the almost conservation law. For sufficiently smooth solutions, the usual energy (4) is shown to be conserved by differentiating in time, integrating by parts, and using the equation (1),

$$\begin{aligned}\partial_t E(\phi) &= \mathrm{Re} \textstyle\int_{\mathbf{R}^3} \overline{\phi_t}(|\phi|^2\phi - \Delta\phi)dx \\ &= \mathrm{Re} \textstyle\int_{\mathbf{R}^3} \overline{\phi_t}(|\phi|^2\phi - \Delta\phi - i\phi_t)dx \\ &= 0.\end{aligned}$$

We begin to estimate $E(I\phi)(t)$ in precisely the same way. We need to pay attention when we use the equation (1) since of course $I\phi$ is not a solution of this nonlinear equation. Repeating our steps above gives,

$$\begin{aligned}\partial_t E(I\phi)(t) &= \mathrm{Re} \textstyle\int_{\mathbf{R}^3} \overline{I(\phi)_t}(|I\phi|^2 I\phi - \Delta I\phi - iI\phi_t)dx \\ &= \mathrm{Re} \textstyle\int_{\mathbf{R}^3} \overline{I(\phi)_t}(|I\phi|^2 I\phi - I(|\phi|^2\phi))dx,\end{aligned}$$

where in the last step we've applied I to (1). When we integrate in time and apply the Parseval formula it remains for us to bound

$$E(I\phi(t)) - E(I\phi(0)) = \int_0^t \int_{\sum_{j=1}^4 \xi_j = 0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)\cdot m(\xi_3)\cdot m(\xi_4)}\right) \widehat{\overline{I\partial_t\phi}}(\xi_1)\widehat{I\phi}(\xi_2)\widehat{\overline{I\phi}}(\xi_3)\widehat{I\phi}(\xi_4) \tag{41}$$

In what follows we drop the complex conjugates as they don't affect the analysis. We may replace the $\partial_t\phi$ factor here using the equation (1). In the following discussion we only mention the contribution of $\Delta\phi$.

Our goal then is to show that

$$|\text{right side of (41)}| \lesssim N^{-1+} \cdot C(Z(t)). \tag{42}$$

This is accomplished by writing ϕ as a sum of dyadic pieces ϕ_i supported in frequency space on $\langle\xi_i\rangle \sim N_i \equiv 2^{k_i}$ where $k_i \in \{0, 1, \ldots\}$. Without loss of generality we may assume $N_2 \geq N_3 \geq N_4$. Also, we may assume each piece has positive Fourier transform. Depending on which ϕ_i are involved, we have different pointwise estimates for the symbol on the right of (41), which we pull out of the integral in L^∞. The remaining factors are handled by undoing the Parseval formula and using Hölder's inequality. We run through two examples of these different frequency interactions now.

Case 1: $N \gg N_2$. According to (11), the symbol $1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)\cdot m(\xi_3)\cdot m(\xi_4)}$ on the right of (41) is in this case identically zero and the bound (42) holds trivially.

Case 2: $N_2 \gtrsim N \gg N_3 \geq N_4$. Since $\sum_i \xi_i = 0$, we have $N_1 \sim N_2$. By the mean value theorem,

$$\left|\frac{m(\xi_2) - m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)}\right| \lesssim \frac{|\nabla m(\xi_2)\cdot(\xi_3+\xi_4)|}{m(\xi_2)} \lesssim \frac{N_3}{N_2}. \tag{43}$$

After estimating the symbol with (43), we can consider the N_3 in the numerator as resulting from a derivative falling on the $I\phi_3$ factor in the integrand. Hence these case two interactions can be estimated using Hölder's inequality and the definition (20) of $Z(t)$,

$$\begin{aligned} |\text{Right Side of (41)}| &\lesssim \frac{N_3}{N_2}\left|\int_0^t\int_{\mathbf{R}^3} \Delta I\phi_1 I\phi_2 I\phi_3 I\phi_4\right| \\ &\leq \frac{1}{N_2}\|\Delta I\phi_1\|_{L^{\frac{10}{3}}_{x,t}} \cdot \|I\phi_2\|_{L^{\frac{10}{3}}_{x,t}} \cdot \|\nabla I\phi_3\|_{L^{\frac{10}{3}}_{x,t}} \cdot \|I\phi_4\|_{L^{10}_{x,t}} \\ &\leq \frac{N_1}{N_2\cdot N_2}\cdot (Z(t))^4 \\ &\leq \frac{1}{N_1}(Z(t))^4, \end{aligned}$$

which is sufficient since $N_1 \geq N$. We still need to sum over all such frequency interactions. This explains why the almost conservation law (3.1) has an increment in $E(I\phi)$ of N^{-1+} over the time interval.

The remaining cases are somewhat similar and will be detailed elsewhere.

Finally, we aim to give at least a sketch of the standard continuity argument one uses to prove that the Morawetz estimate (9) and the almost conservation law (3.1) imply Proposition 2.2. Recall that we have reduced to the case where we have a global smooth solution, our aim is to prove the bounds (14), (15).

The first step is to scale the solution: if ϕ is a solution to (1), then so is

$$\phi^{(\lambda)}(x,t) \equiv \frac{1}{\lambda}\phi\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right). \tag{44}$$

We choose λ so that $E(I\phi_0^{(\lambda)}) \leq \frac{1}{2}$. This is possible since we are working with subcritical s, so long as we choose λ in terms of the parameter[8] N. Roughly speaking,

$$\begin{aligned} E(I\phi_0^{(\lambda)}) &\approx \|\nabla I\phi_0^{(\lambda)}\|_{L^2(\mathbf{R}^3)} \\ &\lesssim N^{1-s}\cdot\|\phi_0^{(\lambda)}\|_{\dot{H}^s(\mathbf{R}^3)} \\ &= N^{1-s}\lambda^{\frac{1}{2}-s}\cdot\|\phi_0\|_{\dot{H}^s(\mathbf{R}^3)}. \end{aligned}$$

[8]The parameter N will be chosen at the very end of the argument, where it is shown to depend only on $\|\phi_0\|_{H^s(\mathbf{R}^3)}$.

Hence we choose

$$\lambda \approx N^{\frac{1-s}{s-\frac{1}{2}}}. \tag{45}$$

We now claim that the set W of all times on which we have (14) is all of $[0,\infty)$. In the process of proving this, we will also show (15) holds on W.

For some universal constant C_1 to be chosen shortly, define

$$W \equiv \left\{ T : \|\phi^{(\lambda)}\|_{L^4([0,T]\times\mathbf{R}^3)} \leq C_1 \lambda^{\frac{3}{8}} \right\} \tag{46}$$

$$= [0,\infty). \tag{47}$$

The set W is clearly closed and nonempty. It suffices then to show it is open. For example, suppose that for some T_0 we have

$$\|\phi^{(\lambda)}\|_{L^4_{x,t}([0,T_0]\times\mathbf{R}^3)} \leq 2C_1\lambda^{\frac{3}{8}}. \tag{48}$$

We claim $T_0 \in W$: by (9),

$$\|\phi^{(\lambda)}\|_{L^4_{x,t}([0,T]\times\mathbf{R}^3)} \lesssim \|\phi_0^{(\lambda)}\|^{\frac{1}{2}}_{L^2_x} \cdot \sup_{0\leq t\leq T_0}\|\phi^{(\lambda)}(t)\|^{\frac{1}{2}}_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^3)} \tag{49}$$

$$\leq \tilde{C}_1\lambda^{\frac{3}{8}} \cdot \sup_{0\leq t\leq T_0}\|I\phi^{(\lambda)}(t)\|^{\frac{1}{4}}_{\dot{H}^1(\mathbf{R}^3)} \tag{50}$$

where the first factor on the right of (49) was bounded using scaling, and the second factor was interpolated between $\|\phi_0^{(\lambda)}(t)\|_{L^2(\mathbf{R}^3)}$ and $\sup_{0\leq t\leq T_0}\|I\phi^{(\lambda)}(t)\|_{\dot{H}^1(\mathbf{R}^3)}$. We conclude $T_0 \in W$ if we establish

$$\sup_{0\leq t\leq T_0}\|I\phi^{(\lambda)}(t)\|^{\frac{1}{4}}_{\dot{H}^1(\mathbf{R}^3)} \lesssim 1 \tag{51}$$

where, as always, the implicit constant is allowed to depend on $\|\phi_0\|_{H^s(\mathbf{R}^3)}$.

By (48) we may divide the time interval $[0,T_0]$ into subintervals $I_j, j = 1,2,\ldots,L$ so that for each j,

$$\|\phi^{(\lambda)}\|_{L^4_{x,t}(I_j\times\mathbf{R}^3)} \leq \epsilon. \tag{52}$$

Apply the almost conservation law in Proposition 3.1 on each of the subintervals I_j to get

$$\sup_{0\leq t\leq T_0}\|\nabla I\phi^{(\lambda)}(t)\|_{L^2(\mathbf{R}^3)} \lesssim E(I\phi_0) + L\cdot N^{-1+}. \tag{53}$$

We get (51) from (53) if we can show

$$L\cdot N^{-1+} \leq \tfrac{1}{4}. \tag{54}$$

Recall L was defined essentially by (52). Since

$$\|\phi^{(\lambda)}\|^4_{L^4_{x,t}([0,T_0]\times\mathbf{R}^3)} \lesssim \lambda^{\frac{3}{2}},$$

we can be certain that $L \approx \lambda^{\frac{3}{2}}$. If we put this together with (54) and (45), we see that we need to be able to choose N so that

$$(N^{\frac{1-s}{s-\frac{1}{2}}})^{\frac{3}{2}} \cdot N^{-1+} \lesssim \tfrac{1}{4}.$$

This is possible since for $s > \frac{4}{5}$ the exponent on the left is negative. Notice that (15) holds on the set W using (51), the definition of I, and L^2 conservation.

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