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Remarks on Carleman estimates and exact controllability of the Lamé system

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Abstract

In this paper we established the Carleman estimate for the two dimensional Lamé system with the zero Dirichlet boundary conditions. Using this estimate we proved the exact controllability result for the Lamé system with with a control locally distributed over a subdomain which satisfies to a certain type of nontrapping conditions.

Introduction

This paper is concerned with a Carleman estimates for the 2-D non stationary Lamé system with the Dirichlet boundary conditions. Starting from the pioneering works of Carleman the theory of Carleman's inequalities has been rapidly developed over the last forty years and now for a single partial differential equations many general results are available (see [Hö, **?**, E2], [**?**].) On the other hand, for the systems of partial differential equations the situation is much less understood. To our best knowledge the most general result in case of systems of P.D.E. is the Calderon's uniqueness theorem (see e.g. [E1, Zui]). The technique, developed by Calderon reduces the system of partial differential equations to the system of pseudodifferential operators of the first order: $\frac{dv}{dx_0} = M(x, D_{x'})v + f$ where $M(x, D_{x'})$ is the matrix pseudodifferential operator. After that by some change of variables $v = Q(x, D_{x'})\tilde{v}$ this matrix P.D.O. M will be reduced to $Q^{-1}MQ$ which consists of blocks of a small size located on the main diagonal, such that in each block the principal symbols of all operators located below the main diagonal are zero. In order, to construct the matrix Q the eigenvalues and eigenvectors of the matrix $M(x, \xi')$ should be the smooth function of the variables x and ξ and eigenvalues should not change the multiplicity. This condition proved to be restrictive, especially in case when we are looking for a Carleman estimate near boundary, and therefore choice for a variable

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x_0 is limited. (For example the non stationary Lamé system in general it does not satisfy to this condition.)

On the other hand, it is well known that thanks to the special structure of the isotropic Lamé system $\operatorname{div} u$, $\operatorname{rot} u$ satisfy to scalar wave equations (modulo lower order terms.) The system of partial differential equations for a functions u, $\operatorname{div} u$, $\operatorname{rot} u$ is coupled via first order terms. This allows to apply the Carleman estimates for a scalar hyperbolic equations in case when the function u has a compact support. (see [EINT]). There are many results on uniqueness of the Cauchy problem for the stationary Lamé system (see [DR, W1, W2]). Therefore the goal of this paper is to obtain the Carleman estimate for u which does not vanish near boundary but satisfies only to the zero Dirichlet boundary condition. The structure of the proof is in principle similar to the paper [Y1]. We work with two hyperbolic equations for the functions $z_{\lambda+2\mu} = e^{s\phi} \operatorname{div} u$ and $z_\mu = e^{s\phi} \operatorname{rot} u$: $P_{\lambda+2\mu}(x, D, s) z_{\lambda+2\mu} = (\operatorname{div} f) e^{s\phi}$; $P_\mu(x, D, s) z_\mu = (\operatorname{rot} f) e^{s\phi}$. The main difficulty one should overcame is that there are no boundary conditions for these functions. This problem is fixed in following way. Microlocally, operators for z_β could be decomposed ,expect for a small set in $T^*(Q)$, into the product of two pseudodifferential operators of the first order $P_\beta(x, D, s) = P_{-,\beta}(x, D_{x'}, s) P_{+,\beta}(x, D_{x'}, s)$ where $P_{\pm,\beta} = D_{x_2} - \Gamma^{\pm}_\beta(x, D_{x'}, s)$, and x_2 is normal to the boundary $\partial\Omega$. Since the principal symbol of the operator $\Gamma^{-}_\beta(x, \xi', s)$ satisfies the inequality $-\operatorname{Im} \Gamma^{-}_\beta(x, \xi', s) \geq C^2 |s|$ we have the a priori estimate for $P_{+,\beta}(x, D_{x'}, s) z_\beta|_{x_2=0}$ in L^2. These estimates combined with zero Dirichlet boundary condition, for stress tensor u provide the H^1 boundary a priori estimates for z_β. The set on which the decomposition at least one of the operators $P_\beta(x, D, s)$in the product of the first order operators is impossible is studied in sections 3,4.

Among the applications of the Carleman estimate obtained in this paper first we mention the controllability results for the Lamé system. First controllability/observability results for the isotropic Lamé system with the constant coefficients were obtained by J.L. Lions in [Li] using the multiplies methods and HUM method. Later controllability and stabilizability properties for isotropic Lamé system and related models were studied by J.L. Lions and J. Lagnese in [LL] and [La]. Also we mention work [Y2], of K.Yamamoto where he studied the dissipation of the energy of Lamé system outside the convex obstacle. The results obtained in this paper could be easily converted in controllability results for the Lamé system using the HUM method. Recently, the technique developed in [Li] were applied by Alabau and Komornik [AK1, AK2] to prove the controllability/observability estimates were obtained for symmetric anisotropic Lamé system with the constant coefficients. In [B3], M. Bellassoued proved the approximate controllability for the isotropic Lamé system with the control distributed on an arbitrary small portion of the boundary.

Another possible application of the Carleman estimate is the inverse problem of the determination of the Lamé coefficients β, μ, λ using the finite number of measurements on a subdomain Q_ω. (The corresponding problem when a finite number of measurements are available on the whole $[0, T] \times \partial\Omega$ was treated in [IIY].)

There are many papers concerning the uniqueness of the Cauchy problem for the Lamé system.([DR, B3, W1]). The survey of resent results on the unique continuation for the stationary Lamé system given in [W2].

1. Main Result

Let us consider the 2-D Lamé system

$$Pu = \rho(x')\frac{\partial^2 u}{\partial x_0^2} - \mu(x')\Delta u - (\mu(x') + \lambda(x'))\nabla \mathrm{div}\, u$$
$$- \mathrm{div}\, u \nabla\lambda(x') - (\nabla u + (\nabla u)^T)\nabla\mu(x') = f \quad \text{in } Q = [0,T]\times\Omega, \quad (1)$$

$$u|_{(0,T)\times\partial\Omega} = 0, \quad u(T,x') = u_{x_0}(T,x') = u(0,x') = u_{x_0}(0,x') = 0, \qquad (2)$$

where $u = (u_1, u_2), f = (f_1, f_2)$ are the vector functions, Ω is a bounded domain in $\mathbb{R}^2$ with $\partial\Omega \in C^3, x = (x_0, x'), x' = (x_1, x_2)$. Coefficients $\rho(x'), \mu(x'), \lambda(x') \in C^2(\bar{Q})$ are assumed to be the positive functions

$$\rho(x') > 0, \quad \mu(x') > 0, \quad \lambda(x') > 0 \quad \text{in } \bar{\Omega}. \qquad (3)$$

The goal of this paper is to establish a Carleman estimate for a system (1), (2). Let $\omega \subset \Omega$ be an arbitrary fixed subdomain. Denote by ν the outward unit normal derivative to $\partial\Omega$. By Q_ω we denote the cylinder $Q_\omega = (0,T)\times\omega$. Let $\xi = (\xi', \xi_2)$, $\xi' = (\xi_0, \xi_1)$. We set

$$p_1(x,\xi) = \rho(x')\xi_0^2 - \mu(x')(|\xi_1|^2 + |\xi_2|^2),$$
$$p_2(x,\xi) = \rho(x')\xi_0^2 - (\lambda(x') + 2\mu(x'))(|\xi_1|^2 + |\xi_2|^2).$$

For an arbitrary smooth functions $\phi(x,\xi), \psi(x,\xi)$ we define the Poisson bracket by the formula $\{\phi,\psi\} = \sum_{i=0}^2 (\frac{\partial\phi}{\partial\xi_i}\frac{\partial\psi}{\partial x_i} - \frac{\partial\phi}{\partial x_i}\frac{\partial\psi}{\partial\xi_i})$. We assume that the Lamé coefficients μ, λ and the domains Ω, ω satisfy the following condition

Condition 1 *There exists a function $\psi \in C^2(\bar{Q})$ such that*

$$\{p_k, \{p_k, \psi\}\}(x,\xi) > 0 \quad \forall k \in \{1,2\}$$

$\forall \xi \in \mathbb{R}^3 \setminus \{0\}, p_k(x,\xi) =< \frac{\partial p_k}{\partial \xi}, \nabla\psi >= 0, x \in \overline{Q \setminus Q_\omega}$.

$$\{p_k(x, \xi - is\nabla\psi(x)), p_k(x, \xi + is\nabla\psi(x))\}/2is > 0$$

for all $\xi \in \mathbb{R}^3 \setminus \{0\}, \ s \neq 0, \ p_k(x, \xi + is\nabla\psi(x)) =< \partial_\xi p_k(x, \xi + is\nabla\psi(x)), \nabla\psi(x) >= 0, \quad x \in \overline{Q \setminus Q_\omega}$.

On the lateral boundary we assume

$$p_1(x, \nabla\psi)|_{\overline{(0,T)\times(\partial\Omega\setminus\partial\omega)}} < 0, \quad \frac{\partial\psi}{\partial\vec{\nu}}|_{\overline{(0,T)\times(\partial\Omega\setminus\partial\omega)}} < 0. \qquad (4)$$

Let $\psi(x)$ be the weight function from Condition 1.1. Using this function we introduce the function $\phi(x)$ by formula

$$\phi(x) = e^{\lambda\psi(x)} \quad \lambda > 1, \qquad (5)$$

where parameter λ will be fixed below.

Now we formulate our main result.

Theorem 1 *Let $f \in L^2(0,T;(H^1(\Omega))^2)$, function ϕ is given by (5) and Lamé coefficients satisfy (3). Then there exist $\hat\lambda$ such that for any $\lambda > \hat\lambda$ exists s_0 such that for any solution $u \in (H^1_0(Q))^2 \cap L^2(0,T;(H^2(\Omega))^2)$ to problem (1), (2) the following estimate holds true*

$$\int_Q (s^2|\nabla u|^2 + s^4|u|^2 + s|\nabla rot\, u|^2 + s^3|rot\, u|^2 + s|\nabla\, div\, u|^2 + s^3|div\, u|^2)e^{2s\phi}dx$$
$$+ s\|\frac{\partial u}{\partial \nu}e^{s\phi}\|^2_{(H^1((0,T)\times\partial\Omega))^2} + s\|\frac{\partial^2 u}{\partial \nu^2}e^{s\phi}\|^2_{(L^2((0,T)\times\partial\Omega))^2} + s^3\|\frac{\partial u}{\partial \nu}e^{s\phi}\|^2_{(L^2((0,T)\times\partial\Omega)))^2}$$
$$\le C_1(s\|fe^{s\phi}\|^2_{(L^2((0,T)\times\partial\Omega))^2}+$$
$$\|rot\, fe^{s\phi}\|^2_{(L^2(Q))^2} + \|div\, fe^{s\phi}\|^2_{(L^2(Q))^2} + \int_{Q_\omega}(s^2|\nabla u|^2 + s^4|u|^2)e^{2s\phi}dx+$$
$$\int_{Q_\omega}(s|\nabla rot\, u|^2 + s^3|rot\, u|^2 + s|\nabla\, div\, u|^2 + s^3|div\, u|^2)e^{2s\phi}dx) \quad \forall s \ge s_0(\lambda), \quad (6)$$

where constant C_1 is independent of s.

For controllability problems we need some variants of Carleman estimate (6). In addition to Condition 1.1 we assume

$$\frac{\partial\phi(T,x')}{\partial x_0} < 0, \quad \frac{\partial\phi(0,x')}{\partial x_0} > 0 \quad \forall x \in \overline{\Omega}. \quad (7)$$

We have

Theorem 2 *Let $f \in (L^2(Q))^2$, function ϕ is given by (5), satisfies (7) and Lamé coefficients satisfy (3). Then there exist $\hat\lambda$ such that for any $\lambda > \hat\lambda$ exists s_0 such that for any solution $u \in (H^1(Q))^2$ to problem (1), (2) the following estimate holds true*

$$\int_Q (|\nabla u|^2 + s^2|u|^2)e^{2s\phi}dx$$
$$\le C_1(\|fe^{s\phi}\|^2_{(L^2(Q))^2} + \int_{Q_\omega}(|\nabla u|^2 + s^2|u|^2)e^{2s\phi}dx) \quad \forall s \ge s_0(\lambda), \quad (8)$$

where constant C_1 is independent of s.

Corollary 1 *Let $f \in L^2(0,T;(H^{-1}(\Omega))^2)$, function ϕ is given by (5) satisfies (7) and Lamé coefficients satisfy (3). Then there exist $\hat\lambda$ such that for any $\lambda > \hat\lambda$ exists s_0 such that for any solution $u \in (L^2(Q))^2$ to problem (1), (2) the following estimate holds true*

$$\int_Q |u|^2 e^{2s\phi}dx$$
$$\le C_1(\|fe^{s\phi}\|^2_{L^2(0,T;(H^{-1}(\Omega))^2)} + \int_{Q_\omega}|u|^2e^{2s\phi}dx) \quad \forall s \ge s_0(\lambda), \quad (9)$$

where constant C_1 is independent of s.

Now we would like to consider the applications of the Carleman estimates to the exact controllability problem of the Lamé system:

$$Pu = f + \chi_\omega v \quad \text{in } Q, \quad u|_{(0,T)\times\partial\Omega} = 0,$$
$$u(0,\cdot) = u_0,\ u_{x_0}(0,\cdot) = u_1,\ u(T,\cdot) = u_{x_0}(T,\cdot) = 0. \quad (10)$$

Here the functions u_0, u_1, f are given and v is a control locally distributed in a domain Q_ω. The following theorem is the direct corollary of the Carleman estimates (8), (9) and the Hilbert Uniqueness Method.

Theorem 3 *Let function ψ satisfies the Condition 1.1 and there exists $\tau \in (0,T)$ such that*

$$\min_{x'\in\bar\Omega} \psi(\tau, x') > \max_{x'\in\bar\Omega}\{\psi(T,x'), \psi(-T,x')\}.$$

Then

A. *If $u_0 \in (H^1_0(\Omega))^2$, $u_1 \in (L^2(\Omega))^2$, $f \in (L^2(Q))^2$ then there exists a solution to problem (5.45) a pair $(u,v) \in (H^1(Q))^2 \times (L^2(Q_\omega))^2$.*

B. *If $u_0 \in (L^2(\Omega))^2$, $u_1 \in (H^{-1}(\Omega))^2$, $f \in L^2(0,T;(H^{-1}(\Omega))^2)$ then there exists a solution to problem (10) a pair $(u,v) \in (L^2(Q))^2 \times L^2(0,T;(H^{-1}(\Omega))^2)$, $supp\, v \subset \overline{Q}_\omega$.*

Proof of Theorem 1.1. Withought the loss of generality we may assume $\rho \equiv 1$. First we note that instead of (6) it suffices to prove more simple estimate

$$\int_Q (s|\nabla \text{rot}\, u|^2 + s^3|\text{rot}\, u|^2 + s|\nabla \text{div}\, u|^2 + s^3|\text{div}\, u|^2)e^{2s\phi}dx$$
$$+ s\|\frac{\partial u}{\partial \nu}e^{s\phi}\|^2_{(H^1((0,T)\times\partial\Omega))^2} + s\|\frac{\partial^2 u}{\partial \nu^2}e^{s\phi}\|^2_{(L^2((0,T)\times\partial\Omega))^2} + s^3\|\frac{\partial u}{\partial \nu}e^{s\phi}\|^2_{(L^2((0,T)\times\partial\Omega))^2}$$
$$\le C_2(s\|fe^{s\phi}\|^2_{(L^2((0,T)\times\partial\Omega))^2} + \|\text{rot} f e^{s\phi}\|^2_{(L^2(Q))^2} + \|\text{div} f e^{s\phi}\|^2_{(L^2(Q))^2}$$
$$+ \int_{Q_\omega} (s|\nabla \text{rot}\, u|^2 + s^3|\text{rot}\, u|^2 + s|\nabla \text{div}\, u|^2 + s^3|\text{div}\, u|^2)e^{2s\phi}dx) \quad \forall s \ge s_0(\hat\lambda). \quad (11)$$

This fact is the simple corollary of the following proposition

Proposition 1 *There exists $\hat\lambda > 1$ such that for any $\lambda > \hat\lambda$ exist $s_0(\lambda)$ that*

$$\int_Q (|\nabla u|^2 + s^2|u|^2)e^{2s\phi}dx \le C_3(\|rot\, u\, e^{s\phi}\|^2_{L^2(Q)} + \|div\, u e^{s\phi}\|^2_{L^2(Q)}$$
$$+ \int_{Q_\omega} (|\nabla u|^2 + s^2|u|^2)e^{2s\phi}dx) \quad \forall s \ge s_0(\lambda), u \in (H^1_0(Q))^2.$$

Denote by $\Box_\beta$ the following hyperbolic operator $\Box_\beta = \frac{\partial^2}{\partial x_0^2} - \beta(x)\Delta$. It is well known that the functions $\text{rot}\, u, \text{div}\, u$ satisfy the equations

$$\Box_\mu \text{rot u} = q_1 \quad \text{in } Q, \quad \Box_{\lambda+2\mu}\text{div u} = q_2 \quad \text{in } Q, \quad (12)$$

$$q_1 = K_1\text{rot}u + K_2\text{div}u + \text{rot}f, \quad q_2 = K_3\text{rot}u + K_4\text{div}u + \text{div}f, \quad (13)$$

where K_i are the first order differential operators with L^∞ coefficients.

We observe that it suffices to prove Theorem 1.1 only locally, assuming

$$\operatorname{supp} u \subset B_\delta, \tag{14}$$

where B_δ is the ball of the radius $\delta > 0$ centered at some point y^*. When $B_\delta \cap (0,T) \times \partial\Omega = \emptyset$ the situation is trivial (see e.g. [Hö]). Therefore, without the loss of generality we may assume that $y^* = (y_0^*, 0, 0)$. Moreover the parameter $\delta > 0$ could be chosen an arbitrary small. Assume that locally near zero the boundary $\partial\Omega$ is given by equation $x_2 - \ell(x_1) = 0$. Moreover since the function $\tilde{u} = \mathcal{O}u(t, \mathcal{O}^{-1}x)$ satisfies the system (1),(2) with $\tilde{f} = \mathcal{O}f(t, \mathcal{O}^{-1}x)$ for any orthogonal matrix $\mathcal{O}$ we may assume that

$$\ell'(0) = 0.$$

Making the change of variables $y_1 = x_1, y_2 = x_2 - \ell(x_1)$ we reduce equations (1) to the form

$$\frac{\partial^2 u_1}{\partial y_0^2} - \mu\Big(\frac{\partial^2 u_1}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 u_1}{\partial y_1 \partial y_2} + (1 + |\ell'(y_1)|^2)\frac{\partial^2 u_1}{\partial y_2^2}\Big) + \mu\ell''(y_1)\frac{\partial u_1}{\partial y_2} \\ - (\lambda+\mu)\frac{\partial}{\partial y_1}\Big(\operatorname{div} u - \frac{\partial u_1}{\partial y_2}\ell'\Big) + (\lambda+\mu)\frac{\partial}{\partial y_2}\Big(\operatorname{div} u - \frac{\partial u_1}{\partial y_2}\ell'\Big)\ell' = f_1,$$

$$\frac{\partial^2 u_2}{\partial y_0^2} - \mu\Big(\frac{\partial^2 u_2}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 u_2}{\partial y_1 \partial y_2} + (1 + |\ell'(y_1)|^2)\frac{\partial^2 u_2}{\partial y_2^2}\Big) + \mu\ell''(y_1)\frac{\partial u_2}{\partial y_2} \\ - (\lambda+\mu)\frac{\partial}{\partial y_2}\Big(\operatorname{div} u - \frac{\partial u_1}{\partial y_2}\ell'\Big) = f_2,$$

where by f_1, f_2 we denote f after the change of variables. After the change of variables equations (12), (13) have the form

$$P_\mu z_1 = \frac{\partial^2 z_1}{\partial y_0^2} - \mu\Big(\frac{\partial^2 z_1}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 z_1}{\partial y_1 \partial y_2} + (1 + |\ell'(y_1)|^2)\frac{\partial^2 z_1}{\partial y_2^2}\Big) \\ + \mu\ell''(y_1)\frac{\partial z_1}{\partial y_2} = m_1 \quad \text{in } \mathcal{G} \triangleq \mathbb{R}^2 \times [0, \hat{\kappa}],$$

$$P_{\lambda+2\mu} z_2 = \frac{\partial^2 z_2}{\partial y_0^2} - (\lambda+2\mu)\Big(\frac{\partial^2 z_2}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 z_2}{\partial y_1 \partial y_2} + (1 + |\ell'(y_1)|^2)\frac{\partial^2 z_2}{\partial y_2^2}\Big) \\ + (\lambda+2\mu)\ell''(y_1)\frac{\partial z_2}{\partial y_2} = m_2 \quad \text{in } \mathcal{G},$$

and m_1, m_2 are the functions q_1, q_2 after the change of variables.

Without the loss of the generality we may assume $\hat{\kappa} = 1$. Next we introduce the operators

$$P_{\mu,s} = e^{|s|\phi} P_\mu e^{-|s|\phi}, \quad P_{\lambda+2\mu,s} = e^{|s|\phi} P_{\lambda+2\mu} e^{-|s|\phi}.$$

Now we note that in order to prove estimate (11) it suffices to establish the following estimate for the function $w = (w_1, w_2)$:

$$\|w\|_*^2 \equiv s\|w\|^2_{(H^1(\mathcal{G}))^2} + s^3\|w\|^2_{(L^2(\mathcal{G}))^2} + s\|\frac{\partial w}{\partial y_2}\|^2_{(L^2(\partial\mathcal{G}))^2} + |w\|^2_{(H^1(\partial\mathcal{G}))^2}$$
$$+ s^3\|\frac{\partial w}{\partial y_2}\|^2_{(L^2(\partial\mathcal{G}))^2} \le C_4(\|P_{\lambda+2\mu,s}w_2\|^2_{L^2(\mathcal{G})} + \|P_{\mu,s}w_1\|^2_{L^2(\mathcal{G})} + s\|g\|^2_{(L^2(\partial\mathcal{G}))^2}$$
$$+ s\|w\|^2_{(H^1(\mathcal{G}_\omega))^2} + s^3\|w\|^2_{(L^2(\mathcal{G}_\omega))^2}) \quad \forall s \ge s_0(\lambda),$$

and for all $w \in H^2(\mathcal{G})$, $w(\cdot, T) = w(\cdot, T) = w_{y_0}(\cdot, 0) = w_{y_0}(\cdot, T) = 0$ such that

$$\frac{\partial w_1}{\partial y_2} = \frac{\lambda+2\mu}{\mu}\frac{\partial w_2}{\partial y_1} + s\phi_{y_2}(y^*)w_1 - s\frac{\lambda+2\mu}{\mu}\phi_{y_1}(y^*)w_2 + g_1, \quad \text{on } \partial\mathcal{G},$$
$$\frac{\partial w_2}{\partial y_2} = -\frac{\mu}{\lambda+2\mu}\frac{\partial w_1}{\partial y_1} + s\phi_{y_2}(y^*)w_2 + s\frac{\mu}{\lambda+2\mu}\phi_{y_1}(y^*)w_1 + g_2, \quad \text{on } \partial\mathcal{G}.$$

We denote as $p_\mu(y, \xi_0, \xi_1, \xi_2)$, $p_{\lambda+2\mu}(y, \xi_0, \xi_1, \xi_2)$ respectively the principal symbols of the operators P_μ, $P_{\lambda+2\mu}$. Then the principal symbols of the operators $P_{s,\mu}$, $P_{s,\lambda+2\mu}$ principal are given by formulas

$$p_\mu(y, \xi_0 + i|s|\phi_{y_0}, \xi_1 + i|s|\phi_{y_1}, \xi_2 + i|s|\phi_{y_2}),$$
$$p_{\lambda+2\mu}(y, \xi_0 + i|s|\phi_{y_0}, \xi_1 + i|s|\phi_{y_1}, \xi_2 + i|s|\phi_{y_2}).$$

In order to prove the Carleman estimate it is convenient for us to introduce new variable σ and consider s as a dual variable to σ. Following [T1, Chapter 14] we consider the pseudodifferential operators

$$\mathbf{P}_\beta(y, D_\sigma, D_{y_0}, D_{y_1}, D_{y_2})v =$$
$$\int_{\mathbb{R}^3} p_\beta(y, \xi_0 + i|s|\phi_{y_0}, \xi_1 + i|s|\phi_{y_1}, D_{y_2} + i|s|\phi_{y_2})\hat{v}(s, \xi_0, \xi_1, y_2)e^{i(<y',\xi'>+\sigma s)}d\sigma d\xi',$$

where $\xi' = (\xi_0, \xi_1)$, $y' = (y_0, y_1)$. Let $\mathbf{v}(\sigma, y) = (v_1(\sigma, y), v_2(\sigma, y))$ be the function with the domain $\mathcal{Q} = \mathbb{R}^2 \times \mathbb{R}^1_+ \times \mathbb{R}^1 (\Sigma = \partial\mathcal{Q}.)$ and $w(s, y) = (w_1(s, y), w_2(s, y))$ be the Fourier transform of $\mathbf{v}$ respect to the variable σ. Let $h(s) = (1 + s^2)^{\frac{1}{4}}$. Using the method developed in [T1] we obtain that in order to prove (11) it suffices to establish the following estimate

$$|||\mathbf{v}|||^2 \triangleq \sum_{j=0}^{1} \|h(D_\sigma)^{3-2j}\mathbf{v}\|^2_{L^2(\mathbb{R}^1;(H^j(\mathcal{G}))^2)} + \|h(D_\sigma)^{3-2j}\mathbf{v}\|^2_{(H^j(\pm))^2} + \|h(D_\sigma)\frac{\partial \mathbf{v}}{\partial y_2}\|_{(L^2(\Sigma))^2}$$
$$\le C_{10}(\|\mathbf{P}_{\lambda+2\mu}(y, D)v_1\|^2_{L^2(\mathcal{Q})} + \|\mathbf{P}_\mu(y, D)v_2\|^2_{L^2(\mathcal{Q})}$$
$$+ \|h(D_\sigma)q\|^2_{(L^2(\Sigma))^2} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}), \quad (15)$$

where $g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} qe^{-is\sigma}d\sigma$, $supp\, \mathbf{v} \subset (-\sigma_0, \sigma_0) \times B_\delta$ with the parameter $\sigma_0 > 0$ which can be chosen an arbitrary small, and

$$B_1(w_1, w_2) \triangleq -\frac{\partial w_1}{\partial y_2} + \frac{\lambda+2\mu}{\mu}\frac{\partial w_2}{\partial y_1} + |s|\phi_{y_2}(y^*)w_1 - |s|\frac{\lambda+2\mu}{\mu}\phi_{y_1}(y^*)w_2 = g_1, \quad \text{on } \Sigma, \quad (16)$$

$$B_2(w_1,w_2) \triangleq -\frac{\partial w_2}{\partial y_2} - \frac{\mu}{\lambda+2\mu}\frac{\partial w_1}{\partial y_1} + |s|\phi_{y_2}(y^*)w_2 + |s|\frac{\mu}{\lambda+2\mu}\phi_{y_1}(y^*)w_1 = g_2, \quad \text{on } \Sigma, \tag{17}$$

where

$$w = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathbf{v}e^{-is\sigma}d\sigma.$$

This fact can be proved in exactly the same way as in [T1, Chapter 14 ,section 2].

Consider the finite covering of the unit sphere $s^2+\xi_0^2+\xi_1^2=1$: $S^2 \subset \cup_{\zeta^*\in S^2}\{\zeta = (s,\xi_0,\xi_1)\in S^2 || \zeta-\zeta^*| < \delta_1\}$ and submitted to this covering partition of unity $\chi_\nu(\zeta)$: $\sum_{\nu=1}^N \chi_\nu(\zeta) = 1$ for any $\zeta \in S^2$ and $supp\,\chi_\nu \subset \{\zeta\in S^2||\zeta-\zeta^*_\nu| < \delta_1\}$. We extend this function on the set $|\zeta|>1$ as the homogeneous function of the order zero in a such a way that $supp\chi_\nu \subset \mathcal{O}(\delta_1) = \{\zeta||\frac{\zeta}{|\zeta|}-\zeta^*| < \delta_1\}$. Let us consider the pseudodifferential operator $\chi_\nu(D)$ and the function $\chi_\nu(D)\mathbf{v}$. Obviously equalities (16), (17) holds true with $w_\nu = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\chi_\nu(D)\mathbf{v}e^{-is\sigma}d\sigma, g_\nu = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\chi_\nu(D)qe^{-is\sigma}d\sigma.$

We claim that instead of (15) it suffices to prove the following estimate

$$|||\chi_\nu(D)\mathbf{v}||| \le C_{10}(\nu)(\|\mathbf{P}_{\lambda+2\mu}\chi_\nu(D)\mathbf{v}\|_{L^2(\mathcal{Q})} + \|\mathbf{P}_\mu\chi_\nu(D)\mathbf{v}\|_{L^2(\mathcal{Q})} + \|h(D_\sigma)\chi_\nu(D)q\|_{L^2(\Sigma)} + \|\mathbf{v}\|_{(H^1(\mathcal{Q}))^2}), \tag{18}$$

where

$$B_1(w_{1,\nu},w_{2,\nu}) = g_{1,\nu}, \quad B_2(w_{1,\nu},w_{2,\nu}) = g_{2,\nu}. \tag{19}$$

The principal symbol of the operator $P_{\beta,s}$ has the form

$$p(y,s,\xi_0,\xi_1) = -(\xi_0+i|s|\phi_{y_0})^2 + \beta[(\xi_1+i|s|\phi_{y_1})^2 - 2\ell'(\xi_1+i|s|\phi_{y_1})(\xi_2+i|s|\phi_{y_2}) + (\xi_2+i|s|\phi_{y_2})^2|G|], \tag{20}$$

where $|G| = 1+(\ell')^2$. The roots of this polynomial respect to variable ξ_2 are

$$\Gamma^\pm_\beta(y,s,\xi_0,\xi_1) = -i|s|\phi_{y_2} + \alpha^\pm_\beta(y,s,\xi_0,\xi_1), \tag{21}$$

$$\alpha^\pm_\beta(y,s,\xi_0,\xi_1) = \frac{(\xi_1+i|s|\phi_{y_1})\ell'(y_1)}{|G|} \pm\sqrt{r_\beta(y,s,\xi_0,\xi_1)}, \tag{22}$$

$$r_\beta(y,\zeta) = \frac{((\xi_0+i|s|\phi_{y_0})^2 - \beta(\xi_1+i|s|\phi_{y_1})^2)|G| + \beta(\xi_1+i|s|\phi_{y_1})^2(\ell')^2}{\beta|G|^2}. \tag{23}$$

Denote $\gamma = (y^*,\zeta^*)$. Suppose that $r_\beta(\gamma)\ne 0$. Now we claim that there exists $\delta_0>0$ such that for all $\delta,\delta_1\in(0,\delta_0)$ there exists a constant $\hat{C}_{14}>0$ such that for one of the roots of the polynomial (20) , which we denote as Γ^-_β we have

$$-\operatorname{Im}\Gamma^-_\beta(y,s,\xi_0,\xi_1) \ge \hat{C}_{14}|s| \quad \forall (y,s,\xi_0,\xi_1)\in B_\delta\times\mathcal{O}(\delta_1).$$

In some situations we may represent the operator $\mathbf{P}_\beta$ as a product of two first order pseudodifferential operators.

Proposition 2 *Let $r_\beta(\gamma) \neq 0$ and supp $\hat{V} \subset \mathcal{O}(\delta_1)$. Then we could decompose the operator $P_{s,\beta}$ into the product of two first order P.D.O.*

$$\mathbf{P}_\beta V = (D_{y_2} - \Gamma^-_\beta(y, D))(D_{y_2} - \Gamma^+_\beta(y, D))V + T_\beta V, \tag{24}$$

where T_β is the continuous operator

$$T_\beta : L^2(0, 1; H^1(\mathbb{R}^3)) \to L^2(0, 1; L^2(\mathbb{R}^3)).$$

Let us consider the equation

$$(D_{y_2} - \Gamma^-_\beta(y, D))V = q, \quad V|_{y_2=1} = 0.$$

For solutions of this problem we have the a priori estimate:

Proposition 3 *There exists a constant $C_{15} > 0$ such that*

$$\|h(D_\sigma)V|_{y_2=0}\|_{L^2(\mathbb{R}^3)} \leq C_{15}\|q\|_{L^2(\mathcal{Q})}.$$

Let $\beta > 0$ and $\tilde{w}(s, y)$ be a function which for almost all $s \in \mathbb{R}^1$ satisfies the scalar second order hyperbolic equation

$$P_{\beta,s}\tilde{w} = q \quad \text{in } \mathcal{G}, \quad \frac{\partial \tilde{w}}{\partial y_2}|_{y_2=1} = \tilde{w}|_{y_2=1} = 0, \quad \text{supp}\, \tilde{w} \subset B_\delta \times \mathbb{R}^1.$$

Let $P^*_{\beta,s}$ be the formally adjoint operator to $P_{s,\beta}$, where $\beta \in [\mu, \lambda + 2\mu]$. Set $L_{+,\beta} = (P_{\beta,s} + P^*_{\beta,s})/2$ and $L_{-,\beta} = (P_{\beta,s} - P^*_{\beta,s})/2$. One may easily check that the operator $L_{-,\beta}$ is given by the formula

$$\begin{aligned} L_{-,\beta}\tilde{w} = &-2|s|\phi_{y_0}\frac{\partial \tilde{w}}{\partial y_0} \\ &+ \beta\Big(2|s|\phi_{y_1}\frac{\partial \tilde{w}}{\partial y_1} - 2|s|\ell'(y_1)\big(\phi_{y_2}\frac{\partial \tilde{w}}{\partial y_1} + \phi_{y_1}\frac{\partial \tilde{w}}{\partial y_2}\big) + 2|s|\big(1 + \ell'(y_1)^2\big)\phi_{y_2}\frac{\partial \tilde{w}}{\partial y_2}\Big). \end{aligned}$$

Obviously $L_{+,\beta}\tilde{w} + L_{-,\beta}\tilde{w} = q$. For almost all $s \in \mathbb{R}^1$ the following equality holds true

$$B_\beta + \|L_{-,\beta}\tilde{w}\|^2_{L^2(\mathcal{G})} + \|L_{+,\beta}\tilde{w}\|^2_{L^2(\mathcal{G})} + \text{Re}([L_{+,\beta}, L_{-,\beta}]\tilde{w}, \tilde{w})_{L^2(\mathcal{G})} = \|q\|^2_{L^2(\mathcal{G})},$$

where

$$\begin{aligned} B_\beta = \text{Re}\int_{\partial\mathcal{G}} &\tilde{p}_\beta(y, \nabla\phi, -\vec{e}_2)(|s|\tilde{p}_\beta(y, \nabla\tilde{w}) - |s|^3\tilde{p}_\beta(y, \nabla\phi, \nabla\phi)\tilde{w}^2)dy_0dy_1 \\ &+ \text{Re}\int_{\partial\mathcal{G}} \tilde{p}_\beta(y, \nabla\tilde{w}, -\vec{e}_2)\overline{L_{-,\beta}\tilde{w}}dy_0dy_1, \end{aligned} \tag{25}$$

$\vec{e} = (0, 0, 1)$ and

$$\tilde{p}_\beta(y, \xi, \tilde{\xi}) = \xi_0\tilde{\xi}_0 - \beta(\xi_1\tilde{\xi}_1 - \ell'(y_1)(\xi_1\tilde{\xi}_2 + \xi_2\tilde{\xi}_1) + (1 + |\ell'(y_1)|^2)\xi_2\tilde{\xi}_2).$$

It is convenient for us to rewrite (25) in the form

$$B_\beta = B_\beta^{(1)} + B_\beta^{(2)},$$

$$B_\beta^{(1)} = \mathrm{Re}\int_{y_2=0} 2|s|\beta\frac{\partial\tilde w}{\partial y_2}\overline{(\beta\frac{\partial\tilde w}{\partial y_1}\phi_{y_1}(y^*) + \beta\frac{\partial\tilde w}{\partial y_2}\phi_{y_2}(y^*) - \frac{\partial\tilde w}{\partial y_0}\phi_{y_0}(y^*))}dy_0dy_1$$
$$+\int_{y_2=0}|s|\beta\phi_{y_2}(y^*)(|\frac{\partial\tilde w}{\partial y_0}|^2 - \beta(|\frac{\partial\tilde w}{\partial y_1}|^2 + |\frac{\partial\tilde w}{\partial y_2}|^2)$$
$$-|s|^2(\phi_{y_0}^2(y^*) - \beta(\phi_{y_1}^2(y^*) + \phi_{y_2}^2(y^*))|\tilde w|^2)dy_0dy_1.$$

and

$$|B_\beta^{(2)}| \le \epsilon_0(s\|\frac{\partial\tilde w}{\partial y_2}\|^2_{L^2(\partial\mathcal{G})} + |s|\|\tilde w\|^2_{H^1(\partial\mathcal{G})} + |s|^3\|\tilde w\|^2_{L^2(\partial\mathcal{G})}). \tag{26}$$

It is known (see e.g. [Im]) that there exists a parameter $\hat\lambda > 1$ such that for any $\lambda > \hat\lambda$ there exists $s_0(\lambda)$ such that

$$\|L_{-,\beta}\tilde w\|^2_{L^2(\mathcal{G})} + \|L_{+,\beta}\tilde w\|^2_{L^2(\mathcal{G})} + \mathrm{Re}([L_{+,\beta}, L_{-,\beta}]\tilde w, \tilde w)_{L^2(\mathcal{G})} \ge$$
$$C_{18}(s\|\tilde w\|^2_{H^1(\mathcal{G})} + |s|^3\|\tilde w\|^2_{L^2(\mathcal{G})}) \quad \forall s \ge s_0(\lambda), \tag{27}$$

where $C_{18} > 0$ is independent of s. Denote by $\varXi_\beta = \int_{-\infty}^{\infty} B_\beta ds$, $\varXi_\beta^{(j)} = \int_{-\infty}^{\infty} B_\beta^{(j)} ds$ $j = 1,2$. Therefore integrating (27) respect to s on $\mathbb{R}^1$ we have

$$C_{19}(\|h(s)\tilde w\|^2_{H^1(\mathcal{Q})} + \|h^3(s)\tilde w\|^2_{L^2(\mathcal{Q})}) + \varXi_\beta \le \|q\|^2_{L^2(\mathcal{Q})} + C_{20}\|\tilde w\|^2_{H^1(\mathcal{Q})}, \tag{28}$$

where $C_{19} > 0$, and by (26)

$$|\varXi_\beta^{(2)}| \le \epsilon\|(\frac{\partial\tilde w}{\partial y_2}, \tilde w)\|^2_X, \tag{29}$$

where

$$\|(\frac{\partial\tilde w}{\partial y_2}, \tilde w)\|^2_X = \|h(s)\frac{\partial\tilde w}{\partial y_2}\|^2_{L^2(\Sigma)} + \|h(s)\tilde w\|^2_{L^2(\mathbb{R}^1;H^1(\mathbb{R}^2))} + \|h(s)\tilde w\|^2_{L^2(\Sigma)}$$

and the parameter ϵ could make an arbitrary small taking δ in (14) sufficiently small.

Later we will need to apply (28), (29) to the functions $w_{1,\nu} = F_\sigma\chi_\nu(D)v_1$, $w_{2,\nu} = F_\sigma\chi_\nu(D)v_{2,\nu}$ since we would like to take the advantage of (18). Formally it is impossible since the condition $\mathrm{supp}\,\chi_\nu(D)\mathbf{v} \subset B_\delta \times \mathbb{R}^1$ in general does not hold true. On the other hand using the fact that

$$\int_{\mathbb{R}^2\setminus B_{2\delta}}\int_{\mathbb{R}^1} h^4(s)\sum_{|\alpha|\le 2}|D^\alpha w_{i,\nu}|^2 dy_0dy_1ds \le C_{21}\|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}.$$

we obtain slightly modified analog of (28), (29):

$$C_{22}(\|h(s)w_{i,\nu}\|^2_{H^1(\mathcal{Q})} + \|h^3(s)w_{i,\nu}\|^2_{L^2(\mathcal{Q})}) + \varXi_\beta \le \|P_{\beta,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + C_{23}\|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}, \tag{30}$$

where $C_{22} > 0$, and $\beta = \mu$ for $i = 1$, $\beta = \lambda + 2\mu$ for $i = 2$

$$|\varXi_\beta^{(2)}| \le \epsilon\|(\frac{\partial w_{i,\nu}}{\partial y_2}, w_{i,\nu})\|^2_X + C_{23}\|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}. \tag{31}$$

2. Case $r_\mu \neq 0$ and $r_{\lambda+2\mu} \neq 0$.

In this section we consider the conic neighborhood $\mathcal{O}(\delta_1)$ of the point (y^*, ζ^*) such that

$$|r_\mu(y^*, \zeta^*)| \neq 0 \quad \text{and } |r_{\lambda+2\mu}(y^*, \zeta^*)| \neq 0. \tag{32}$$

In that case thanks to (32) and Proposition 1.2 decomposition (24) holds true for $\beta = \mu$ and $\beta = \lambda + 2\mu$. Therefore we have

$$(D_{y_2} - \Gamma^+_\mu(y, D))v_{1,\nu}|_{y_2=0} = V^+_\mu(\cdot, 0), \tag{33}$$

$$(D_{y_2} - \Gamma^+_{\lambda+2\mu}(y, D))v_{2,\nu}|_{y_2=0} = V^+_{\lambda+2\mu}(\cdot, 0). \tag{34}$$

By Proposition 1.3 we have the a priori estimate

$$\|h(D_\sigma)V^+_\mu(\cdot, 0)\|^2_{L^2(\Sigma)} + \|h(D_\sigma)V^+_{\lambda+2\mu}(\cdot, 0)\|^2_{L^2(\Sigma)} \\ \leq C_1(\|\mathbf{P}_{\lambda+2\mu}v_2\|^2_{L^2(\mathcal{Q})} + \|\mathbf{P}_\mu v_1\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}). \tag{35}$$

Using (19) we may rewrite (33), (34) as

$$\frac{\lambda + 2\mu}{\mu}\Big(\frac{\partial v_{2,\nu}}{\partial y_1} - |D_\sigma|\phi_{y_1}(y^*)v_{2,\nu}\Big) - i\alpha^+_\mu(y, D)v_{1,\nu} = V^+_\mu(\cdot, 0) - q_{1,\nu},$$

$$\frac{\mu}{\lambda + 2\mu}\Big(-\frac{\partial v_{1,\nu}}{\partial y_1} + |D_\sigma|\phi_{y_1}(y^*)v_{1,\nu}\Big) - i\alpha^+_{\lambda+2\mu}(y, D)v_{2,\nu} = V^+_{\lambda+2\mu}(\cdot, 0) - q_{2,\nu}.$$

Let $\mathbf{B}(y, D)$ be the matrix P.D.O. with the symbol

$$\mathbf{B}(y, \zeta) = \begin{pmatrix} -i\alpha^+_\mu(y, \zeta) & \frac{\lambda+2\mu}{\mu}(i\xi_1 - |s|\phi_{y_1}) \\ \frac{\mu}{\lambda+2\mu}(-i\xi_1 + |s|\phi_{y_1}) & -i\alpha^+_{\lambda+2\mu}(y, \zeta) \end{pmatrix}.$$

By (22), (23) the set $\{\zeta^* \in S^2 | det\, \mathbf{B}(y^*, \zeta^*) = 0\}$ is empty.

Then there exists a parametrix of the operator $\mathbf{B}(y, D)$ (see [T2]) which we denote as $\mathbf{B}^{-1}(y, D)$ such that

$$(v_{1,\nu}, v_{2,\nu}) = \mathbf{B}^{-1}(y, D)(V^+_\mu(\cdot, 0) - q_{1,\nu}, V^+_{\lambda+2\mu}(\cdot, 0) - q_{2,\nu}) + K(v_{1,\nu}, v_{2,\nu}), \tag{36}$$

where

$$K : (L^2(\mathcal{Q})) \rightarrow (H^1(\mathcal{Q}))^2,$$

By (35), (36)

$$|\Xi_\mu| + |\Xi_{\lambda+2\mu}| \leq C_2(\|\mathbf{P}_\mu v_1\|^2_{L^2(\mathcal{Q})} + \|\mathbf{P}_{\lambda+2\mu}v_2\|^2_{L^2(\mathcal{Q})} + \|h(s)g\|^2_{(L^2(\Sigma))^2} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}). \tag{37}$$

By (37), (28), (29) we obtain (18).

3. Case $r_\mu = 0$.

In this section mainly we treat the case when supp $\chi_\nu \subset \mathcal{O}(\delta_1)$, and $\gamma = (y^*, \zeta^*)$ be a point on $\Sigma \times S^2$ such that $r_\mu(\gamma) = 0$.

We remind that by (30), (31) there exists $C_1 > 0, C_2 > 0$ such that

$$C_1(\|h(s)w_{1,\nu}\|^2_{H^1(\mathcal{Q})} + \|h^3(s)w_{1,\nu}\|^2_{L^2(\mathcal{Q})}) + \Xi^{(1)}_\mu \le C_2\|\mathbf{P}_\mu v_1\|^2_{L^2(\mathcal{Q})} + \epsilon(\delta)\|(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu})\|^2_X, \tag{38}$$

and the parameter ϵ could be taken sufficiently small, if we decrease δ. Note that $\Xi^{(1)}_\mu$ could be written in the form

$$\Xi^{(1)}_\mu = \int_\Sigma (|s|\mu^2\phi_{y_2}(y^*)|\frac{\partial w_{1,\nu}}{\partial y_2}|^2 + |s|^3\mu^2\phi^3_{y_2}(y^*)|w_{1,\nu}|^2)d\Sigma$$
$$+ Re\int_\Sigma 2|s|\mu\frac{\partial w_{1,\nu}}{\partial y_2}\overline{(\mu\phi_{y_1}(y^*)\frac{\partial w_{1,\nu}}{\partial y_1} - \phi_{y_0}(y^*)\frac{\partial w_{1,\nu}}{\partial y_0})}d\Sigma$$
$$+ \int_\Sigma |s|\mu\phi_{y_2}(y^*)(\xi_0^2 - \mu\xi_1^2 - s^2\phi^2_{y_0}(y^*) + s^2\mu\phi^2_{y_1}(y^*))|\hat{v}_{1,\nu}|^2 d\Sigma =$$
$$J_1 + J_2 + J_3. \tag{39}$$

We shall consider the two cases. Let us introduce the set $\mathcal{M}$ by formula

$$\mathcal{M} = \{\zeta = (s, \xi_0, \xi_1) \in \mathcal{O}(\delta_1)|\frac{\mu}{2}\phi_{y_2}(y^*)\hat{C}s^2 >$$
$$4\mu^2\frac{\phi^2_{y_1}(y^*)}{|\phi_{y_2}(y^*)|}\xi_1^2 + 4\frac{\phi^2_{y_0}(y^*)}{|\phi_{y_2}(y^*)|}\xi_0^2 + (|\xi_0|^2 + |\xi_1|^2)\}, \tag{40}$$

where $\hat{C} = \min_{y\in B_\delta}\{-p_1(y, \nabla\phi)\}$. From (4) it follows that $\hat{C}$ is positive.

Case A. Assume that

$$\text{supp}\, \hat{\mathbf{v}}_\nu \subset \mathcal{O}(\delta_1) \subset \mathcal{M}.$$

Applying the Cauchy-Bunyakovskii inequality and using (40) and (4) we obtain that there exists a constant $C_3 > 0$ such that

$$\Xi^{(1)}_\mu \ge \int_\Sigma |s|\mu^2\phi_{y_2}(y^*)|\frac{\partial w_{1,\nu}}{\partial y_2}|^2 - |s|^3\mu\phi_{y_2}(y^*)p(y^*, \nabla\phi(y^*))|w_{1,\nu}|^2\, d\Sigma$$
$$- \int_\Sigma \frac{1}{2}|s|\mu^2\phi_{y_2}(y^*)|\frac{\partial w_{1,\nu}}{\partial y_2}|^2 + 4|s|\mu^2\frac{\phi^2_{y_1}(y^*)}{|\phi_{y_2}(y^*)|}|\frac{\partial w_{1,\nu}}{\partial y_1}|^2 + 4|s|\frac{\phi^2_{y_0}(y^*)}{|\phi_{y_2}(y^*)|}|\frac{\partial w_{1,\nu}}{\partial y_0}|^2\, d\Sigma$$
$$\ge C_3\int_\Sigma \frac{1}{2}|s|\mu^2\phi_{y_2}(y^*)|\frac{\partial w_{1,\nu}}{\partial y_2}|^2 + |s||\frac{\partial w_{1,\nu}}{\partial y_1}|^2 + |s||\frac{\partial w_{1,\nu}}{\partial y_0}|^2 + \frac{1}{2}|s|^3\mu\phi_{y_2}(y^*)\hat{C}|w_{1,\nu}|^2\, d\Sigma. \tag{41}$$

We remind that by (19) we have the equality

$$\frac{\partial w_{2,\nu}}{\partial y_2} - |s|\phi_{y_2}(y^*)w_{2,\nu} = -\frac{\mu}{\lambda + 2\mu}(\frac{\partial w_{1,\nu}}{\partial y_1} - |s|\phi_{y_1}(y^*)w_{1,\nu}) + g_{2,\nu}.$$

Taking the L^2 norm of the left and right hand sides of this equality and using the estimate (35) we obtain

$$\int_\Sigma h(s)|\frac{\partial w_{2,\nu}}{\partial y_2}|^2 + h^3(s)\phi^2_{y_2}(y^*)|w_{2,\nu}|^2\, d\Sigma$$
$$\le C_4\Big(\Xi^{(1)}_\mu + \|h(s)g\|^2_{(L^2(\Sigma))^2} + \epsilon(\sigma_0)\|(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu})\|^2_X$$
$$+ \int_\Sigma |\frac{\partial w_{2,\nu}}{\partial y_2}|^2 + s^2\phi^2_{y_2}(y^*)|w_{2,\nu}|^2\, d\Sigma\Big),$$

where $\epsilon(\sigma_0)\to 0$ as $\sigma_0 \to 0$. By (40), (19)

$$\int_\Sigma h(s)\big(|\frac{\partial w_{2,\nu}}{\partial y_1}|^2 + |\frac{\partial w_{2,\nu}}{\partial y_0}|^2\big)\, d\Sigma \le C_5\Big(\Xi^{(1)}_\mu + \|h(s)g\|^2_{(L^2(\Sigma))^2}$$
$$+ \int_\Sigma |\frac{\partial w_{2,\nu}}{\partial y_2}|^2 + s^2\phi^2_{y_2}(y^*)w^2_{2,\nu}\, d\Sigma\Big). \quad (42)$$

If we consider (30) with $\beta = \lambda + 2\mu$ then (38), (41), (42) immediately imply (18).

Now we assume that supp $\mathbf{v}_\nu \subset \mathcal{O}(\delta_1)$ and $r_\mu(\gamma) = 0$. The parameter $\delta_1 > 0$ we'll fix later. By (21)-(23) there exists $C_6 > 0$ such that

$$|\xi_0^2 - s^2\phi^2_{y_0}(y^*) - \mu\xi_1^2 + \mu s^2\phi^2_{y_1}(y^*)| + |\xi_0 s\phi_{y_0}(y^*) - \mu s\xi_1\phi_{y_1}(y^*)|$$
$$\le \delta_1 C_6(|\xi_1|^2 + |\xi_0|^2 + s^2), \quad \forall\zeta \in \mathcal{O}(\delta_1). \quad (43)$$

Now we suppose that δ_1 is assumed to be a sufficiently small, such that there exists a constant $C_7 > 0$ such that

$$|\xi_0|^2 \le C_7(|\xi_1|^2 + s^2) \quad \forall\zeta \in \mathcal{O}(\delta_1).$$

Next we introduce the set $\tilde{\mathcal{M}}$ by formula

$$\tilde{\mathcal{M}} = \{\zeta = (s,\xi_0,\xi_1) \in \mathcal{O}(\delta_1) | \frac{\mu}{4}\phi_{y_2}(y^*)\hat{C}s^2 <$$
$$4\mu^2\frac{\phi^2_{y_1}(y^*)}{|\phi_{y_2}(y^*)|}\xi_1^2 + 4\frac{\phi^2_{y_0}(y^*)}{|\phi_{y_2}(y^*)|}\xi_0^2 + (|\xi_0|^2 + |\xi_1|^2)\}.$$

Obviously $\mathcal{O}(\delta_1) \subset \mathcal{M} \cup \tilde{\mathcal{M}}$.

By (43)

$$|J_3| \le \delta_1\mu\phi_{y_2}(y^*)\|(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu})\|^2_X. \quad (44)$$

Case B. Assume that

$$\text{supp}\,\hat{\mathbf{v}}_\nu \subset \tilde{\mathcal{M}}.$$

We observe that decomposition (24) with $\beta = \lambda + 2\mu$ holds true. We set $V^+_{\lambda+2\mu} = (D_{y_2} - \Gamma^+_{\lambda+2\mu}(y,D))v_{2,\nu}$. Then

$$\mathbf{P}_{\lambda+2\mu}v_{2,\nu} = (D_{y_2} - \Gamma^-_{\lambda+2\mu}(y,D))V^+_{\lambda+2\mu} + T_{\lambda+2\mu}v_{2,\nu},$$

where $T_{\lambda+2\mu} \in \mathcal{L}(H^1(\mathcal{Q}), L^2(\mathcal{Q}))$. This decomposition and the Proposition 1.2 immediately imply

$$\left\| h(D_\sigma)\big(D_{y_2} - \Gamma^+_{\lambda+2\mu}(y,D)\big) v_{2,\nu}|_{y_2=0} \right\|_{L^2(\Sigma)} \\ \le C_5\big(\|P_{\lambda+2\mu,s} w_{2,\nu}\|_{L^2(\mathcal{Q})} + \|\mathbf{v}\|_{(H^1(\mathcal{Q}))^2}\big).$$

Now we need again to estimate $\Xi^{(1)}_\mu$. In view of (44) it suffices to estimate the term J_2.

Let us consider the equation

$$-\frac{\mu}{\lambda+2\mu}\Big(\frac{\partial v_{1,\nu}}{\partial y_1} - |D_\sigma|\phi_{y_1}(y^*) v_{1,\nu}\Big) - i\alpha^+_{\lambda+2\mu}(y,D) v_{2,\nu} = iV^+_{\lambda+2\mu}(\cdot,0) - \frac{\mu}{\lambda+2\mu} q_{2,\nu}. \tag{45}$$

Since $r_{\lambda+2\mu}(\gamma) \ne 0$ then $\alpha^+_{\lambda+2\mu}(\gamma) \ne 0$. Therefore by Proposition from [T2] there exists a parametrix of the operator $\alpha^+_{\lambda+2\mu}(y,D)$ which we denote as $(\alpha^+_{\lambda+2\mu}(y,D))^{-1}$. From (45) we obtain

$$v_{2,\nu} = -\frac{1}{i}(\alpha^+_{\lambda+2\mu}(y,D))^{-1}\Big(\frac{\mu}{\lambda+2\mu}\Big(\frac{\partial v_{1,\nu}}{\partial y_1} - |D_\sigma|\phi_{y_1}(y^*) v_{1,\nu}\Big) \\ + iV^+_{\lambda+2\mu}(\cdot,0) - \frac{\mu}{\lambda+2\mu} q_{2,\nu}\Big) + T_0 v_{2,\nu}, \tag{46}$$

where $T_0 \in \mathcal{L}(L^2(\Sigma), H^1(\Sigma))$. Using (46), (19) we may transform J_2 to the form

$$J_2 = \mathrm{Re}\int_\Sigma -2|D_\sigma|\mu\frac{\mathrm{sign}(\xi_1^*)}{i}\sqrt{\frac{\lambda+\mu}{\lambda+2\mu}}\Big(\frac{\partial v_{1,\nu}}{\partial y_1} - |D_\sigma|\phi_{y_1}(y^*) v_{1,\nu}\Big)\overline{\Big(\mu\frac{\partial v_{1,\nu}}{\partial y_1}\phi_{y_1}(y^*)} \\ \overline{-\frac{\partial v_{1,\nu}}{\partial y_0}}\phi_{y_0}(y^*)\Big)d\Sigma + Re\kappa_3.$$

where

$$|\kappa_3| \le \epsilon\Big\|\Big(\frac{\partial w_\nu}{\partial y_2}, w_\nu\Big)\Big\|^2_X + C_{10}\big(\|h(s)g\|^2_{(L^2(\Sigma))^2} + \|P_{\lambda+2\mu,s} w_{2,\nu}\|^2_{L^2(\mathcal{Q})}\big)$$

Since $J_2 = Re\kappa_3$ we have

$$|J_2| \le \epsilon\|(\frac{\partial w_\nu}{\partial y_2}, w_\nu)\|^2_X + C_{11}(\|h(s)g\|^2_{(L^2(\Sigma))^2} + \|P_{\lambda+2\mu,s} w_{2,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}), \tag{47}$$

Next we observe that there exists $C > 0$ such that

$$\|(\frac{\partial w_\nu}{\partial y_2}, w_\nu)\|^2_X \le C_{12}\Big(\int_\Sigma (h(s)|\frac{\partial w_{1,\nu}}{\partial y_2}|^2 + h^3(s)|w_{1,\nu}|^2)d\Sigma + \|h(s)g\|^2_{(L^2(\Sigma))^2}\Big). \tag{48}$$

Inequalities (38), (39), (44) (47), (48) imply

$$\|(\frac{\partial w_\nu}{\partial y_2}, w_\nu)\|^2_X + \|h(s)w_{1,\nu}\|^2_{H^1(\mathcal{Q})} + \|h^3(s)w_{1,\nu}\|^2_{L^2(\mathcal{Q})} \le \epsilon\|(\frac{\partial w_\nu}{\partial y_2}, w_\nu)\|^2_X + \\ C_{14}(\|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2} + \|h(s)g_\nu\|^2_{(L^2(\Sigma))^2} + \|P_{\mu,s} w_{2,\nu}\|^2_{L^2(\mathcal{Q})}).$$

From this inequality and (30) with $\beta = \lambda + 2\mu$ we obtain (18). ∎

4. Case $r_{\lambda+2\mu} = 0$.

Let $\gamma = (y^*, \zeta^*)$ be a point on $\Sigma \times S^2$ such that $r_{\lambda+2\mu}(\gamma) = 0$ and $\operatorname{supp} \chi_\nu \subset \mathcal{O}(\delta_1)$. We consider several cases.

Case A. Assume that

$$s^* = 0 \quad \textit{and} \quad \lim_{\zeta \to \zeta^*} \operatorname{Im} r_\mu(y^*, \zeta)/|s| = 0.$$

In that case there exists a constant $C_2 > 0$ such that

$$-\operatorname{Im} \Gamma_\mu^\pm(y, \zeta) \geq C_2|s| \quad \forall (y, \zeta) \in B_\delta \times \mathcal{O}(\delta_1)$$

provided that $|\delta| + |\delta_1|$ is sufficiently small. We set $V_\mu^\pm = (D_{y_2} - \Gamma_\mu^\pm(y, D))v_{1,\nu}$. Then

$$\mathbf{P}_\mu v_{1,\nu} = (D_{y_2} - \Gamma_\mu^\mp(y, D))V_\mu^\pm + T_\mu^\pm v_{1,\nu},$$

where $T_\mu^\pm \in \mathcal{L}(H^1(\mathcal{Q}), L^2(\mathcal{Q}))$. This decomposition and the Proposition 1.2 immediately imply

$$\|h(D_\sigma)(D_{y_2} - \Gamma_\mu^\pm(y, D))v_{1,\nu}|_{y_2=0}\|_{L^2(\Sigma)} \leq C_4(\|\mathbf{P}_\mu v_{1,\nu}\|_{L^2(\mathcal{Q})} + \|\mathbf{v}\|_{(H^1(\mathcal{Q}))^2}). \quad (49)$$

Obviously

$$V_\mu^+(\cdot, 0) - V_\mu^-(\cdot, 0) = (\alpha_\mu^+(y, D) - \alpha_\mu^-(y, D))v_{1,\nu} \quad \text{on } \Sigma.$$

Since $\alpha_\mu^+(y^*, \zeta^*) - \alpha_\mu^-(y^*, \zeta^*) = 2\sqrt{r_\mu(y^*, \zeta^*)} \neq 0$ by (49) and Garding's inequality

$$\int_\Sigma (h(s)(|\frac{\partial w_{1,\nu}}{\partial y_1}|^2 + |\frac{\partial w_{1,\nu}}{\partial y_0}|^2) + h^3(s)|w_{1,\nu}|^2)d\Sigma \leq C_5(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(Q))^2}). \quad (50)$$

From this inequality and (49) we obtain the estimate for $\frac{\partial w_{1,\nu}}{\partial y_2}$:

$$\int_\Sigma h(s)|\frac{\partial w_{1,\nu}}{\partial y_2}|^2 d\Sigma \leq C_6(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(Q))^2}). \quad (51)$$

And finally (50), (51) combined with (19) give the estimate

$$\|(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu})\|^2_X \leq C_7(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(Q))^2} + \|h(s)g\|^2_{(L^2(\Sigma))^2}). \quad (52)$$

By (50)-(52), (30), (31) we obtain (18).

Case B. Assume that

$$s^* = 0 \quad and \quad \lim_{\zeta\to\zeta^*} \operatorname{Im} r_\mu(y^*,\zeta)/|s| \neq 0.$$

In that case first we note that since $s^* = 0$ then $Re\, r_\mu(y^*,\zeta^*) > 0$. Set $\mathbf{I} = sign \lim_{\zeta\to\zeta^*} \operatorname{Im} r_\mu(y^*,\zeta)/|s|$. For all $(y,\zeta) \in B_\delta \times \mathcal{O}(\delta_1)$ we have

$$\Gamma^+_\mu(y,\zeta) = \mathbf{I}\sqrt{Re\, r_\mu(y^*,\zeta)} + \tilde{b}_1(y,\zeta), \tag{53}$$

where for the P.D.O. $\tilde{b}_1(y,D)$ we have the estimate

$$\|h(s)\tilde{b}_1(y,D)w_{1,\nu}\|_{L^2(\Sigma)} \le \epsilon(\delta,\delta_1)\|(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu})\|^2_X, \tag{54}$$

where $\epsilon(\delta,\delta_1) \to 0$ as $|\delta| + |\delta_1| \to 0$.

We may assume that $|\delta| + |\delta_1|$ is so small that there exists $C_8 > 0$

$$\mathbf{I} \operatorname{Im} r_\mu(y^*,\zeta)/|s| \ge C_8|\zeta| \quad \forall\zeta \in \mathcal{O}(\delta_1). \tag{55}$$

Let us consider formula (32) from the previous section. One may easily see that the term J_3 is nonnegative. On the other hand (53)-(54) imply

$$J_2 \ge \int_\Sigma 2\mu\sqrt{Re\, r_\mu(y^*,\zeta)}|\operatorname{Im} r_\mu(y^*,\zeta)||\hat{v}_{1,\nu}|^2 d\Sigma - \\ C_9\epsilon(\delta,\delta_1)\|(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu})\|^2_X - C_{10}(\delta,\delta_1)(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}). \tag{56}$$

Inequalities (55), (56) imply

$$J_2 \ge C_{11}\int_\Sigma h(s)(|\frac{\partial w_{1,\nu}}{\partial y_1}|^2 + |\frac{\partial w_{1,\nu}}{\partial y_0}|^2) + h^3(s)|w_{1,\nu}|^2\, d\Sigma \\ - C_9\epsilon(\delta,\delta_1)\|(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu})\|^2_X - C_{10}(\delta,\delta_1)(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}).$$

By (56) we have that there exists a constant $C_{14} > 0$

$$\Xi^{(1)}_\mu \ge C_{14}\|(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu})\|^2_X - C_{10}(\delta,\delta_1)(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}).$$

This inequality and (19)

$$\Xi^{(1)}_\mu \ge C_{15}\|(\frac{\partial w_\nu}{\partial y_2}, w_\nu)\|^2_X - C_{16}(\delta,\delta_1)(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|h(s)g\|^2_{(L^2(\Sigma))^2} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}). \tag{57}$$

From (57), (30), (31) we obtain (18).

Case C. Assume that $s^* \neq 0$. Then if $\delta_1 > 0$ is small enough there exists a constant $C_{17} > 0$ such that

$$|\xi_0\phi_{y_1}(y^*) - (\lambda+2\mu)\xi_1\phi_{y_1}(y^*)|^2 \le \delta_1^2 C_{18}(|\xi_1|^2 + s^2). \tag{58}$$

By (30) there exists $C_{18} > 0$ such that

$$\Xi^{(1)}_{\lambda+2\mu} + C_{18}\big(\|h(s)w_{2,\nu}\|^2_{H^1(\mathcal{Q})} + \|h^3(s)w_{2,\nu}\|^2_{L^2(\mathcal{Q})}\big) \\ \le C_{19}\big(\|\mathbf{P}_{\lambda+2\mu}v_2\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}\big) + \epsilon\big\|\big(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu}\big)\big\|^2_X.$$

Note that $\Xi^{(1)}_{\lambda+2\mu}$ could be written in the form

$$\Xi^{(1)}_{\lambda+2\mu} = \tilde J_1 + \tilde J_2 + \tilde J_3,$$
$$\tilde J_1 = \int_\Sigma |s|(\lambda+2\mu)^2\phi_{y_2}(y^*)|\frac{\partial w_{2,\nu}}{\partial y_2}|^2 + |s|^3(\lambda+2\mu)^2\phi^3_{y_2}(y^*)|w_{2,\nu}|^2\, d\Sigma,$$
$$\tilde J_2 = \mathrm{Re}\int_\Sigma 2|s|(\lambda+2\mu)\frac{\partial w_{2,\nu}}{\partial y_2}\overline{\Big((\lambda+2\mu)\phi_{y_1}(y^*)\frac{\partial w_{2,\nu}}{\partial y_1} - \phi_{y_0}(y^*)\frac{\partial w_{2,\nu}}{\partial y_0}\Big)}\, d\Sigma,$$
$$\tilde J_3 = \int_\Sigma |s|(\lambda+2\mu)\phi_{y_2}(y^*)\big(\xi_0^2 - (\lambda+2\mu)\xi_1^2 - s^2\phi^2_{y_0}(y^*) \\ + s^2(\lambda+2\mu)\phi^2_{y_1}(y^*)\big)|\hat v_{2,\nu}|^2\, d\Sigma. \tag{59}$$

By (58), (**??**) we have

$$|\tilde J_2 + \tilde J_3| \le C_{21}\delta_1\big\|\big(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu}\big)\big\|^2_X. \tag{60}$$

By (60) we obtain from (59) that there exists a constant $C_{22} > 0$ such that

$$\Xi^{(1)}_{\lambda+2\mu} \ge -\epsilon\big\|\big(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu}\big)\big\|^2_X \\ + C_{22}\int_\Sigma h(s)(\lambda+2\mu)^2\phi_{y_2}(y^*)|\frac{\partial w_{2,\nu}}{\partial y_2}|^2 + h^3(s)(\lambda+2\mu)^2\phi^3_{y_2}(y^*)|w_{2,\nu}|^2\, d\Sigma. \tag{61}$$

From (61), inequality (49) for $V^+_\mu(\cdot,0)$ and (19) we obtain the estimate

$$\Xi^{(1)}_{\lambda+2\mu} \ge C_{27}\big\|\big(\frac{\partial w_\nu}{\partial y_2}, w_\nu\big)\big\|^2_X \\ - C_{28}(\delta,\delta_1)\big(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|h(s)g\|^2_{(L^2(\Sigma))^2} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}\big), \tag{62}$$

where $C_{27} > 0$. From (62), (30), (31) we obtain (18). ■

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