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Expansions and eigenfrequencies for damped wave equations

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Abstract

We study eigenfrequencies and propagator expansions for damped wave equations on compact manifolds. In the strongly damped case, the propagator is shown to admit an expansion in terms of the finitely many eigenmodes near the real axis, with an error exponentially decaying in time. In the presence of an elliptic closed geodesic not meeting the support of the damping coefficient, we show that there exists a sequence of eigenfrequencies converging rapidly to the real axis. In the case of Zoll manifolds, the set of all eigenfrequencies is shown to exhibit a cluster structure determined by the Morse index of the closed geodesics and the damping coefficient averaged along the geodesic flow. We then show that the propagator can be expanded in terms of the clusters of eigenfrequencies in the entire spectral band.

1. Introduction and statement of results

The purpose of this note is to describe some recent results [7], [8] concerning the asymptotic distribution of eigenfrequencies and propagator expansions for damped wave equations.

Let M be a compact connected Riemannian manifold of dimension n, and let Δ be the corresponding Laplace-Beltrami operator. In some aspects of control theory (see [10]), one is interested in the long time behaviour of solutions to the Cauchy problem for the wave equation with a dissipative term,

$$\begin{cases} \left(\partial_t^2 - \Delta + 2a(x)\partial_t\right) u = 0, \quad (t,x) \in \mathbf{R} \times M, \\ u|_{t=0} = u_0 \in H^1(M), \\ \partial_t u|_{t=0} = u_1 \in L^2(M). \end{cases} \tag{1.1}$$

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Here the damping coefficient a is a bounded nonnegative function on M, and we shall assume that $a \in C^\infty(M)$. Associated with the evolution problem (1.1) is the solution operator $\mathcal{U}(t) = e^{it\mathcal{A}}$ acting in the Hilbert space $H^1 \times L^2$, where the infinitesimal generator $\mathcal{A}$ is the operator

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -\Delta & 2ia(x) \end{pmatrix} : H^1 \times L^2 \to H^1 \times L^2,$$

with the domain $D(\mathcal{A}) = H^2 \times H^1$. Here $H^s = H^s(M)$ is the standard Sobolev space on M.

The energy of the solution $u(x,t)$,

$$E(u,t) = \frac{1}{2}\int_M \left(|\nabla_x u|^2 + |u_t'|^2\right)\, dx \tag{1.2}$$

is nonincreasing as $t \to \infty$, and relations between the rate of the decay of the energy and the spectrum (i.e. the set of eigenvalues) of $\mathcal{A}$ were studied in [10] and [14]–below we shall recall these results. The asymptotic distribution of the eigenvalues of $\mathcal{A}$ has been studied by J. Sjöstrand [15]. In this note we shall mainly be interested in relations between the long time behaviour of the operator $\mathcal{U}(t)$ and the eigenvalues of $\mathcal{A}$. Here the techniques of [15] will play an essential role.

We notice that $\tau \in \mathbf{C}$ is an eigenvalue of $\mathcal{A}$ precisely when the equation

$$\left(-\Delta + 2i\tau a(x) - \tau^2\right) u(x) = 0 \tag{1.3}$$

has a nonvanishing smooth solution u. The eigenvalues τ will also be called the eigenfrequencies. It is well-known and follows easily from (1.3) that if τ is an eigenvalue, then

$$0 \leq \operatorname{Im}\tau \leq 2\|\, a\,\|_\infty,$$

and when $\tau \neq 0$ is in $\operatorname{Spec}(\mathcal{A})$, then $\operatorname{Im}\tau > 0$. We also have that the spectrum of $\mathcal{A}$ is symmetric with respect to the reflection in the imaginary axis.

We introduce now the *propagator* $U(t) : L^2 \to L^2$ obtained by taking the first component of

$$\mathcal{U}(t)\begin{pmatrix} 0 \\ f \end{pmatrix},$$

where $f \in L^2$. In the self-adjoint case, i.e. when $a \equiv 0$, then $U(t) = i\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}$, and we have a generalized Fourier expansion of the wave group,

$$\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} f(x) = \sum_{\lambda_j^2 \in \operatorname{Spec}(-\Delta)} \frac{\sin \lambda_j t}{\lambda_j}\varphi_j(x),$$
$$-\Delta\varphi_j(x) = \lambda_j^2\varphi_j(x),$$

where the convergence is absolute in the case of smooth data. We are interested in expansions of the propagator $U(t)$ in the case of the nonvanishing damping term. Notice that this problem is closely related to the problem of completeness and summability of generalized eigenfunction expansions of the operator $\mathcal{A}$, see [11].

We have also been to a large extent motivated by recent studies of expansions for propagators on unbounded domains in terms of the resonance states–see [20] for an overview of this work and further references. Our Theorems 1.2 and 1.3 below complement the corresponding results for resonances in a situation when the decay of the states is caused by a direct dissipative mechanism on a compact manifold. In Theorem 1.4 we state a result which, to the best of our knowledge, has no analogue in the theory of resonances.

When $p(x,\xi)=\xi^2$ is the principal symbol of $-\Delta$ defined on T^*M, we introduce the Hamilton vector field H_p and recall that the Hamilton flow $\exp{(tH_p)}:p^{-1}(1)\to p^{-1}(1)$ can be identified with the geodesic flow on the sphere bundle of M. When $T>0$, we put

$$\langle a\rangle_T(x,\xi)=\frac{1}{T}\int_0^T a(\exp{(tH_p)}(x,\xi))\,dt,\quad \text{on } p^{-1}(1).$$

We define also

$$A(T)=\inf_{p^{-1}(1)}\langle a\rangle_T,$$

and we set

$$A(\infty):=\sup_{T>0}A(T)=\lim_{T\to\infty}A(T). \tag{1.4}$$

Here the second equality follows from subadditivity arguments, see [10].

In what follows we shall say that the *Rauch-Taylor condition* holds if there exists a time $T_0>0$ such that any geodesic of length $\geq T_0$ meets the open set $\{x;a(x)>0\}$. In this case it has been established by J. Rauch and M. Taylor [13] that the energy $E(u,t)$ decays exponentially, and this result was extended to the case of manifolds with boundary by C. Bardos, G. Lebeau, and J. Rauch [2].

The starting point is the following result, which is essentially due to [10], and which was also proved in [15].

Theorem 1.1. *Assume that the Rauch-Taylor condition holds. Then $A(\infty)>0$, and for every $\varepsilon>0$, the strip*

$$\Sigma_\varepsilon=\{\tau\ ;0\leq \operatorname{Im}\tau\leq A(\infty)-\varepsilon\}$$

contains at most finitely eigenfrequencies.

Our first result gives an expansion of the propagator $U(t)$ in terms of the eigenfrequencies in the "dynamical" strip Σ_ε, with an error term decaying exponentially in time. It is analogous to a result in scattering theory giving a similar propagator expansion in nontrapping situations.

Theorem 1.2. *Assume that the Rauch-Taylor condition holds. Then for every $\varepsilon>0$ there exists $C=C(\varepsilon)>0$ such that*

$$U(t)f=\sum_{\operatorname{Im}\tau_j<A(\infty)-\varepsilon}\operatorname{Res}\left(e^{it\tau}R(\tau),\tau_j\right)f+E(t)f,\quad f\in L^2,$$

where

$$||\,E(t)\,||_{\mathcal{L}(L^2)}\leq Ce^{-(A(\infty)-\varepsilon)t}.$$

Here $R(\tau)=P(\tau)^{-1}$, and $P(\tau)=-\Delta+2ia\tau-\tau^2$.

We remark that $R(\tau)$ is a meromorphic family of operators on L^2, and that $\mathrm{Res}\,(R(\cdot),\tau_j)$ denotes the residue of this family at the eigenfrequency τ_j.

In order to motivate the second result, we recall that in [10], G. Lebeau has determined the optimal rate of the exponential decay of the energy $E(u,t)$. See also [14]. Namely, if we set

$$\alpha = \sup\{\beta \geq 0;\ \exists B > 0,\ \forall u \in H^1 \times L^2, \forall t \geq 0, E(u,t) \leq Be^{-\beta t}E(u,0)\},$$

and

$$D = \inf\{\mathrm{Im}\,\tau; \tau \in \mathrm{Spec}\,(\mathcal{A}),\ |\tau| > 0\},$$

then it was proved in [10] that

$$\alpha = 2\min(D, A(\infty)). \tag{1.5}$$

In particular, it follows from (1.5) that if there exists a geodesic not intersecting the support of a, then $\alpha = 0$. In [10], Lebeau also gives an explicit example of a surface of revolution where there exists a closed geodesic not meeting $\mathrm{supp}(a)$, while $D > 0$, so that the eigenfrequencies are bounded away from the real axis. In this example the geodesic in question is *hyperbolic*. Our next result shows that when there exists a closed geodesic away from the support of a, which is *elliptic*, then we have a sequence of eigenvalues converging rapidly to the real axis. We remark here that analogous results are known in the theory of resonances, where they are due to P. Stefanov–G. Vodev [17], S. H. Tang–M. Zworski [18], and P. Stefanov [16].

In the formulation of the result we shall use the following terminology: given an elliptic nondegenerate closed geodesic γ, with the eigenvalues of the corresponding linearized Poincaré map P_γ of the form $e^{\pm i\alpha_j}$, $j = 1,\dots n-1$, and $0 < \alpha_j < \pi$, we say that P_γ is N-elementary if

$$\sum_{j=1}^{n-1} k_j\alpha_j \neq 0 \qquad \mathrm{mod}\ 2\pi\mathbf{Z}$$

for each $k = (k_1,\dots k_{n-1}) \in \mathbf{Z}^{n-1}$ with $0 < \sum_{j=1}^{n-1}|k_j| \leq N$.

Theorem 1.3. *Suppose that there exists a closed geodesic $\gamma : [0,T] \to p^{-1}(1)$, which is nondegenerate, elliptic, does not meet $\mathrm{supp}(a)$, and such that the associated linearized Poincaré map is 4-elementary. Then there exists a sequence (τ_j) of eigenfrequencies such that $|\mathrm{Re}\,\tau_j| \to \infty$, and $0 < \mathrm{Im}\,\tau_j \leq C_N\,|\mathrm{Re}\,\tau_j|^{-N}$ for any $N \in \mathbf{N}$.*

Remark. It is known that under the assumptions on the geodesic γ in Theorem 1.3, one can construct quasimodes for the Laplace-Beltrami operator, concentrated near γ, see [12] and [6]. The construction of the latter paper gives quasimodes of positive mass with exponentially small errors.

Remark. In the situation of Theorem 1.3, by adapting the methods of Stefanov [16] to our setting, one could also obtain a sharp lower bound for the counting function $N(r)$ for the eigenfrequencies τ_j,

$$N(r) = \sharp\{\tau_j; 0 \leq \mathrm{Re}\,\tau_j \leq r, 0 < \mathrm{Im}\,\tau_j \leq |\mathrm{Re}\,\tau_j|^{-N}\} \geq r^n/\mathcal{O}_N(1),$$

for any $N \in \mathbf{N}$. The sharpness of this bound is confirmed by the general Weyl law for the distribution of the real parts of the eigenfrequencies: the number of eigenfrequencies τ with $0 \leq \operatorname{Re}\tau \leq r$ is equal to

$$\left(\frac{r}{2\pi}\right)^n \iint_{p^{-1}([0,1])} dx\, d\xi + \mathcal{O}(r^{n-1}), \quad r \to \infty.$$

This result is due to A. Markus and V. Matsaev, see [11], and also [15]. Moreover, the results of [11] indicate that the generalized eigenvectors of the operator $\mathcal{A}$ form a Riesz basis in the energy space $H^1 \times L^2$ in dimension one only.

In our last theorem we present a result where a propagator expansion in terms of the eigenfrequencies in the entire spectral band is valid in any dimension. This is the case when the geodesic flow is periodic, and the expansion then holds in terms of the disjoint clusters of eigenfrequencies.

Theorem 1.4. *Let M be a Zoll manifold,* i.e. *assume that the geodesic flow is periodic,* $\exp(\pi H_p)(x,\xi) = (x,\xi)$ *on $p^{-1}(1)$. Then*

1. There exists $C > 0$ such that all eigenfrequencies τ, with $\operatorname{Re}\tau > 0$, except for finitely many, are contained in the union of the rectangles

$$I_k = \left[k + \frac{\alpha}{4} - \frac{C}{k}, k + \frac{\alpha}{4} + \frac{C}{k}\right] + i[0, \mathcal{O}(1)], \quad k = 1, 2, \ldots,$$

where α is the common Maslov index of the closed H_p-trajectories in $p^{-1}(1)$. Moreover, if τ is an eigenfrequency with $\operatorname{Re}\tau \sim k$, then we have

$$\inf_{p^{-1}(1)} \langle a \rangle_\pi - \mathcal{O}\left(\frac{1}{k}\right) \leq \operatorname{Im}\tau \leq \sup_{p^{-1}(1)} \langle a \rangle_\pi + \mathcal{O}\left(\frac{1}{k}\right), \quad k \to \infty.$$

2. When $f \in H^2(M)$, then

$$U(t)f = \sum_{k=-\infty}^{\infty} \left(\sum_{\tau_j \in I_k} \operatorname{Res}\left(e^{it\tau} R(\tau), \tau_j\right) f \right), \quad t > 0.$$

The outer sum converges absolutely in L^2 for each fixed $t > 0$.

Remark. The first part of Theorem 1.4 is a well-known and in general optimal result in the self-adjoint case — see Chapter 29 of [9].

Remark. In the second part of Theorem 1.4 we have given an expansion result which is valid only in a very special situation of Zoll manifolds. It is however quite likely that weaker expansions hold on general manifolds, and it is our intention to look further into this question. The methods of [5] should prove useful in establishing results of this kind, such as time-dependent expansions. Some new features are present in our case due to the fact that there is no underlying self-adjoint operator.

2. An outline of the proofs of Theorems 1.2–1.4

We shall begin by sketching the proofs of Theorem 1.2 and 1.4. The proof of Theorem 1.3 is briefly described at the end of the section.

The starting point is the following well-known relation between the operator $\mathcal{U}(t)$ and the resolvent of $\mathcal{A}$,

$$e^{it\mathcal{A}}x = \frac{1}{2\pi i}\int_{-\infty-i\alpha}^{\infty-i\alpha} e^{it\tau}\left(\tau-\mathcal{A}\right)^{-1}x\,d\tau,\quad t>0,\quad \alpha>0. \tag{2.1}$$

and here we shall assume for simplicity that $x\in D(\mathcal{A}^2)$. Specializing to the propagator $U(t)$ we get that

$$U(t)f = \frac{i}{2\pi}\int_{-\infty-i\alpha}^{\infty-i\alpha} e^{it\tau}R(\tau)f\,d\tau,\quad f\in H^2(M). \tag{2.2}$$

The idea is now to perform a contour deformation in the spectral decomposition formula (2.2). The expansions then come from a residue calculation, and the main issue becomes estimating the resolvent $R(\tau)$ in the spectral strip.

The key point in the proof of Theorem 1.2 is the following result.

Proposition 2.1. *Assume that the Rauch-Taylor condition holds. Then for every $\varepsilon>0$ there exists $C=C(\varepsilon)>0$ such that if $\operatorname{Im}\tau\in[-1,A(\infty)-\varepsilon]$ and $|\operatorname{Re}\tau|\geq C$ then*

$$\|\,R(\tau)\,\|_{\mathcal{L}(L^2)}\leq \frac{C}{|\operatorname{Re}\tau|}. \tag{2.3}$$

Proof. Following [15], we begin by performing a semiclassical reduction. Write $\tau=\sqrt{z}/h$, $0<h\ll 1$, where z belongs to the fixed domain $[\alpha,\beta]+i[-\gamma,\gamma]$, with $0<\alpha<1<\beta<\infty$ and $\gamma>0$. We are then led to study the operator

$$\mathcal{P}=\mathcal{P}(h)=P+ihQ(z),$$

where $P=-h^2\Delta$, and $Q(z)=2a(x)\sqrt{z}$. If we introduce the semiclassical resolvent $R(z,h)=(\mathcal{P}-z)^{-1}$, we see that we have to prove the following statement: for every $\varepsilon>0$ there exists $C>0$ and $h_0>0$ such that for all $h\in(0,h_0]$ and all $z\in\Omega(h)=\{\operatorname{Re}z\in[\alpha,\beta],-2\sqrt{\operatorname{Re}z}\leq\operatorname{Im}z/h\leq 2\sqrt{\operatorname{Re}z}\,(A(\infty)-\varepsilon)\}$, it is true that

$$\|\,R(z,h)\,\|_{\mathcal{L}(L^2)}\leq\frac{C}{h}. \tag{2.4}$$

The proof of (2.4) is now a contradiction argument. Assuming that (2.4) does not hold, we see that there exists a sequence $(h_j)_{j=1}^{\infty}$ of positive numbers with $h_j\to 0$ as $j\to\infty$, a sequence $(u_j)_{j=1}^{\infty}\in L^2$ with $\|\,u_j\,\|_{L^2}=1$, and a sequence $(z_j)_{j=1}^{\infty}$ with $z_j\in\Omega(h_j)$, such that

$$(\mathcal{P}(h_j)-z_j)\,u_j=o(h_j)\quad\text{in }L^2. \tag{2.5}$$

Passing to a subsequence, if necessary, we may assume that $\operatorname{Re}z_j\to e\in[\alpha,\beta]$, and to simplify notation we may take $e=1$. Then $\operatorname{Im}z_j/h_j\to f\in[-2,2\,(A(\infty)-\varepsilon)]$ as $j\to\infty$. We then get from (2.5) that

$$(P(h_j)+ih_jQ(\operatorname{Re}z_j)-z_j)\,u_j=o(h_j)\quad\text{in }L^2. \tag{2.6}$$

Let now ν be a semiclassical measure associated with the bounded sequence $(u_j)_{j=1}^{\infty}$ — see [3] for an exposition of the theory of semiclassical and microlocal defect measures. Here we recall that up to an extraction of a subsequence, the positive (Radon) measure ν on T^*M is characterized by the following property:

$$\lim_{j\to\infty} (\mathrm{Op}_{h_j}(q)u_j|u_j) = \int q(x,\xi)\, d\nu(x,\xi), \quad q \in C_0^\infty(T^*M). \tag{2.7}$$

Here $\mathrm{Op}_h(q)$ is an h-pseudodifferential operator with principal symbol q. Using (2.6) it is then easy to see that ν is supported by $p^{-1}(1)$, and the total mass of ν is one. Moreover, we also find that ν satisfies the following transport equation,

$$\int \Big(H_p g + 4\,(a - f/2)\, g \Big)\, d\nu = 0, \quad g \in C_0^\infty(T^*M). \tag{2.8}$$

Notice that in the case when $a = f = 0$, (2.8) merely says that ν is invariant under the geodesic flow. In the general case, integrating (2.8) we get that when $E \subset p^{-1}(1)$ is a Borel subset of $p^{-1}(1)$ such that $\nu(E) \neq 0$, then

$$\nu(\exp(tH_p)(E)) = \int_E \exp\left(4t(\langle a\rangle_t - f/2)\right) d\nu. \tag{2.9}$$

Now the right-hand side of (2.9) is, for $t > 0$, greater than or equal to

$$\exp\left(4t(A(t) - A(\infty) + \varepsilon)\right)\nu(E), \quad t > 0,$$

and since $A(t) \to A(\infty)$ as $t \to \infty$, and $\varepsilon > 0$ is fixed, we see the the right-hand side of (2.9) is unbounded as $t \to \infty$, whereas the left-hand side is uniformly bounded in t since $\exp(tH_p)(E) \subset p^{-1}(1)$ for all t, and ν is a Radon measure. This contradiction completes the proof. $\square$

Remark. The idea of arguing by contradiction and using semiclassical (or microlocal defect) measures when proving energy and resolvent estimates is due to G. Lebeau in [10]. See also [1] and [4] for further recent applications of this approach.

We now come to discuss the proof of Theorem 1.4. Here, too, the main point is resolvent estimates, which are now valid in the gaps between the spectral clusters. We have the following result.

Proposition 2.2. *Let M be a manifold with periodic geodesic flow. Then it is true that*

$$\mathrm{Spec}\,(\mathcal{A}) \cap \{\tau; \mathrm{Re}\,\tau > 0\} \subset \cup_{k=1}^{\infty} I_k,$$

where

$$I_k = \left[k + \frac{\alpha}{4} - \frac{\mathcal{O}(1)}{k}, k + \frac{\alpha}{4} + \frac{\mathcal{O}(1)}{k}\right] + i[0, \mathcal{O}(1)], \quad k = 1, 2, \ldots.$$

Furthermore, there exists $C > 0$ such that when $\mathrm{Im}\,\tau \in [-1, 2||\,a\,||_{L^\infty} + 1]$ we have

$$||\,R(\tau)\,||_{\mathcal{L}(L^2)} \le \frac{C}{|\mathrm{Re}\,\tau|}, \quad |\mathrm{Re}\,\tau| \ge C, \quad \mathrm{dist}\left(|\mathrm{Re}\,\tau|\,, \mathrm{Spec}\,(\sqrt{-\Delta})\right) \ge \frac{1}{C}. \tag{2.10}$$

With this proposition at hand, it is straightforward to prove Theorem 1.4 by deforming the contour in (2.2) through the spectral gaps.

Proof. As before, we work in the semiclassical setting, and then we see that in order to prove (2.10) we have to show the existence of $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0]$ we have that

$$|| R(z,h) ||_{\mathcal{L}(L^2)} \leq \frac{C}{h}, \tag{2.11}$$

when $\operatorname{Im} z = \mathcal{O}(h)$ and $\operatorname{dist}(\operatorname{Re} z, \operatorname{Spec}(P)) \geq h/C$. We remark here that the spectrum of P in $[\alpha, \beta]$ consists of clusters which are of width $\mathcal{O}(h^2)$ and are separated from each other by a distance which is of the order of h. The proof of (2.11) is based on a version of the averaging method, which has a long tradition in the study of operators with periodic Hamilton flows–see [19] and Section 2 of [15]. We shall describe here a simplified argument which is sufficient to establish (2.11), while it does not allow one to get the optimal width of the clusters I_k, $\mathcal{O}(1/k)$. We refer to [7] for more details on this matter.

Conjugating the operator $\mathcal{P}$ by an elliptic self-adjoint h-pseudodifferential operator $A = \operatorname{Op}_h(e^g)$, with $g \in S^0(T^*M)$, we get

$$A^{-1}(P + ihQ(z))A = P + ih\operatorname{Op}_h(q(\operatorname{Re} z) - H_p g) + h^2 R,$$

where $q(\operatorname{Re} z) = 2a\sqrt{\operatorname{Re} z}$ and R is a bounded h-pseudodifferential operator. The idea is now to choose g (depending also on $\operatorname{Re} z$) so that

$$q - H_p g = \langle q \rangle_T \quad \text{on} \quad p^{-1}(\operatorname{Re} z). \tag{2.12}$$

Here we write $q = q(\operatorname{Re} z)$, and $T = T(\operatorname{Re} z)$ is the common period of the closed H_p-trajectories in $p^{-1}(\operatorname{Re} z)$. To solve (2.12) we choose $g \in S^{-1}$ such that

$$g(x,\xi) = \frac{1}{T}\int_0^T tq(\exp(tH_p)(x,\xi))\, dt \quad \text{on} \quad p^{-1}(\operatorname{Re} z).$$

Thus, after a conjugation of $\mathcal{P}$ by a bounded invertible h-pseudodifferential operator, we may work with the operator

$$\widetilde{\mathcal{P}} = P + ih\widehat{Q} + h^2 R,$$

where $\widehat{Q} = \operatorname{Op}_h(\hat{q})$, with $\hat{q} = \langle q \rangle_T$ on $p^{-1}(\operatorname{Re} z)$. It is then possible to estimate the resolvent of $\widetilde{\mathcal{P}}$, and let us just mention that the main point is that we are now able to handle the commutator term $ih[P, \widehat{Q}]$. Indeed, the fact that p and $\hat{q}$ Poisson commute on $p^{-1}(\operatorname{Re} z)$ shows that

$$ih[P, \widehat{Q}] = h^2 K(P - \operatorname{Re} z) + h^3 T, \tag{2.13}$$

for some bounded h-pseudodifferential operators K and T. Using (2.13) it is then straightforward to show that

$$|| (\widetilde{\mathcal{P}} - z)u || \geq h/C || u ||,$$

when $\operatorname{Im} z = \mathcal{O}(h)$, $\operatorname{dist}(\operatorname{Re} z, \operatorname{Spec}(P)) \geq h/C$, $u \in H^2$, and $h > 0$ is sufficiently small. A similar estimate is then true for the operator $\mathcal{P} - z$, and this completes the proof of (2.11). □

We shall finally mention how the proof of Theorem 1.3 proceeds. Here one needs general a priori bounds of the resolvent $R(\tau)$, and as usual it is more convenient to work in the semiclassical setting. We then have the following result.

Proposition 2.3. *For every $C > 0$ and $E \in [\alpha + 1/\mathcal{O}(1), \beta - 1/\mathcal{O}(1)]$ there exists $A > 0$ and $h_0 > 0$ such that for $0 < h < h_0$ we have*

$$|| R(z,h) ||_{\mathcal{L}(L^2)} \leq \frac{A}{h} \exp\left(A h^{1-n} \log \frac{h}{g(h)} \right), \tag{2.14}$$

for $|z - E| < Ch$, $|z - z_j| \geq g(h)$, and $0 < g(h) \ll h$. Here z_j, $j = 1, \ldots N$ are the eigenvalues of $\mathcal{P}$ in the set $\{z; |z - E| < Ch\}$.

The proof of this result is based on the idea of [11] and [15] to use a finite rank perturbation of trace class norm $\mathcal{O}(h^{2-n})$ to create a gap in the spectrum of P of width $\mathcal{O}(h)$ around E.

With Proposition 2.3 available, we are in the position to apply the general argument of [18] relating the quasimodes of the operator $\mathcal{P}$, localized near the geodesic away from the damping region, to the existence of eigenfrequencies near the real axis.

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