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Resolvent estimates and the decay of the solution to the wave equation with potential

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Abstract

We prove a weighted L^∞ estimate for the solution to the linear wave equation with a smooth positive time independent potential. The proof is based on application of generalized Fourier transform for the perturbed Laplace operator and a finite dependence domain argument. We apply this estimate to prove the existence of global small data solution to supercritical semilinear wave equations with potential.

In this work we study the following Cauchy problem

$$\begin{aligned} \partial_t^2 u + Au &= F_p(u), \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) &= u_1(x), \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^3$ and $F_p(u)$ behaves like $|u|^p$ for some $p > 1$ and A is a self-adjoint non-negative operator in $L^2(\mathbf{R}^3)$.

In the case $A = -\Delta$ the classical results due to F.John in [6] show that (1) has a global solution when $p > p_0(3) = 1 + \sqrt{2}$ and the initial data u_0, u_1 have compact support and small Sobolev norms; this critical value $p_0 = 1 + \sqrt{2}$. The exponent $p_0 = 1+\sqrt{2}$ is critical in the sense that solutions in general blow up when $1 < p < p_0$ ([6]).

We shall consider a potential type perturbation of the flat Laplace operator

$$A = -\Delta + V(x),$$

where $V(x)$ is a smooth non-negative potential with compact support:

$$V(x) \geq 0 \quad \text{for } x \in \mathbf{R}^3. \tag{2}$$

Our main goal will be to show global existence of small solutions to the semilinear equation

$$\partial_t^2 u - \Delta u + V(x)u = F_p(u) \quad \text{in } [0,\infty) \times \mathbf{R}^3, \tag{3}$$

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when $p > p_0 = 1 + \sqrt{2}$.

Our essential tool will be a suitable L^∞ weighted estimate for the linear equation

$$\partial_t^2 u - \Delta u + V(x)u = F \quad \text{in } [0,\infty) \times \mathbf{R}^3. \tag{4}$$

We recall the following well known estimate for the solution $u = L_0(F)$ to the wave equation $u_{tt} - \Delta u = f$with zero initial data:

$$\|\tau_+\tau_-^\lambda L_0(F)\|_{L^\infty([0,T]\times\mathbf{R}^3)} \leq C_1\|\tau_+^\mu\tau_- F\|_{L^\infty([0,T]\times\mathbf{R}^3)}, \tag{5}$$

where the weights $\tau_\pm$ are defined by

$$\tau_\pm = 1 + |t \pm |x||,$$

$T > 0$ is arbitrary, and C_1 is a positive constant depending only on the positive parameters λ and μ satisfying

$$\lambda < 1, \quad \mu > 2 + \lambda. \tag{6}$$

From the results in ([3]) we can assert that this estimate is fulfilled in the case of short range potential with compact support. Moreover we have the following variant of (5): if $m > 2$ and $0 < \lambda \leq m - 1$, then

$$\|\tau_+\tau_-^\lambda L_0(F)\|_{L^\infty([0,T]\times\mathbf{R}^3)} \leq C_2\|\langle x\rangle^m\tau_+\tau_-^\lambda F\|_{L^\infty([0,T]\times\mathbf{R}^3)}, \tag{7}$$

where $\langle x\rangle = \sqrt{1+|x|^2}$ and C_2 is a positive constant depending only on λ and m.

When $V(x)$ is small, no sign condition is needed. In [12] W.Strauss and K. Tsutaya proved the existence of small data solutions in the supercritical case ($p > p_0 = 1+\sqrt{2}$) for the semilinear wave equation with small potential of arbitrary sign, and also blow-up in the subcritical case. Actually, it is not difficult to apply the estimate (5) to this case. Indeed, let $u = L(F)$ be the solution to the perturbed wave equation

$$\partial_t^2 u - \Delta u + Vu = F \quad \text{in } [0,\infty) \times \mathbf{R}^3, \tag{8}$$

$$u(0,x) = 0, \quad \partial_t u(0,x) = 0 \quad \text{in } \mathbf{R}^3. \tag{9}$$

If the potential V is small, we can rewrite the above equation in the form

$$\partial_t^2 u - \Delta u = -V(x)u + F. \tag{10}$$

The smallness assumption on $V(x)$, $x \in \mathbf{R}^3$ means that

$$|V(x)| \leq \delta_0(1+|x|)^{-m},$$

where $\delta_0 > 0$ is sufficiently small and $m > 2$. More precisely, we choose δ_0 so small that $C_2\delta_0 \leq 1/2$ holds. Here C_2 is the constant in (7).

Applying estimates (5), (7) to (10) and using the smallness of V we easily obtain the estimate

$$\|\tau_+\tau_-^\lambda L(F)\|_{L^\infty([0,T]\times\mathbf{R}^3)} \leq 2C_1\|\tau_+^\mu\tau_- F\|_{L^\infty([0,T]\times\mathbf{R}^3)}, \tag{11}$$

provided $0 < \lambda < 1$ and $\mu > 2 + \lambda$. Here C_1 is the constant in (5).

The main goal of this work is to relax the smallness assumption on V and to show that (11) is still true, when V is arbitrary non-negative smooth function with compact support.

On the basis of this estimate we prove the existence of a global solution for the supercritical case $p > p_0 = 1 + \sqrt{2}$.

Before stating the main a priori estimate we show the existence of a classical solution $u = L(F)$, provided $F \in C^2$. More precisely, we have the following

Lemma 1 *Assume $V \in C_0^4(\mathbf{R}^3)$ and $F \in C(\mathbf{R}; C^2(\mathbf{R}^3))$ satisfies*

$$suppF \subset K_M := \{(t,x) \in \mathbf{R}^4 : |x| \leq |t| + M\} \tag{12}$$

with some $M > 0$. Then there exists a unique solution $u \in C^2(\mathbf{R}^4)$ to the equation $\partial_t^2 u - \Delta u + Vu = F$ with zero initial data so that

$$supp\ u \subset K_M. \tag{13}$$

The main tool to establish existence of small data solution is the following a priori estimate for the solution $u = L(F)$ to (8) and (9).

Theorem 1 *Assume $V \in C_0^\infty(\mathbf{R}^3)$ is non - negative and*

$$F \in C^2([0,T] \times \mathbf{R}^3)$$

satisfies

$$suppF \cap \{0 \leq t \leq T\} \subset K(M,T) = \{(t,x) \in \mathbf{R}^4 : |x| \leq t + M, 0 \leq t \leq T\} \tag{14}$$

for $T > 0$ and $M > 0$. Then for any λ and μ with

$$0 < \lambda < 1, \quad \mu > 2 + \lambda, \tag{15}$$

we have

$$\|\tau_+ \tau_-^\lambda L(F)\|_{L^\infty([0,T]\times\mathbf{R}^3)} \leq C_3 \|\tau_+^\mu \tau_- F\|_{L^\infty([0,T]\times\mathbf{R}^3)}. \tag{16}$$

Here $C_3 = C_3(\lambda, \mu, V)$ is a positive constant independent of T.

Moreover, we have the following estimate, corresponding to (7).

Theorem 2 *Let the assumptions of Theorem 1.1 be satisfied. If $0 < \lambda < 1$ and $m > 2$, then we have*

$$\|\tau_+ \tau_-^\lambda L(F)\|_{L^\infty([0,T]\times\mathbf{R}^3)} \leq C_4 \ln^3(2+T) \|\langle x \rangle^m \tau_+ \tau_-^\lambda F\|_{L^\infty([0,T]\times\mathbf{R}^3)}. \tag{17}$$

Here $C_4 = C_4(\lambda, m, V)$ is a positive constant independent of T.

Since no explicit representation of the fundamental solution is available, in order to prove the above theorem we express the solution to (4) (with zero initial data) by

$$u(t,x) = \int\int K(t-s,x,y)F(s,y)dyds, \tag{18}$$

where $K(t,x,y)$ is the kernel of the operator

$$\frac{\sin(t\sqrt{-\Delta+V})}{\sqrt{-\Delta+V}}. \tag{19}$$

We have the following (formal) oscillatory integral

$$K(t,x,y) = \sum_{\pm}\int \mathrm{e}^{i(x-y)\xi \pm it|\xi|} a_{\pm}(x,\xi)d\xi. \tag{20}$$

This construction, typical for microlocal analysis, is quite simple, when t, x are bounded. Unfortunately, for large t, x some anomalies in the behaviour of the amplitude $a_{\pm}(x,\xi)$ arise; in particular, the amplitude is not a classical symbol. Such non-standard symbols were studied by H. Isozaki (see [5]). To overcome this difficulty we introduce a new class of (degenerate) symbols.

Definition 1 *Let $A(x,\xi) \in C(\mathbf{R}^3_x; C^\infty(\mathbf{R}^3_\xi))$ and $m \in \mathbf{R}$. We say $A(x,\xi) \in \Sigma^m$ if and only if for any non-negative integer k, there is a positive number $C = C(k)$ such that*

$$|\partial_\xi^\alpha A(x,\xi)| \le C\langle\xi\rangle^m \quad \text{for } 0 \le |\alpha| \le k.$$

We use only derivatives with respect to ξ, since we prove an uniform L^∞ bound of the solution by estimating the integral (20). A direct integration by parts argument will not assure convergence in (20), since $a_{\pm}(x,\xi) \in \Sigma^{-1}$ and any ξ derivative of $a_{\pm}(x,\xi)$ can not improve the decay in ξ ; hence, the above class of symbols is quite delicate and needs some new specific tools in order to establish the desired decay.

Our main idea is to apply weighted resolvent estimates that have been intensively studied for the case of potential perturbation of Schrödinger equation (see [4], [1]). The result of H. Isozaki in [5] shows that the weighted resolvent

$$\langle x\rangle^{-s}(z+\Delta-V)^{-1}\langle x\rangle^{-s},$$

is a bounded operator in $L^2(\mathbf{R}^3)$ for any real $s > 1/2$ and for any complex z with $\mathrm{Re}z > 0$. Moreover, the L^2 norm of the operator

$$\langle x\rangle^{-s}(z+\Delta-V)^{-l}\langle x\rangle^{-s}, \quad l = 1,2,\cdots$$

is bounded from above by $C/z^{l/2}$ for $s > 1/2 + l - 1$(See also C. Morawetz in [10] for $l = 1$ and [9]). The corresponding estimate, obtained in [3] plays a crucial role in the proof of the fact that the amplitude has high frequency part in the class of symbols introduced in Definition 1. Moreover, we have a corresponding asymptotic symbol expansion

$$a_{\pm}(x,\xi) \sim a_{-1}(x,\xi) + a_{-2}(x,\xi) + \cdots$$

Here the symbols $a_k(x,\xi) \in \Sigma^k, k = -1, -2, \cdots$, have explicit representation involving the unperturbed resolvent $(z + \Delta)^{-1}$.

The second point in the proof of (16) is a decomposition of the kernel $K(t, x, y)$ of the operator

$$\frac{\sin(t\sqrt{-\Delta + V})}{\sqrt{-\Delta + V}}.$$

Namely, we have

$$K(t, x, y) = K_{free}(t, x, y) + K_{pert}(t, x, y),$$

where $K_{free}(t, x, y)$ is the fundamental solution of the free wave equation, while $K_{pert}(t, x, y)$ is an oscillatory integral with symbols in $\Sigma^k, k \leq -1$. Using the asymptotic expansion of Theorem 2.1 in [3] and stationary phase method, we obtain a pointwise estimate of the kernel $K_{pert}(t, x, y)$ for points far away from the cone $\{t = |x - y|\}$. From this estimate and standard finite dependence domain argument we arrive at (16).

With our a priori estimate in hand, we can now consider the global existence of small solutions to the supercritical perturbed equation (3). More precisely, we assume that $F_p \in C^2(\mathbf{R})$ satisfies

$$F_p(0) = F_p^{'}(0) = F_p^{''}(0) = 0, \tag{21}$$

and that there is a positive constant $\Lambda > 0$ such that for $|u| \leq 1$, $|u^*| \leq 1$

$$|F_p^{''}(u) - F_p^{''}(u^*)| \leq \Lambda G_p(u, u^*) \tag{22}$$

where

$$G_p(u, u^*) = \begin{cases} |u - u^*|^{p-2} & \text{if} \quad 2 < p \leq 3 \\ |u - u^*|(|u| + |u^*|)^{p-3} & \text{if} \quad\quad p > 3 \end{cases}.$$

Typical examples are $F_p(u) = |u|^p$ and $F_p(u) = |u|^{p-1}u$ with $p > 2$.

Further, we assume that the initial data,

$$u(0, x) = \varepsilon f(x), \;\; \partial_t u(0, x) = \varepsilon g(x) \tag{23}$$

satisfy $f \in C^3(\mathbf{R}^3)$, $g \in C^2(\mathbf{R}^3)$ and

$$\text{supp} f, \;\; \text{supp} g \subset \{x \in \mathbf{R}^3 : \;\; |x| \leq M\} \tag{24}$$

with some $M > 0$.

The next step is to introduce a function space X of solutions to (3); given any positive number T this space is

$$X = X(T) = \{u \in C([0, T]; C^2(\mathbf{R}^3)) : \sum_{|\alpha| \leq 2} \|\partial_x^\alpha u\|_T < +\infty\}, \tag{25}$$

where

$$\|u\|_T = \|\tau_+ \tau_-^\lambda u\|_{L^\infty([0,T] \times \mathbf{R}^3)},$$

and λ is a real number satisfying

$$1/p \leq \lambda < p-2, \quad \lambda < 1. \tag{26}$$

Note that the interval $(1/p, p-2)$ is non - empty if and only if $p > p_0$.

The norm in X is obviously

$$\sum_{|\alpha|\leq 2} \|\partial_x^\alpha u\|_T.$$

The main existence result is the following.

Theorem 3 *Let $V(x)$ be a smooth compact supported non - negative potential and let $F_p \in C^2(\mathbf{R})$ satisfy (21), (22) with $p > p_0 = 1+\sqrt{2}$. Assume that (24) is fulfilled for $f \in C^3(\mathbf{R}^3)$, $g \in C^2(\mathbf{R}^3)$. Then there exist $\varepsilon_0 > 0$ and $C > 0$, such that for any $T > 0$ and ε with $0 < \varepsilon \leq \varepsilon_0$, the problem (3), (23) has a unique classical solution*

$$u \in X(T) \cap C^2([0,T] \times \mathbf{R}^3)$$

such that u is supported in K_M and the inequalities

$$\|u\|_T \leq 2C\varepsilon, \qquad \sum_{1\leq|\alpha|\leq 2} \|\partial_x^\alpha u\|_T \leq C\varepsilon \ln^6(2+T)$$

hold for $0 < \varepsilon \leq \varepsilon_0$ with some constant $C > 0$ independent of $T > 0$.

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