Journées Équations aux dérivées partielles

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Journées Équations aux dérivées partielles (2000), p. 1-12
<http://www.numdam.org/item?id=JEDP_2000____A19_0>

Journées *Équations aux dérivées partielles*
Nantes, 5–9 juin 2000
GDR 1151 (CNRS)

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Abstract

We survey some recent results for the KP-II equation. We also give an idea for treating the "bad frequency interactions" of the bilinear estimates in the Fourier transform restriction spaces related to the KP-I equation.

1. Introduction

The Kadomtsev-Petviashvili (KP) equations occur naturally in many physical contexts as "universal" models for the propagation of weakly nonlinear dispersive long waves which are essentially one-directional, with weak transverse effects. The KP equations are two dimensional extensions of the Korteweg-de Vries (KdV) equation, which is the first known soliton equation. The soliton structure of the KdV equation is not broken down by the transverse perturbation and therefore the KP equations are two dimensional soliton equations. Thus the inverse scattering technique could be applied to the Cauchy problem associated to the KP equations under appropriate decay assumptions on the initial data. Our goal here is to study the KP equations with harmonic analysis techniques developed in the context of KdV and NLS equations mainly by J. Bourgain, C. E. Kenig, G. Ponce, L. Vega, in order to obtain local or global well-posedness results for the initial value problem associated to the KP equations. Consider the Cauchy problem associated to the KP equations

$$(u_t + u_{xxx} + uu_x)_x \mp u_{yy} = 0, \quad u(0,x,y) = \phi(x,y). \tag{1}$$

The initial data ϕ is supposed to belong to a low order Sobolev type space and x and y are on the real line or the circle. The KP-I equation corresponds to sign $-$ in (1), while the KP-II equation to sign $+$. The role of the sign is transparent when considering (1) in the context of water waves. The KP-II equation occurs when the surface tension is small or absent (Bond number $< 1/3$). The KP-I equation corresponds to the case when the surface tension dominates as in very shallow water (Bond number $> 1/3$). In the critical case when the Bond number is near to $1/3$ higher order terms should be taken into account in (1) (the fifth order

MSC 2000 : 35Q53, 35Q51, 35A07.
Keywords : Fourier transform restriction method, KP equations, Cauchy problem.

KP equations should be considered). There is a big difference between the KP-I (the focusing case) and KP-II (the defocusing case) equations from mathematical point of view. When studying the KP models by harmonic analysis techniques there are satisfactory local or global well-posedness results for the KP-II equation, obtained in the last years starting from the work of J. Bourgain [2]. Unfortunately at the present moment the initial value problem for the KP-I equation is not so well-understood. Here we shall give an idea for the proof of a bilinear estimate in the Fourier transform restriction spaces associated to the KP-I equation[1]. Since the KP models are infinite dimensional integrable Hamiltonian systems there exists an infinite number of conserved by the time evolution quantities. These conservation laws may be useful for obtaining global solutions, when they have positively defined quadratic parts, i.e. a Sobolev type norm is controlled trough the flow. In the case of the KP-II equation the quadratic part of the conservation laws is not positively defined and the only "globalizing" conservation law seems to be the L^2 norm. On the other hand the conservation laws for the KP-I equation have "good" signs and hence one may expect that higher Sobolev norms are controlled through the flow. In the periodic setting global well-posedness for the KP-I equation using the "good" signs of the conservation laws is obtained in [19]. In the real case an additional difficulty appears, since the conservation laws contain anti-derivatives and hence their justification is not a trivial issue.

Notations. We denote by $\widehat{\cdot}$ or $\mathcal{F}$ the Fourier transform and by $\mathcal{F}^{-1}$ the inverse transform. $\|\cdot\|_{L^p}$ denotes the norm in the Lebesgue space L^p. $A \sim B$ means that there exists a constant $c \geq 1$ such that $\frac{1}{c}|A| \leq |B| \leq c|A|$. $A \approx B$ means that $\frac{1}{2}|A| \leq |B| \leq 2|A|$. For any positive A and B the notation $A \lesssim B$ (resp. $A \gtrsim B$) means that there exists a positive constant c such that $A \leq cB$ (resp. $A \geq cB$). The notation $a\pm$ means $a \pm \varepsilon$ for arbitrary small $\varepsilon > 0$. By mes(A) or $|A|$ we denote the Lebesgue measure of a set A.

2. The KP-II equation

Consider the Cauchy problem for the KP-II equation

$$(u_t + u_{xxx} + uu_x)_x + u_{yy} = 0, \quad u(0,x,y) = \phi(x,y). \tag{2}$$

Using energy methods one can obtain local well-posedness results for sufficiently smooth initial data ϕ. The regularity assumptions on the data are in order to control the L^∞ norm of the solution, which makes the L^2 conservation law useless for proving global well-posedness. Unfortunately most of the local smoothing properties of the KdV equations are lost when considering transverse effects. For example the sharp version of the Kato smoothing effect for the linearized KdV equation is used in order to gain regularity for the KdV equation in [11, 12]. The point is that one controls the $L^\infty_x(L^2_t)$ norm of the gradient of the solution to the linearized KdV equation by the L^2_x norm of the initial data. In order to prove this (sharp) version of Kato smoothing effect one changes the role of the time variable t and the space variable x in the oscillatory integral representing the solution of the linearized KdV equation.

[1]Some very recent progress in the context of the KP-I equation is done in [4, 5].

Then an application of Plancherel identity provides the needed bound. If we try to use the above argument for the linearized KP equations we should choose one of the space variables x or y to be changed with the time variable and hence we lose the symmetry of the estimate.

In [2] local (and hence global) well-posedness of (2) for data in $L^2(\mathbb{R}^2)$ or $L^2(\mathbb{T}^2)$ is obtained. The proof uses the methods based on analysis of multiple Fourier series, first introduced in [1] in the context of the NLS and KdV equation. An essential ingredient is a L^2 convolution estimate. This estimate can be regarded as a localized in the frequency space version of the $L^4 - L^2$ Strichartz inequality for the periodic KP-II equation. The proof is typical for the periodic setting but can be performed in the continuous case too. When the data is defined on $\mathbb{R}^2$ a global version of the Strichartz inequality for (2) holds (cf. [17] [2]). The Strichartz estimates can be easily injected into the framework of the Fourier transform restriction spaces associated to the KP-II equation. Combining that estimates with some simple calculus techniques due to C. E. Kenig, G. Ponce and L. Vega one can obtain local well-posedness results for (2) with data below L^2 (cf. [21]).

Now we introduce the functional spaces where the initial data is expected to belong. Let $H^{s_1,s_2}_{x,y}(\mathbb{R}^2)$ be an anisotropic Sobolev space equipped with the norm

$$\|\phi\|_{H^{s_1,s_2}_{x,y}} = \|\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\widehat{\phi}(\xi,\eta)\|_{L^2_{\xi,\eta}}.$$

The spaces $H^{s_1,s_2}_{x,y}(\mathbb{R}^2)$ are a natural set for the initial data of (2) since their homogeneous versions are invariant under the scale transformations preserving the KP equations. The pair (s_1, s_2) is critical if $s_1+2s_2+1/2 = 0$, i.e. for $s_1+2s_2+1/2 = 0$ the space $\dot{H}^{s_1,s_2}_{x,y}(\mathbb{R}^2)$ is invariant under the scale transformation which preserves the KP equations. It seems that similarly to the KdV equation (and other dispersive models) in the case of the KP-II equation the critical for the local well-posedness Sobolev exponent differs from the scaling one. We have the following local well-posedness result.

Theorem 2.1 *([21]) Let $s_1 > -1/3$ and $s_2 \geq 0$. Then for any $\phi \in H^{s_1,s_2}_{x,y}(\mathbb{R}^2)$, such that $|\xi|^{-1}\widehat{\phi}(\xi,\eta) \in \mathcal{S}'(\mathbb{R}^2)$ there exist a positive $T = T(\|\phi\|_{H^{s_1,s_2}_{x,y}})$ ($\lim_{\rho\to 0} T(\rho) = \infty$) and a unique solution $u(t,x,y)$ of the initial value problem (1) on the time interval $I = [-T,T]$ satisfying $u \in C(I, H^{s_1,s_2}_{x,y}(\mathbb{R}^2)) \cap B^{\frac{1}{2}+,\frac{1}{6}+,\frac{1}{3}}_{s_1,s_2}(I)$ (cf. (3) and (4) below for the definition of the spaces $B^{b,b_1,b_2}_{s_1,s_2}(I)$).*

Actually we solve an integral equation corresponding to (2) for any data $\phi \in H^{s_1,s_2}_{x,y}(\mathbb{R}^2)$. The condition $|\xi|^{-1}\widehat{\phi}(\xi,\eta) \in \mathcal{S}'(\mathbb{R}^2)$ is imposed in order to insure that the solution of the integral equation solves also (2) in distribution sense.

Now we introduce the Fourier transform restriction spaces related to the KP-II equation with data defined on $\mathbb{R}^2$. For $b, b_1, b_2, s_1, s_2 \in \mathbb{R}$ we define $B^{b,b_1,b_2}_{s_1,s_2}$ as a Bourgain type space associated to the KP-II equation equipped with the norm

$$\|u\|_{B^{b,b_1,b_2}_{s_1,s_2}} = \left\| \left\langle \tau - \xi^3 + \frac{\eta^2}{\xi} \right\rangle^b \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2} \left(1 + \frac{\langle \tau - \xi^3 + \frac{\eta^2}{\xi}\rangle^{b_1}}{\langle\xi\rangle^{b_2}} \right) \widehat{u}(\tau,\xi,\eta) \right\|_{L^2_{\tau,\xi,\eta}} \quad (3)$$

[2] It seems that similarly to the 2D NLS equation the exact periodic version of the Strichartz inequality obtained in [17] fails.

and $\langle\cdot\rangle = (1+|\cdot|^2)^{\frac{1}{2}}$. Let $I \subset \mathbb{R}$ be an interval. Then we define the space $B^{b,b_1,b_2}_{s_1,s_2}(I)$ equipped with the norm

$$\|u\|_{B^{b,b_1,b_2}_{s_1,s_2}(I)} = \inf_{w \in B^{b,b_1,b_2}_{s_1,s_2}} \left\{ \|w\|_{B^{b,b_1,b_2}_{s_1,s_2}}, \quad w(t) = u(t) \text{ on } I \right\}. \tag{4}$$

The proof of Theorem 2.1 relies on the following bilinear estimate.

Theorem 2.2 *Let $s_1 > -1/3$, $s_2 \geq 0$. Then the following estimate holds*

$$\|\partial_x(uv)\|_{B^{-\frac{1}{2}+,\frac{1}{6}+,\frac{1}{3}}_{s_1,s_2}} \lesssim \|u\|_{B^{\frac{1}{2}+,\frac{1}{6}+,\frac{1}{3}}_{s_1,s_2}} \|v\|_{B^{\frac{1}{2}+,\frac{1}{6}+,\frac{1}{3}}_{s_1,s_2}}. \tag{5}$$

Once we obtain (5) the proof of Theorem 2.1 follows from some general arguments similarly to other dispersive models (cf. [8]).

Now we give some ideas of the proof of Theorem 2.2. Let $\zeta = (\xi,\eta)$, $\zeta_1 = (\xi_1,\eta_1)$ and $\sigma := \sigma(\tau,\zeta) = \tau - \xi^3 + \eta^2/\xi$, $\sigma_1 := \sigma(\tau_1,\zeta_1)$, $\sigma_2 := \sigma(\tau-\tau_1,\zeta-\zeta_1)$. Then Theorem 2.2 can be reformulated in the following way via a duality and a polarization argument.

Proposition 1 *Let $s_1 > -1/3$, $s_2 \geq 0$ and u, v, w be positive functions in $L^2(\mathbb{R}^3)$. Then the following inequality holds*

$$\int K(\tau,\zeta,\tau_1,\zeta_1) u(\tau_1,\zeta_1) v(\tau-\tau_1,\zeta-\zeta_1) w(\tau,\zeta)\, d\tau_1 d\zeta_1 d\tau d\zeta$$
$$\lesssim \|u\|_{L^2}\|v\|_{L^2}, \|w\|_{L^2}, \tag{6}$$

where

$$K(\tau,\zeta,\tau_1,\zeta_1) = \frac{|\xi| \left\langle \frac{\langle\sigma\rangle^{\frac{1}{6}+}}{\langle\xi\rangle^{\frac{1}{3}}} \right\rangle \langle\xi\rangle^{s_1}\langle\xi_1\rangle^{-s_1}\langle\xi-\xi_1\rangle^{-s_1}}{\langle\sigma\rangle^{\frac{1}{2}-}\langle\sigma_1\rangle^{\frac{1}{2}+}\langle\sigma_2\rangle^{\frac{1}{2}+} \left\langle \frac{\langle\sigma_1\rangle^{\frac{1}{6}+}}{\langle\xi_1\rangle^{\frac{1}{3}}} \right\rangle \left\langle \frac{\langle\sigma_2\rangle^{\frac{1}{6}+}}{\langle\xi-\xi_1\rangle^{\frac{1}{3}}} \right\rangle} \frac{\langle\eta\rangle^{s_2}}{\langle\eta_1\rangle^{s_2}\langle\eta-\eta_1\rangle^{s_2}}.$$

We can assume $s_2 = 0$, since for $s_2 \geq 0$ one has $\langle\eta\rangle^{s_2} \lesssim \langle\eta_1\rangle^{s_2}\langle\eta-\eta_1\rangle^{s_2}$. There are two main tools in the proof of (6), the simple calculus argument due to C. E. Kenig, G. Ponce and L. Vega and the global Strichartz inequality for the KP equations injected into the framework of the Fourier transform restriction spaces $B^{b,b_1,b_2}_{s_1,s_2}$.

The Kenig-Ponce-Vega argument. Denote by J the left-hand side of (6). Then using twice the Cauchy-Schwarz inequality we obtain

$$J \lesssim \int \left\{ \int K^2 d\tau_1 d\zeta_1 \right\}^{1/2} \left\{ \int |u(\tau_1,\zeta_1)v(\tau-\tau_1,\zeta-\zeta_1)|^2 d\tau_1 d\zeta_1) \right\}^{1/2} w(\tau,\zeta) d\tau d\zeta$$
$$\lesssim \|K\|_{L^\infty_{\tau\zeta}(L^2_{\tau_1\zeta_1})}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Hence the difficulty is to prove $\|K\|_{L^\infty_{\tau\zeta}(L^2_{\tau_1\zeta_1})} < \infty$. There is a canonical way to integrate K with respect to τ_1 (the simple calculus inequalities). The integration with respect to ξ_1 can be done by using the change of variables $\xi_1 \mapsto \sigma_1 + \sigma_2$. The integration with respect to η_1 is the additional difficulty for the KP equations.

The Strichartz inequalities. The following form of the Strichartz inequality for the KP-II equation holds

$$\|u\|_{L^4} \lesssim \|u\|_{B_{0,0}^{\frac{1}{2}+,0,0}}$$

or equivalently

$$\|\mathcal{F}^{-1}(\langle\sigma\rangle^{-\frac{1}{2}-}u)\|_{L^4(\mathbb{R}^3)} \lesssim \|u\|_{L^2(\mathbb{R}^3)}. \tag{7}$$

One can obtain L^p $(2 \leq p \leq 4)$ versions of (7) by interpolation with the Plancherel identity. Note that J can be written in the form $J = \int u_1(\tau,\zeta)(u_2 \star u_3)(\tau,\zeta)d\tau d\zeta$. Using Plancherel identity and Hölder inequality we obtain

$$J \lesssim \prod_{j=1}^{3} \| \mathcal{F}^{-1}(u_j)\|_{L^{q_j}},$$

where $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$. In the favorable cases u_j have the form

$$u_1 = \langle\sigma\rangle^{-\alpha}w, \quad u_2 = \langle\sigma_1\rangle^{-\alpha_1}u, \quad u_3 = \langle\sigma_2\rangle^{-\alpha_2}v$$

and an application of the L^p version of (7) completes the proof.

The smoothing relation. In order to compensate the loss of a derivative in the nonlinear term we use the relation (cf. [2])

$$\sigma_1 + \sigma_2 - \sigma = 3\xi_1\xi(\xi - \xi_1) + \frac{(\xi_1\eta - \xi\eta_1)^2}{\xi_1\xi(\xi - \xi_1)}$$

and hence

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq |\xi_1\xi(\xi - \xi_1)|. \tag{8}$$

Here we essentially use the KP-II nature. A bad sign in the corresponding relation in the KP-I case is the main obstruction to perform the argument for the KP-I equation. There are three main cases to be considered taking into account which terms dominates in the left-hand side of (8). By symmetry arguments we can assume that $|\sigma_1| \geq |\sigma_2|$. The case when $|\sigma|$ dominates is easier due to an additional smoothing coming from some Jacobian matrixes when performing the Kenig-Ponce-Vega argument. There are two reasons to introduce the extra factor in the definition of the Fourier transform restriction spaces B_{s_1,s_2}^{b,b_1,b_2} : 1. The small frequency cases. 2. The cases when $|\sigma_1|$ dominates in (8).

The small frequency cases. Let $M \leq 1$ and $K \geq 1$ be dyadic. The small frequency cases consist of bounding the contributions to J to the following sets

$$A^{KM} = \left\{(\tau_1, \zeta_1, \tau, \zeta) \ : \ |\xi_1| \approx M, \quad \left|\tau_1 - \xi_1^3 + \frac{\eta_1^2}{\xi_1}\right| \approx K\right\}.$$

Denote by J^{KM} the contribution of A^{KM} to J. Then by using the Strichartz inequalities we can obtain the estimate

$$J^{KM} \lesssim \frac{1}{M^{\delta_1} K^{\delta_2}} \|u\|_{L^2} \|v\|_{L^2} \|w\|_{L^2}, \quad \delta_1 > 0,\ \delta_2 > 0.$$

On the other hand the Kenig-Ponce-Vega argument can provide the following bound for J^{KM}

$$J^{KM} \lesssim M^{\frac{1}{4}} K^{\frac{1}{12}-} \|u\|_{L^2} \|v\|_{L^2} \|w\|_{L^2}, \quad \delta_1 > 0,\ \delta_2 > 0.$$

A suitable interpolation completes the proof.

Counterexample. The result of Theorem 2.2 seems to be optimal when fixing $s_2 = 0$.

Theorem 2.3 *([21]) The estimate*

$$\|\partial_x(uv)\|_{B^{b-1,b_1,b_2}_{s_1,0}} \lesssim \|u\|_{B^{b,b_1,b_2}_{s_1,0}} \|v\|_{B^{b,b_1,b_2}_{s_1,0}} \tag{9}$$

fails for $s_1 < -1/3$.

In order to prove Theorem 2.3 we choose the functions u, v and w in (6) as characteristic functions of suitable sets in $\mathbb{R}^3$. There are two main steps : 1. The necessity for $b + b_1 \geq 2/3$. 2. The necessity for $s_1 > b + b_1 - 1$. In the first step the sections of the needed sets with the planes $\{\eta = \text{const}\}$ are essentially the sets used in [11] in the KdV case. In the second step we need a different construction. In the next table we compare the sets needed in the KdV and in the KP case (note that the set W is not symmetric with respect to the origin in the KP-II case).

KdV	KP-II
There exist $A,\ B,\ W \subset \mathbb{R}^2_{\tau,\xi}$ such that : $\|A\| \sim \|B\| \sim \|W\| \sim N^{-1/2}$. For (τ, ξ) on the support of $A \cup B$ one has : $\|\tau - \xi^3\| \lesssim 1,\ \|\xi\| \sim N$. For (τ, ξ) on the support of W one has : $\|\tau - \xi^3\| \lesssim N^{3/2},\ \|\xi\| \sim N^{-1/2}$. In addition $\chi_A \star \chi_B \gtrsim N^{-1/2} \chi_W$.	There exist $A,\ B,\ W \subset \mathbb{R}^3_{\tau,\xi,\eta}$ such that : $\|A\| \sim \|B\| \sim \|W\| \sim 1$. For (τ, ξ) on the support of $A \cup B$ one has : $\|\tau - \xi^3 + \frac{\eta^2}{\xi}\| \lesssim 1,\ \|\xi\| \sim N$. For (τ, ξ) on the support of W one has : $\|\tau - \xi^3 + \frac{\eta^2}{\xi}\| \lesssim N^2,\ \|\xi\| \sim 1$. In addition $\chi_A \star \chi_B \gtrsim \chi_W$.

Global well-posedness below L^2. In [3] J. Bourgain developed a new method to prove global well-posedness for nonlinear evolution equations when the conservation laws are not directly available. The method was recently applied in different contexts (cf [6, 7, 9, 10, 14, 16, 20, 23]). In [23] we prove that the Cauchy problem for the KP-II equation is globally well-posed for data in $H^{s_1,s_2}_{x,y}(\mathbb{R}^2)$, $s_1 > -1/310$, $s_2 \geq 0$. The method of [23] is further developed in [9] where the restriction on s_1 is removed to $s_1 > -1/64$ (cf. also [20] where global well-posedness for the KP-II equation is shown with initial data belonging to a homogeneous Sobolev space of negative index with respect x).

The KP-II equation with data on $\mathbb{R}\times\mathbb{T}$. In [22] we study the KP-II equation when the data is defined on $\mathbb{R} \times \mathbb{T}$. We observe that better versions of the crucial convolution estimate hold comparing to the purely periodic case. In particular we prove the local well-posedness of the KP-II equation with data below $L^2(\mathbb{R} \times \mathbb{T})$.

3. The KP-I equation

Consider the Fourier transform restriction spaces $X^{b,b_1,b_2}_{s_1,s_2}(\mathbb{R}^3)$ associated to the KP-I equation, equipped with the norm

$$\|u\|_{X^{b,b_1,b_2}_{s_1,s_2}} = \left\|\langle\theta\rangle^b\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\left(1+\langle\theta\rangle^{b_1}\langle\xi\rangle^{-b_2}\right)\widehat{u}(\tau,\xi,\eta)\right\|_{L^2_{\tau,\xi,\eta}},$$

where $\theta := \theta(\tau,\xi,\eta) = \tau+\xi^3+\eta^2/\xi$, $\theta_1 := \theta(\tau_1,\xi_1,\eta_1)$, $\theta_2 := \theta(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1)$. Unfortunately in the case of the KP-I equation the relation for the symbol

$$\theta_1+\theta_2-\theta = -3\xi\xi_1(\xi-\xi_1)+\frac{(\xi_1\eta-\xi\eta_1)^2}{\xi\xi_1(\xi-\xi_1)} \tag{10}$$

does not yield an inequality of type (8) because of the "bad signs" in the right-hand side of (10). Our goal here is to show that a bilinear estimate in the Bourgain spaces associated to the KP-I equation holds despite the "bad signs" in the relation (10). More precisely we have the following inequality.

Theorem 3.1 *([15]) Let $s_1 > 1/2$, $s_2 \geq 0$. Then the following inequality holds*

$$\||D_x|^{\frac12}(uv)\|_{X^{-\frac12+,\frac{7}{32},\frac14}_{s_1,s_2}} \lesssim \|u\|_{X^{\frac12+,\frac{7}{32},\frac14}_{s_1,s_2}}\|v\|_{X^{\frac12+,\frac{7}{32},\frac14}_{s_1,s_2}}. \tag{11}$$

Note that we choose $b_1 = \frac{7}{32}$ and $b_2 = \frac14$ in (11) similarly to [23] but in fact there is a large range for the parameters b_1 and b_2 such that (11) holds. Consider the following region in $\mathbb{R}^6_{\tau,\zeta,\tau_1,\zeta_1}$ (recall that $\zeta = (\xi,\eta)$, $\zeta_1 = (\xi_1,\eta_1)$)

$$R = \left\{(\tau,\zeta,\tau_1,\zeta_1)\ :\ 2\frac{(\xi_1\eta-\xi\eta_1)^2}{|\xi_1\xi(\xi-\xi_1)|} \geq |3\xi_1\xi(\xi-\xi_1)| \geq \frac12\frac{(\xi_1\eta-\xi\eta_1)^2}{|\xi_1\xi(\xi-\xi_1)|}\right\}.$$

The region $(\tau,\zeta,\tau_1,\zeta_1) \notin R$ can be handled similarly to [21, 23] in the case of the KP-II equation since for $(\tau,\zeta,\tau_1,\zeta_1) \notin R$ one has

$$\max\{|\theta|,|\theta_1|,|\theta_2|\} \gtrsim |\xi_1\xi(\xi-\xi_1)|$$

and therefore one can perform arguments similar to the KP-II case.

The region $(\tau,\zeta,\tau_1,\zeta_1) \in R$. Using a duality argument we can rewrite the estimate (11) in the following form

$$\int K(\tau,\zeta,\tau_1,\zeta_1)u(\tau_1,\zeta_1)v(\tau-\tau_1,\zeta-\zeta_1)w(\tau,\zeta)d\tau_1 d\zeta_1 d\tau d\zeta \lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}, \tag{12}$$

where u, v, w are positive and

$$K(\tau,\zeta,\tau_1,\zeta_1) = \frac{|\xi|^{\frac12}\left\langle\frac{\langle\theta\rangle^{\frac{7}{32}}}{\langle\xi\rangle^{\frac14}}\right\rangle\langle\xi\rangle^{s_1}\langle\xi_1\rangle^{-s_1}\langle\xi-\xi_1\rangle^{-s_1}}{\langle\theta\rangle^{\frac12-}\langle\theta_1\rangle^{\frac12+}\langle\theta_2\rangle^{\frac12+}\left\langle\frac{\langle\theta_1\rangle^{\frac{7}{32}}}{\langle\xi_1\rangle^{\frac14}}\right\rangle\left\langle\frac{\langle\theta_2\rangle^{\frac{7}{32}}}{\langle\xi-\xi_1\rangle^{\frac14}}\right\rangle}\frac{\langle\eta\rangle^{s_2}}{\langle\eta_1\rangle^{s_2}\langle\eta-\eta_1\rangle^{s_2}}.$$

For $s_2 \geq 0$, we have $\langle \eta \rangle^{s_2} \lesssim \langle \eta_1 \rangle^{s_2} \langle \eta - \eta_1 \rangle^{s_2}$ and hence we can assume $s_2 = 0$ hereafter. We also assume $|\xi| \geq 1$ since when $|\xi| \leq 1$ one can bound the kernel $K(\tau, \zeta, \tau_1, \zeta_1)$ by $\langle \theta_1 \rangle^{-\frac{1}{2}-} \langle \theta_2 \rangle^{-\frac{1}{2}-}$ and the Strichartz inequality approach described in the previous section is available (note that the same Strichartz inequalities hold in both KP-I or KP-II cases). Denote by J the contribution of R to the left-hand side of (12). Consider the dyadic levels in $\mathbb{R}^6_{\tau,\zeta,\tau_1,\zeta_1}$

$$\begin{aligned} D^{KK_1K_2}_{MM_1M_2} &= \{(\tau, \zeta, \tau_1, \zeta_1) \ : \ \langle \theta \rangle \approx K, \ \langle \theta_1 \rangle \approx K_1, \ \langle \theta_2 \rangle \approx K_2, \\ &\qquad\qquad |\xi| \approx M, \ \langle \xi_1 \rangle \approx M_1, \ \langle \xi - \xi_1 \rangle \approx M_2\}, \end{aligned}$$

where K, K_1, K_2, M_1, M_2, M are the dyadic integers. Denote by $J^{K,K_1,K_2}_{M,M_1,M_2}$ the contribution of $D^{KK_1K_2}_{MM_1M_2} \cap R$ to J. Then

$$J \lesssim \sum_{K,K_1,K_2,M,M_1,M_2} J^{K,K_1,K_2}_{M,M_1,M_2}.$$

Let

$$u_{K_1M_1}(\tau_1, \zeta_1) = \begin{cases} u(\tau_1, \zeta_1), & \text{when } \langle \theta_1 \rangle \approx K_1, \quad \langle \xi_1 \rangle \approx M_1 \\ 0, & \text{elsewhere.} \end{cases}$$

Similarly we define $v_{K_2M_2}(\tau - \tau_1, \zeta - \zeta_1)$ and $w_{KM}(\tau, \zeta)$ to be localized $v(\tau - \tau_1, \zeta - \zeta_1)$ and $w(\tau, \zeta)$ respectively to the regions $\{\langle \theta_2 \rangle \approx K_2, \ \langle \xi - \xi_1 \rangle \approx M_2\}$ and $\{\langle \theta \rangle \approx K, \ |\xi| \approx M \geq 1\}$. We are not going to use the additional factor in the definition of the Fourier transform restriction spaces associated to the KP-I equation when estimating $J^{K,K_1,K_2}_{M,M_1,M_2}$. Note that

$$\frac{\left\langle \frac{\langle \theta \rangle^{\frac{7}{32}}}{\langle \xi \rangle^{\frac{1}{4}}} \right\rangle}{\langle \theta \rangle^{\frac{1}{2}-} \left\langle \frac{\langle \theta_1 \rangle^{\frac{7}{32}}}{\langle \xi_1 \rangle^{\frac{1}{4}}} \right\rangle \left\langle \frac{\langle \theta_2 \rangle^{\frac{7}{32}}}{\langle \xi - \xi_1 \rangle^{\frac{1}{4}}} \right\rangle} \lesssim \frac{1}{\langle \theta \rangle^{0+}}$$

and hence

$$J^{K,K_1,K_2}_{M,M_1,M_2} \lesssim \frac{M^{\frac{1}{2}} \langle M \rangle^{s_1} M_1^{-s_1} M_2^{-s_1}}{K^{0+} K_1^{\frac{1}{2}+} K_2^{\frac{1}{2}+}} \int u_{K_1M_1}(\tau_1, \zeta_1) v_{K_2M_2}(\tau - \tau_1, \zeta - \zeta_1) w_{KM}(\tau, \zeta),$$

where the integration is on $D^{KK_1K_2}_{MM_1M_2} \cap R$. It remains to bound the expression

$$\int u_{K_1M_1}(\tau_1, \zeta_1) v_{K_2M_2}(\tau - \tau_1, \zeta - \zeta_1) w_{KM}(\tau, \zeta).$$

Applying Cauchy-Schwarz inequality in (τ_1, ζ_1), using the support properties of $u_{K_1M_1}$, $v_{K_2M_2}$ and another use of the Cauchy-Schwarz inequality in (τ, ζ) yield

$$J^{K,K_1,K_2}_{M,M_1,M_2} \lesssim \frac{M^{\frac{1}{2}+s_1} M_1^{-s_1} M_2^{-s_1}}{K^{0+} K_1^{\frac{1}{2}+} K_2^{\frac{1}{2}+}} \sup_{(\tau, |\xi| \approx M, \eta)} |A_{\tau\xi\eta}|^{\frac{1}{2}} \|u_{K_1M_1}\|_{L^2} \|v_{K_2M_2}\|_{L^2} \|w_{KM}\|_{L^2}, \quad (13)$$

where $A_{\tau\xi\eta} \subset \mathbb{R}^3$ is defined as follows

$$A_{\tau\xi\eta} = \big\{(\tau_1, \xi_1, \eta_1) \ : \ (\tau_1, \xi_1, \eta_1) \in \ \text{supp} \ u_{K_1 M_1}, \\ (\tau - \tau_1, \xi - \xi_1, \eta - \eta_1) \in \ \text{supp} \ v_{K_2 M_2}, \ (\tau, \xi, \eta, \tau_1, \xi_1, \eta_1) \in R\big\} \\ = \big\{(\tau_1, \xi_1, \eta_1) \ : \ \langle \xi_1 \rangle \approx M_1, \ \langle \xi - \xi_1 \rangle \approx M_2, \ \langle \theta_1 \rangle \approx K_1, \ \langle \theta_2 \rangle \approx K_2, \\ (\tau, \xi, \eta, \tau_1, \xi_1, \eta_1) \in R\big\}.$$

We first eliminate τ_1. Using that for $(\tau_1, \xi_1, \eta_1) \in A_{\tau\xi\eta}$ one has $\langle \theta_1 \rangle \approx K_1$, $\langle \theta_2 \rangle \approx K_2$, we obtain via the triangle inequality

$$\text{mes}(A_{\tau\xi\eta}) \lesssim \min\{K_1, K_2\} \text{mes}(B_{\tau\xi\eta}), \tag{14}$$

where $B_{\tau\xi\eta} \subset \mathbb{R}^2$ is the following set

$$B_{\tau\xi\eta} \ = \ \{(\xi_1, \eta_1) \ : \ \langle \xi_1 \rangle \approx M_1, \quad \langle \xi - \xi_1 \rangle \approx M_2, \ |\theta_1 + \theta_2| \lesssim \max\{K_1, K_2\}, \\ 2\frac{(\xi_1 \eta - \xi \eta_1)^2}{|\xi_1 \xi (\xi - \xi_1)|} \geq |3\xi_1 \xi (\xi - \xi_1)| \geq \frac{1}{2} \frac{(\xi_1 \eta - \xi \eta_1)^2}{|\xi_1 \xi (\xi - \xi_1)|}\}.$$

We have that for $(\tau, \zeta, \tau_1, \zeta_1) \in R$

$$\left| \frac{\partial}{\partial \eta_1} (\theta_1 + \theta_2) \right| \sim \left| \frac{\xi_1 \eta - \xi \eta_1}{\xi_1 (\xi - \xi_1)} \right| \sim |\xi| \sim M \tag{15}$$

since for $(\tau, \zeta, \tau_1, \zeta_1) \in R$ one has

$$\frac{(\xi_1 \eta - \xi \eta_1)^2}{|\xi \xi_1 (\xi - \xi_1)|} \approx |3\xi \xi_1 (\xi - \xi_1)|.$$

The measure of the projection of the set $B_{\tau\xi\eta}$ on the ξ_1 axis is bounded by $\min\{M_1, M_2\}$, since for $(\xi_1, \eta_1) \in B_{\tau\xi\eta}$ one has $\langle \xi_1 \rangle \approx M_1$ and $\langle \xi - \xi_1 \rangle \approx M_2$. Fix now ξ_1. Then the measure of η_1 such that $(\xi_1, \eta_1) \in B_{\tau\xi\eta}$ is bounded by

$$\frac{\max\{K_1, K_2\}}{\inf \left| \frac{\partial}{\partial \eta_1} (\theta_1 + \theta_2) \right|} \lesssim \frac{\max\{K_1, K_2\}}{M}$$

since for $(\xi_1, \eta_1) \in B_{\tau\xi\eta}$ one has $|\theta_1 + \theta_2| \lesssim \max\{K_1, K_2\}$. Hence we obtain that that the maximum of the measures of the sections of $B_{\tau\xi\eta}$ with lines parallel to the η_1 axis is bounded by $M^{-1} \max\{K_1, K_2\}$ (the factor M^{-1} gives the smoothing effect). Hence

$$\text{mes}(B_{\tau\xi\eta}) \lesssim \frac{\max\{K_1, K_2\} \min\{M_1, M_2\}}{M}. \tag{16}$$

Using (14) and (16) we arrive at

$$\text{mes}(A_{\tau\xi\eta}) \lesssim \frac{K_1 K_2 \min\{M_1, M_2\}}{M}. \tag{17}$$

By the triangle inequality we have that $M \lesssim \max\{M_1, M_2\}$. Using a symmetry argument we can suppose that $M_1 \geq M_2$ and therefore $M \lesssim M_1$. Let $M_1 = 2^l M$, where $l \in \mathbb{Z}$, $l \geq -l_0$ (l_0 is fixed, positive and independent of M). Then substituting (17) in (13) we obtain

$$J^{K,K_1,K_2}_{M,2^l M,M_2} \lesssim \frac{1}{K^{0+}K_1^{0+}K_2^{0+}M_2^{s_1-\frac{1}{2}}2^{ls_1}} \|u_{K_1 2^l M}\|_{L^2}\|v_{K_2 M_2}\|_{L^2}\|w_{KM}\|_{L^2}. \tag{18}$$

It remains to sum (18) over K, K_1, K_2, M, M_2, l. Since $s_1 > 1/2$ we can easily sum (18) over K, K_1, K_2, M_2

$$\sum_{K,K_1,K_2,M_2} J^{K,K_1,K_2}_{M,2^l M,M_2} \lesssim \frac{1}{2^{ls_1}}\|u_{2^l M}\|_{L^2}\|v\|_{L^2}\|w_M\|_{L^2},$$

where $u_{2^l M}(\tau_1, \zeta_1)$ and $w_M(\tau, \zeta)$ are localized $u(\tau_1, \zeta_1)$ and $w(\tau, \zeta)$ respectively to the regions $\{\langle \xi_1 \rangle \approx 2^l M\}$ and $\{|\xi| \approx M \geq 1\}$. Next we sum over M and l via the Cauchy-Schwarz inequality

$$\begin{aligned} J &\lesssim \sum_{K,K_1,K_2,M_2,M,l} J^{K,K_1,K_2}_{M,2^l M,M_2} \\ &\lesssim \sum_l \frac{1}{2^{ls_1}} \left\{\sum_M \|u_{2^l M}\|^2_{L^2}\right\}^{1/2} \left\{\sum_M \|w_M\|^2_{L^2}\right\}^{1/2} \|v\|_{L^2} \\ &\lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \end{aligned}$$

Finally we note that if we prove an analogue of Theorem 3.1, with a gain of one derivative then one could expect to obtain finite energy solutions for the KP-I equation (finite energy solutions for the fifth order KP-I equations are obtained in [18]). However, since the energy density for the KP-I equation contains anti-derivatives, in order to obtain the local well-posedness in the energy space, some additional bilinear estimates would be needed.

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