

*Journées*

# **ÉQUATIONS AUX DÉRIVÉES PARTIELLES**

Port d'Albret, 7 juin–11 juin 2010

Laure Saint-Raymond

**Lecture notes : Mathematical study of singular perturbation problems  
Applications to large-scale oceanography**

*J. É. D. P.* (2010), Exposé n° I, 49 p.

<[http://jedp.cedram.org/item?id=JEDP\\_2010\\_\\_\\_\\_A1\\_0](http://jedp.cedram.org/item?id=JEDP_2010____A1_0)>

**cedram**

*Article mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>*

# LECTURE NOTES : Mathematical study of singular perturbation problems Applications to large-scale oceanography

Laure Saint-Raymond

## CONTENTS

1. Oceanic motions. Orders of magnitude and scalings	3
1.1. Physical observations	3
1.1.1. Static description	3
1.1.2. Kinematic description	4
1.1.3. Dynamic description	5
1.2. Mathematical models	6
1.2.1. The Navier-Stokes equations	6
1.2.2. About viscous dissipation	7
1.2.3. Boundary conditions	7
1.3. Orders of magnitude	8
1.3.1. Geometric approximations	8
1.3.2. About the free surface	9
1.3.3. About the Coriolis force	9
1.4. Mathematical theories for simplified models	10
1.4.1. The incompressible Navier-Stokes equations	10
1.4.2. The Saint-Venant equations	13
References	14
2. Rotating fluids. Weak and strong asymptotics	16
2.1. Heuristic study of rotating fluids	16
2.1.1. A simple model	16
2.1.2. The linear propagator	16
2.1.3. The nonlinear coupling	17
2.2. Compensated compactness and weak convergence	17
2.2.1. The constraint equation	18
2.2.2. Regularity of the geostrophic motion	19
2.2.3. Compensated compactness	20
2.3. Filtering methods and strong convergence	21
2.3.1. Filtering the oscillations	21
2.3.2. Poincaré waves	23

2.3.3. The modulated energy	24
References	26
3. Boundary layers and Ekman pumping	27
3.1. Scalings and asymptotic expansion for the Ekman layers	27
3.1.1. A simple example	27
3.1.2. Balance between viscous and rotating terms	27
3.1.3. Resulting vertical motion	28
3.2. Two-scale analysis and weak convergence	29
3.2.1. Uniform bounds	30
3.2.2. Weak formulation of the Navier-Stokes equations with boundary conditions	31
3.2.3. The trace equation at $z = 0$	32
3.2.4. The asymptotic motion	33
3.3. Oscillating boundary layer terms and strong convergence	34
3.3.1. The energy method	34
3.3.2. Anomalous boundary layers : a systematic approach	34
References	35
4. Rossby waves and oceanic eddies	37
4.1. Mathematical models for oceanic eddies	37
4.1.1. About oceanic eddies, their formation and their propagation	37
4.1.2. Mathematical description of oceanic motions	38
4.1.3. Orders of magnitude and scaling	38
4.2. Mathematical study of wave propagation	39
4.2.1. A simple case with explicit spectral resolution	39
4.2.2. Tools for asymptotic analysis	40
4.2.3. Main difficulties	41
4.3. Semiclassical analysis of the linear propagation	41
4.3.1. Polarization of Poincaré and Rossby waves	41
4.3.2. Dispersion of Poincaré waves	42
4.3.3. Trapping of Rossby waves	43
4.4. Case of a weak coupling	44
4.4.1. A symmetrizable hyperbolic system	45
4.4.2. Propagation of regularity	45
4.4.3. Linear approximation	46
References	47
Index	49

## 1. Oceanic motions. Orders of magnitude and scalings

In this first chapter, we will introduce both from the physical and the mathematical points of view some models which are usually considered to describe oceanic motions.

For large-scale motions, physical considerations based on orders of magnitude lead to some simplifications and therefore to better understood mathematical models.

### 1.1. Physical observations

#### 1.1.1. Static description

Oceans are huge masses of water delimited by continents, and rotating together with the earth. The domain occupied by oceans evolves with time (a little bit).

- At the bottom, water is stopped by the earth crust. Because of this fluid-structure interaction, the *bottom topography* is expected to play a crucial role in the dynamics.

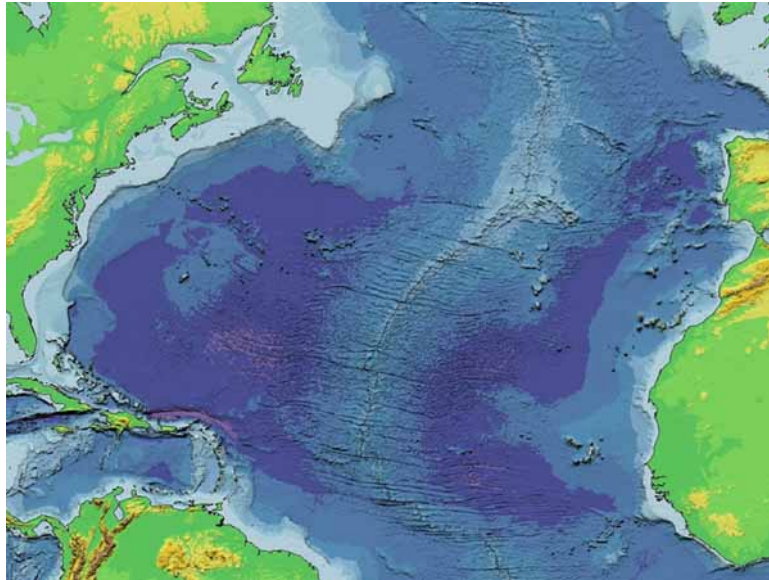


Figure 1.1: Bottom topography

- At the surface, water is surrounded by air. The interface is a *free surface* with negligible surface tension, constrained by *wind forcing*. In all the sequel, we will assume that the wind forcing is known a priori. Considering coupled models for oceans and atmosphere would be more relevant, but is too much difficult at the present time.

The density of oceans has very small variations, at least outside from the *pycnocline*. In any case, we will neglect the compressibility of water. Under such an assumption, thermodynamics can be decoupled from dynamics.

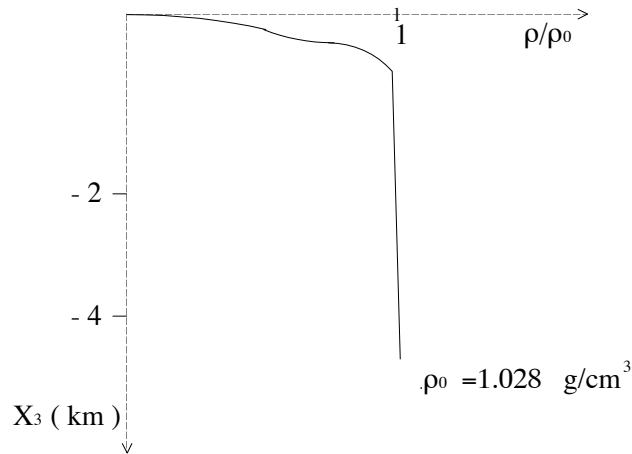


Figure 1.2: Density of water

### 1.1.2. Kinematic description

The movement of oceans can be described as a superposition of various fluctuations with respect to the rigid body rotation.

- Oscillating motions with small period (1-10 seconds) : *ripples* and *swell*  
Ripples are created by some local phenomenon (obstacle or wind for instance), while swell is the response to some distant or switched-off excitation. The am-



Figure 1.3: Swell

plitude of such waves decreases as depth increases, and is essentially negligible at a depth equal to one half of the wave length. On lateral boundaries, these

waves break when the bottom rises abruptly, whereas they go flat in shallow water.

- Oscillating motions with larger period (1-10 hours) : *tides*, *tsunamis*, *storm waves*

Storm waves are created by some decrease in atmospheric pressure, then amplified by wind and Coriolis force (resonance).

Tsunamis are linked with tectonic phenomena in deep water, they carry a lot of energy and propagate very fast.

Tides are long waves generated by the moon gravitation, depending also on the Coriolis force and on the configuration of the coasts (and on the depth).

- Non oscillating motions, namely *oceanic currents* (independent from tides)

Surface currents are created by wind, then transmitted by *Ekman pumping* (to be explained in Chapter 3) around one kilometer deep, and damped by friction.

In deep water, currents are due mainly to temperature and salinity gradients.

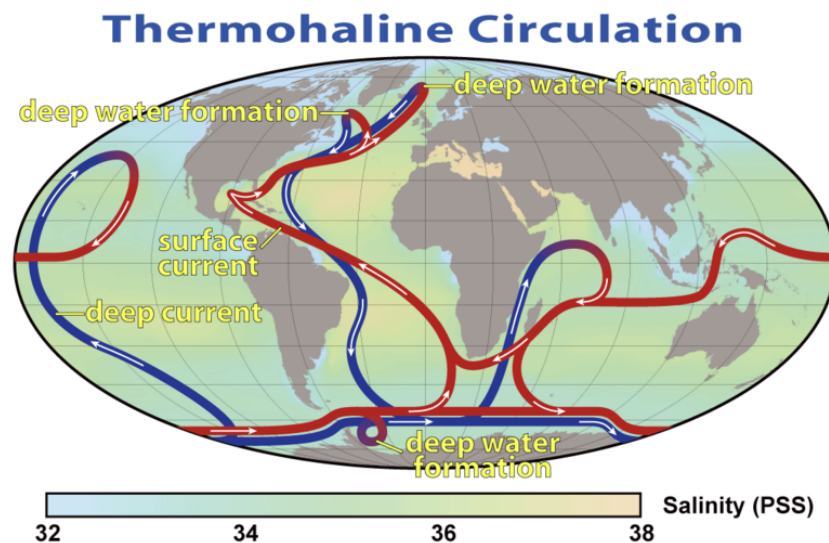


Figure 1.4: Oceanic currents

### 1.1.3. Dynamic description

Forces which are responsible for the ocean dynamics have been already mentioned in the previous paragraph. Conservative forces can be classified as follows :

- Gravity, namely earth gravity and - in weaker measure - moon gravity.

In the absence of relative motion, it has to be balanced by the pressure  $p$ , so that the pressure is given by the *hydrostatic law*

$$\frac{\partial p}{\partial z} = -\rho g \text{ with } g \sim 9.8 \text{ ms}^{-2}.$$

- Coriolis and centrifugal forces

Because the reference frame is rotating, Coriolis and centrifugal forces appear, the significance of which is measured by the Rossby number

$$\text{Ro} = \frac{U}{2|\Omega|L} \text{ with } |\Omega| \sim 7.3 \times 10^{-5} \text{ s}^{-1},$$

denoting by  $U$  and  $L$  the typical velocity and length scales of the flow to be considered.

- Wind forcing

The coupling with the atmosphere is the cause both of surface currents and of many oscillating motions. As a first approximation, we will assume that wind is known and affects the motion through the boundary condition at the surface (Navier boundary condition).

- Temperature and salinity gradients

Such data seem to be crucial to understand the global *thermohaline* circulation, but not for the response to the wind. They will be neglected in all the sequel.

## 1.2. Mathematical models

Given the assumptions and simplifications presented in the previous section, it is natural to describe the ocean dynamics using the Navier-Stokes equations with free surface.

### 1.2.1. The Navier-Stokes equations

The incompressibility constraint states

$$\nabla \cdot u = 0,$$

while the conservation of momentum provides the evolution equation

$$\partial_t u + (u \cdot \nabla)u + \frac{1}{\rho} \nabla p = g - 2\Omega \wedge u + \frac{\mathcal{F}}{\rho}$$

where  $\rho = \rho_0$  is the (constant) density,  $g$  is the gravity (of constant modulus),  $\Omega$  is the earth rotation vector,  $p$  is the pressure, defined as the Lagrange multiplier associated to the incompressibility constraint, and  $\mathcal{F}$  is the viscous dissipation.

Note that the main inadequacy of this model is actually related to the absence of temperature as an important parameter to describe the state of water.

Temperature is essentially transported by the velocity field, but it is also expected to be involved in the mechanism governing the thermohaline circulation.

### 1.2.2. About viscous dissipation

Even though friction may be weak compared with other forces, its dissipative nature, qualitatively distinct from the conservative nature of the inertial forces, requires its consideration.

Let us first consider the dissipation due to the interactions at microscopic level.  $\mathcal{F}$  is then proportional to the spatial derivative of the stress tensor

$$\frac{\mathcal{F}}{\rho} \sim \nu_{molecular} \nabla \cdot (\nabla u + (\nabla u)^T).$$

The ratio of the frictional force to the Coriolis force acceleration is then measured by the Ekman number

$$E = \frac{\nu_{molecular}}{2\Omega L^2} \sim 10^{-14}$$

for  $L = 1000km$  and  $\nu_{molecular} = 10^{-6}m^2s^{-1}$ . The molecular dissipation is therefore too small to compensate the energy income, due for instance to solar heating.

Another - supposedly much more efficient - dissipative mechanism is turbulence. It should result from the energy transfer associated to the nonlinear interaction between waves. There is indeed a possibility that small scale motions (which are not the focus of our interest) may yet influence the large scale motions, smoothing and mixing properties by processes analogous to molecular diffusive transport :

$$\frac{\mathcal{F}}{\rho} \sim \nu_{turbulent} \nabla \cdot (\nabla u + (\nabla u)^T).$$

Note that this model of turbulent dissipation is completely ad hoc, and has even no heuristic derivation.

### 1.2.3. Boundary conditions

Because of the incompressibility constraint, the normal velocity has to be prescribed on the boundary, and it is rather natural to ask that the flux is zero.

With the previous choice of dissipation operator, one has further to prescribe either the tangential velocity or the normal stress.

At the bottom  $B$ , the fluid-structure interaction imposes some stopping condition, namely the homogeneous *Dirichlet boundary condition*

$$u|_B = 0.$$

On the free surface  $\Sigma$ , as we neglect the surface tension, we have some slipping condition referred to as *Navier boundary condition*

$$n \cdot u|_\Sigma = 0, \quad n \cdot (p - \nu(\nabla u + (\nabla u)^T)) = \tau,$$

where  $n$  denotes the outwards unit normal. These continuity conditions have to be supplemented by a *kinematic condition* defining the moving domain  $\mathcal{D}(t)$

$$\partial_t \mathbf{1}_{\mathcal{D}} + \nabla \cdot (\mathbf{1}_{\mathcal{D}} u) = 0.$$



Even though the system we obtain by gathering together all the previous equations seems to satisfy all the conditions required to be well-posed, many mathematical difficulties arise when studying the Cauchy problem. For instance, if we consider weak solutions, the interface is not defined (note that even for strong solutions, the interface is not necessarily a graph). Furthermore, singularities are expected to appear in the vicinity of any point of  $\Sigma \cap B$ .

At the present time, there is therefore - to our knowledge - no mathematical result concerning the existence of solutions for the complete system.

### 1.3. Orders of magnitude

In order to further simplify the above model, a common method in physics is to compare the contributions of the different terms, introducing the orders of magnitude of the physical parameters.

#### 1.3.1. Geometric approximations

Oceans are thin layers of fluid, located at the surface of the earth.

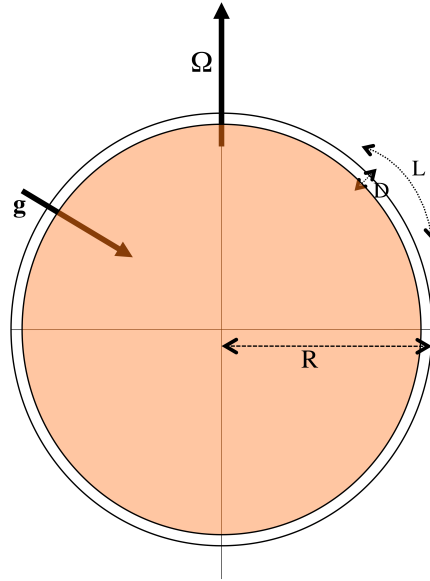


Figure 1.5: Geometric parameters

Typically, denoting by  $D$  the depth and by  $L$  the horizontal extent, one has

$$D \sim 1 - 5km, \quad L \sim 100 - 1000km.$$

The *aspect ratio* being very small, it is then natural to use some *shallow-water approximation*.

In addition, for the sake of simplicity, the earth curvature is usually neglected, i.e. spherical coordinates are considered as cartesian coordinates. Such an approximation is justified if  $L \ll R$  where  $R$  is the earth radius

$$R \sim 6400km.$$

More generally, we expect this approximation to provide a rough description of qualitative behaviours.

### 1.3.2. About the free surface

Except in the vicinity of coasts and islands, the fluctuations of the surface height  $\delta h$  are negligible compared to the depth  $D$  of oceans :

$$\delta h \sim 1 - 10m, \quad D \sim 1 - 5km.$$

When considering the global circulation, it seems actually relevant to neglect the variations of the surface elevation, which leads to the so-called *rigid lid approximation*.

However, in some situations, the fluctuations of the surface height have to be considered insofar as they introduce some kind of compressible effects (of a different nature from both the physical and the mathematical points of view). For instance, such effects are crucial for the understanding of surface waves :  $\delta h$  has then to be compared to the wave length  $\lambda$ .

### 1.3.3. About the Coriolis force

Our main interest in this work lies in large scale motions, that are by definition motions for which the Rossby number  $Ro = U/2|\Omega|L$  is small.

In such a regime, we expect the **mean motion** to be constrained, in order that the dominating forces, namely the pressure gradient and the Coriolis force, balance one another. This constraint is referred to as the geostrophic constraint, and the corresponding simplification of the mathematical model is the so-called *geostrophic approximation*.

Note that the geostrophic constraint may be incompatible with the boundary conditions, in which case we expect to observe some boundary layers matching these two different constraints. These boundary layers actually transfer some energy from the boundary inside the domain, possibly a macroscopic part of the energy. The feedback of the boundary layers on the mean motion is a crucial mechanism in the global circulation, known as *Ekman pumping*, which will be explained in Chapter 3.

At this stage, we have therefore raised two important issues concerning the description of large-scale oceanic motions, namely

- the derivation of an evolution equation for the geostrophic motion,
- and the stability of boundary layers.

**Departures from geostrophy** can then be described by a superposition of waves, depending on the time and space scales to be considered. The linear propagator involves the Coriolis term and the pressure gradient. The dispersion relation

therefore depends on the rotation frequency, on the inhomogeneities of the rotation vector and on the stratification.

Physicists use to classify waves into two types according to the dominating phenomenon :

Regime	Period (typical value)	Dynamical structure
Rotating (Poincaré waves)	1 day 1 week	Frequency increases with wave number
Quasigeostrophic (Rossby waves)	1 month	Geostrophic balance between fluid velocity and pressure. Eastward propagation Frequency decreases as wave number increases

Figure 1.6: Wave classification

- Poincaré waves which may be gravity or rotation waves depending on the wave length and of the buoyancy frequency

$$N = -g \frac{\partial \log \rho}{\partial z};$$

- Rossby waves which are quasi-geostrophic waves, propagating much slower, with an eastwards group velocity.

A natural problem is then to understand the interaction between waves via the nonlinear couplings, and to obtain the slow dynamics of the amplitudes referred to as envelope equations.

All these questions will be studied in the sequel starting from simplified mathematical models.

## 1.4. Mathematical theories for simplified models

In this last section, we present with more details the mathematical properties of two simplified models. Of course many other simplified models are used by oceanographers, but these two can be considered as prototypes regarding their mathematical structure.

### 1.4.1. The incompressible Navier-Stokes equations

Here we consider that the free surface is so turbulent with foam and waves that it can be replaced at first sight by its average (rigid lid approximation), and we neglect the bottom topography. The fluid is then contained in some horizontal layer  $\mathcal{D}$  that we assume to have no lateral boundary for the sake of simplicity.

Since the density  $\rho$  is essentially constant, dynamics decouples from thermodynamics. More precisely, the velocity  $u$  is governed by the Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla)u + \nabla p = 2u \wedge \Omega + \nu \Delta u, \quad \nabla \cdot u = 0, \quad (1.1)$$

where the pressure  $p$  is the Lagrange multiplier associated to the incompressibility constraint.

These equations have to be supplemented by boundary conditions, namely the stopping condition at the bottom  $B$ , and some inhomogeneous Navier condition at the surface  $\Sigma$  stating

$$u|_B = 0, \quad u_{3|\Sigma} = 0, \quad \nu \partial_3 u_{h|\Sigma} = \tau, \quad (1.2)$$

where  $\tau$  is the wind forcing.

A fundamental property of this system is the **energy estimate**

$$\frac{1}{2} \|u(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla u(s)\|_{L^2}^2 ds + \int_0^t \int_\Sigma \tau u_h(s, x_h) dx_h ds \leq \frac{1}{2} \|u_0\|_{L^2}^2$$

obtained formally by multiplying (1.1) by  $u$  and integrating by parts using (1.2).

Because  $u$  is divergence-free, we further have the trace estimate

$$\|u_{h|\Sigma}\|_{L^2(\Sigma)} \leq C \|u\|_{H^{1/2}}^2 \leq C \|u\|_{L^2}^{1/2} \|u\|_{\dot{H}^1}^{1/2}$$

which provides - together with Gronwall's inequality - the a priori estimate

$$\frac{1}{2} \|u(t)\|_{L^2}^2 + \frac{\nu}{2} \int_0^t \|\nabla u(s)\|_{L^2}^2 ds \leq \frac{1}{2} \|u_0\|_{L^2}^2 e^{Ct} + C \nu^{-1/2} \int_0^t \|\tau(s)\|_{L^2(\Sigma)}^2 e^{C(t-s)} ds \quad (1.3)$$

Note that this estimate does not provide any uniform bound with respect to the viscosity  $\nu$ .

This estimate is the crucial tool to prove the global existence of weak solutions to (1.1)-(1.2) :

**Theorem 1.1** (Leray). *Let  $u_0 \in L^2(\mathcal{D})$  be any divergence-free vector field satisfying the zero mass flux condition*

$$u_{0,3|\Sigma} = u_{0,3|B} = 0,$$

*and  $\tau$  some smooth 2D vector field on  $\Sigma$ . Then there exists  $u \in L_{loc}^\infty(\mathbf{R}^+, L^2(\mathcal{D})) \cap L_{loc}^2(\mathbf{R}^+, H^1(\mathcal{D}))$  weak solution to the Navier-Stokes equations (1.1)-(1.2), and satisfying the energy inequality (1.3).*

*Sketch of proof.* Any solution to (1.1)-(1.2) can be decomposed as the sum of a solution  $v$  to the Stokes equation with inhomogeneous boundary condition

$$\begin{aligned} \partial_t v + \nabla p &= \nu \Delta v, \quad \nabla \cdot v = 0, \\ v|_B &= 0, \quad v_{3|\Sigma} = 0, \quad \nu \partial_3 v_{h|\Sigma} = \tau, \end{aligned}$$

and a solution  $w$  of the following modified Navier-Stokes equations with homogeneous boundary conditions

$$\begin{aligned} \partial_t w + (w \cdot \nabla) w + \nabla p - \nu \Delta w &= 2(v + w) \wedge \Omega - ((v + w) \cdot \nabla) v - v \cdot \nabla w, \quad \nabla \cdot w = 0, \\ w|_B &= 0, \quad w_{3|\Sigma} = 0, \quad \nu \partial_3 w_{h|\Sigma} = 0, \end{aligned}$$

Since the linear and source terms in the right-hand side of these modified Navier-Stokes equations can be dealt with without difficulty, we will restrict our attention - without loss of generality - to the usual Navier-Stokes equations with homogeneous boundary conditions ( $\tau = 0$  and  $\Omega = 0$ ).

We then define  $J_n$  as the projection on the  $n$  first modes of the Stokes operator with homogeneous boundary conditions if the spectrum is discrete (and more generally as a suitable spectral truncation of the Stokes operator with homogeneous boundary conditions).

The Cauchy-Lipschitz theorem provides the existence of a unique solution  $u_n \in C^1([0, T_n], J_n L^2(\mathcal{D}))$  to the regularized equations

$$\partial_t u_n + J_n \nabla \cdot (u_n \otimes u_n) - \nu J_n \Delta u_n = 0, \quad (1.4)$$

with initial data  $J_n u_0$ . By the energy estimate (1.3), we further have a global  $L^2$  control on  $u_n$ , so that  $T_n = +\infty$ .

This same energy estimate gives uniform bounds on  $u_n$  with respect to  $n$ . In particular, up to extraction of a subsequence,

$$u_n \rightharpoonup u \text{ weakly in } L^2_{loc}(\mathbf{R}^+, L^2(\mathcal{D})).$$

By Sobolev's embeddings, we also have some strong compactness on  $(u_n)$  with respect to space variables :

$$\|u_n\|_{L^2([0, T], H^1(\mathcal{D}))} \leq C_T.$$

The evolution equation provides then some control on the time derivative

$$\|\partial_t u_n\|_{L^1([0, T], H^{-3/2}(\mathcal{D}))} \leq C_T.$$

By interpolation, we finally obtain the strong convergence

$$u_n \rightarrow u \text{ weakly in } L^2_{loc}(\mathbf{R}^+ \times \mathcal{D})$$

Taking limits in (1.4) shows that  $u$  is a weak solution to the Navier-Stokes equations.

Note that such a proof does not provide any uniqueness, and this remains a challenging open problem.  $\square$

Another important feature of the Navier-Stokes equations is the **scaling invariance**. Actually, if  $u \equiv u(t, x)$  is a solution to the Navier-Stokes equations (1.1) set for instance in the whole space  $\mathbf{R}^3$ , then  $u_\lambda = \lambda u(\lambda^2 t, \lambda x)$  is also a solutions to (1.1) for any  $\lambda > 0$ . Functional spaces which are invariant under these transformations are referred to as "scaling invariant spaces" :  $L_t^\infty(L_x^3)$ ,  $L_t^\infty(\dot{H}_x^{1/2})$ ,  $L_t^2(\dot{H}_x^{3/2})$ ....

In scaling invariant functional spaces, precised energy estimates give some stability and therefore the existence and uniqueness of local smooth solutions :

**Theorem 1.2** (Fujita-Kato). *Let  $u_0 \in H^{1/2}(\mathcal{D})$  be any divergence-free vector field satisfying the boundary conditions (1.2) and  $\tau$  some smooth 2D vector field on  $\Sigma$ . Then there exists a unique local solution  $u \in C([0, T^*), \dot{H}^{1/2}) \cap L^2_{loc}([0, T^*), \dot{H}^{3/2})$  to the Navier-Stokes equations (1.1).*

*If  $\nabla \Omega \neq 0$ , the lifespan of the solution depends in particular on the Rossby number*

*Sketch of proof.* In the absence of forcing and of (inhomogeneous) Coriolis force, the precised energy estimate

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{\dot{H}^{1/2}}^2 + \nu \|\nabla u\|_{\dot{H}^{1/2}}^2 &= \langle (u \cdot) u | u \rangle_{\dot{H}^{1/2}} \\ &\leq \|u\|_{L^3} \|\nabla u\|_{L^3}^2 \leq C \|u\|_{\dot{H}^{1/2}} \|\nabla u\|_{\dot{H}^{1/2}}^2 \end{aligned}$$

gives the global existence of strong solutions for small data.

Indeed, if  $\|u_0\|_{\dot{H}^{1/2}} \leq \nu/2C$ , then

$$\sup\{t \in \mathbf{R}^+ / \|u(t)\|_{\dot{H}^{1/2}} \leq \frac{\nu}{C}\} = +\infty.$$

With a forcing term or an (inhomogeneous) Coriolis force, or for large data, the idea is to introduce the same decomposition  $u = v + w$  as previously, with

$$\begin{aligned} \partial_t v + \nabla p &= \nu \Delta v, \quad \nabla \cdot v = 0, \\ v|_B &= 0, \quad v_{3|\Sigma} = 0, \quad \nu \partial_3 v_h|_\Sigma = \tau, \quad v_0 = u_0, \end{aligned}$$

and

$$\begin{aligned} \partial_t w + (w \cdot \nabla)w + \nabla p - \nu \Delta w &= 2(v + w) \wedge \Omega - ((v + w) \cdot \nabla)v - v \cdot \nabla w, \quad \nabla \cdot w = 0, \\ w|_B &= 0, \quad w_{3|\Sigma} = 0, \quad \nu \partial_3 w_h|_\Sigma = 0, \quad w_0 = 0. \end{aligned}$$

Using a variant of the precised energy estimate and Gronwall's lemma, we obtain that

$$T^* = \sup\{t \in \mathbf{R}^+ / \|w(t)\|_{\dot{H}^{1/2}} \leq \frac{\nu}{C}\} > 0.$$

Note that if  $\nabla \Omega = 0$ , the Coriolis term  $\Omega \wedge u$  does not appear in the precised energy estimate since it is skew-symmetric for any  $H^s$  scalar product. The lifespan  $T^*$  of the solution is then independent of the Rossby number.  $\square$

#### 1.4.2. The Saint-Venant equations

In some situations, the influence of the free surface is dominating (for instance when considering the propagation of surface wave). A very crude mathematical model to account for these features can be obtained in the shallow-water approximation.

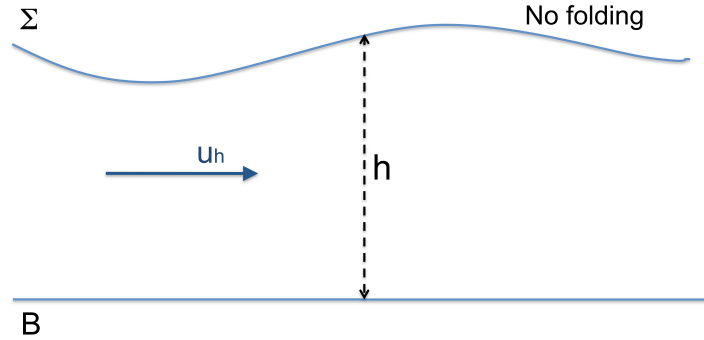


Figure 1.7: Shallow-water approximation

As previously, we consider that the density  $\rho$  is constant, and we further assume that, because of the small aspect ratio, the motion depends essentially only on the horizontal variables  $x_h$ . Note that the incompressibility constraint and the zero mass flux condition then imply that the vertical velocity  $u_3$  is zero.

If the free surface has no folding, integrating formally the incompressibility relation and the 3D Navier-Stokes equations with respect to the vertical variable  $z$ , we get the Saint-Venant equations

$$\begin{aligned}\partial_t h + \nabla \cdot (hu_h) &= 0, \\ \partial_t(hu_h) + \nabla \cdot (hu_h \otimes u_h) + \omega \wedge hu_h + h\nabla p &= \mathcal{F} \\ h\partial_z p &= hg,\end{aligned}\tag{1.5}$$

where  $h \equiv h(t, x_h)$  is the local height of water,  $\omega$  is the local vertical component of the rotation vector  $\Omega$ , and  $\mathcal{F}$  is the viscous dissipation.

- In the inviscid case, namely when  $\mathcal{F} = 0$ , the classical theory of symmetrizable hyperbolic systems gives the existence and uniqueness of local strong solutions in  $L^\infty([0, T^*), H_x^s)$  for  $s > 2$ . Defining the sound speed  $u_0$  by

$$u_0 = 2(\sqrt{\rho} - 1),$$

we indeed obtain that (1.5) is equivalent to

$$\partial_t U + AU + S_1(U)\partial_1 U + S_2(U)\partial_2 U = 0, \quad U = (u_0, u_1, u_2) \tag{1.6}$$

where  $A$  is the linear propagator

$$A = \begin{pmatrix} 0 & \partial_1 & \partial_2 \\ \partial_1 & 0 & -\omega \\ \partial_2 & \omega & 0 \end{pmatrix}, \tag{1.7}$$

and

$$S_1(U) := \begin{pmatrix} u_1 & \frac{1}{2}u_0 & 0 \\ \frac{1}{2}u_0 & u_1 & 0 \\ 0 & 0 & u_1 \end{pmatrix}, \quad S_2(U) := \begin{pmatrix} u_2 & 0 & \frac{1}{2}u_0 \\ 0 & u_2 & 0 \\ \frac{1}{2}u_0 & 0 & u_2 \end{pmatrix}. \tag{1.8}$$

- In the viscous case, and more precisely if we choose  $\mathcal{F} = \nu \Delta u_h$ , we can build global weak solutions  $(h, u_h) \in L_{loc}^2(\mathbf{R}^+, L_x^2 \times H_x^1)$  starting from the energy inequality

$$\frac{1}{2} \int (hu_h^2 + gh^2)(t, x) dx + \nu \int_0^t \|\nabla u_h(s)\|_{L^2}^2 ds \leq \frac{1}{2} \int (hu_h^2 + gh^2)(0, x) dx.$$

For the sake of simplicity, we will consider such global weak solutions in the sequel. Note however that this choice of  $\mathcal{F}$  does not seem to be physically relevant insofar as the dissipation is independent of  $h$ . In particular, this is not the dissipation obtained by integration of the Navier-Stokes equations.

## References

- [1] D. Bresch and B. Desjardins, Existence of global weak solutions for a 2D viscous shallow water equation and convergence to the quasi-geostrophic model. *Commun. Math. Phys.* **238** (2003), pages 211 – 223.
- [2] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Basics of Mathematical Geophysics*, to appear in *Oxford University Press*, 2006.
- [3] F. C. Fuglister, *Gulf Stream '60*, Progress in Oceanography I, Pergamon Press, 1963.
- [4] H. Fujita and T. Kato, On the Navier–Stokes initial value problem I, *Archive for Rational Mechanics and Analysis*, **16** (1964) pages 269–315.

- [5] I. Gallagher & L. Saint-Raymond, On the influence of the Earth's rotation on geophysical flows, *Handbook of Mathematical Fluid Dynamics*, S. Friedlander and D. Serre Editors Vol 4, Chapter 5, 201-329, 2007.
- [6] Gerbeau, J.-F.; Perthame, B. Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation. *Discrete Contin. Dyn. Syst. Ser. B* 1 (2001), no. 1, 89–102.
- [7] A. E. Gill, Atmosphere-Ocean Dynamics, *International Geophysics Series*, **Vol. 30**, 1982.
- [8] A. E. Gill and M. S. Longuet-Higgins, Resonant interactions between planetary waves, *Proc. Roy. Soc. London*, **A 299** (1967), pages 120–140.
- [9] J. Leray, Essai sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Mathematica*, **63** (1933), pages 193-248.
- [10] J. Leray, Etude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique. *Journal de Mathématiques Pures et Appliquées* **12** (1933), pages 1–82.
- [11] P.-L. Lions, *Mathematical Topics in Fluid Mechanics*, Vol. II, Compressible Models, Oxford Science Publications, 1997.
- [12] A. Majda, *Introduction to PDEs and waves for the atmosphere and ocean*. Courant Lecture Notes in Mathematics, 9. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.
- [13] A. Mellet and A. Vasseur, On the isentropic compressible Navier-Stokes equation, *Preprint*, 2005.
- [14] E. Palmén & C.W. Newton, *Atmospheric Circulation Systems*, Academic Press, 1969.
- [15] J. Pedlosky, *Geophysical fluid dynamics*, Springer, 1979.
- [16] R. Temam and M. Ziane, Some mathematical Problems in Geophysical Fluid Dynamics, *Handbook of Mathematical Fluid Dynamics*, vol. III, eds S. Friedlander and D. Serre, 535–657, 2004.



## 2. Rotating fluids. Weak and strong asymptotics

The aim of this second chapter is to present some classical methods to study singular perturbation problems (in the absence of boundary).

For rotating fluids such as oceans, the dynamics is understood as a superposition of waves propagating under both the Coriolis force and the gravity.

We will state two types of mathematical results describing this approximation.

### 2.1. Heuristic study of rotating fluids

For the sake of simplicity, we will present all the arguments on the viscous Saint-Venant model, for which waves are rather simple to describe.

More complex models can be dealt with using the same methods provided that one has a good knowledge of the spectral structure of the propagator.

#### 2.1.1. A simple model

In addition to the Ekman number measuring the influence of viscous effects, the Saint-Venant equations involve two nondimensional parameters

- the Rossby number measuring the influence of rotation  $Ro = U/2|\Omega|L$ ;
- the Froude number measuring the influence of gravity  $Fr = \sqrt{U^2/gD}$ .

For large oceanic motions, both effects are comparable.

$$\begin{cases} \partial_t h + \nabla \cdot (hu) = 0, \\ \partial_t (hu) + \nabla \cdot (hu \otimes u) + \frac{1}{\varepsilon} \omega(hu)^\perp + \frac{1}{\varepsilon^2} \nabla \frac{h^2}{2} = \nu \Delta u. \end{cases}$$

With such a scaling, we expect the fluctuations of the water height to be of the order of  $\varepsilon$

$$\begin{cases} \partial_t \eta + \frac{1}{\varepsilon} \nabla \cdot ((1 + \varepsilon \eta)u) = 0, \\ \partial_t u + u \cdot \nabla u + \frac{1}{\varepsilon} \omega u^\perp + \frac{1}{\varepsilon} \nabla \eta + \nabla \frac{\eta^2}{2} = \frac{\nu}{1 + \varepsilon \eta} \Delta u. \end{cases} \quad (2.1)$$

which can be rewritten in a more abstract way

$$\partial_t U + \frac{1}{\varepsilon} L U + Q(U) = 0,$$

where  $L$  is a skew-symmetric (pseudo)-differential operator.

#### 2.1.2. The linear propagator

For small  $\varepsilon$ , the dynamics should be dominated by the linear propagation

$$U_\varepsilon \sim \exp\left(-\frac{tL}{\varepsilon}\right) V_0$$

which depends crucially on the spectral structure of  $L$ .

For instance, we have here

$$L : \begin{pmatrix} \eta \\ u \end{pmatrix} \mapsto \begin{pmatrix} \nabla \cdot u \\ \omega u^\perp + \nabla \eta \end{pmatrix}.$$

Neglecting the variations of the Coriolis parameter  $\omega$ , which is relevant at mid-latitudes, we can express  $L$  as a Fourier multiplier

$$L_k = \begin{pmatrix} 0 & ik_1 & ik_2 \\ ik_1 & 0 & -\omega \\ ik_2 & \omega & 0 \end{pmatrix},$$

which can be diagonalized in orthonormal basis :

$$L_k = P_k \begin{pmatrix} 0 & 0 & 0 \\ 0 & i\sqrt{k_1^2 + k_2^2 + \omega^2} & 0 \\ 0 & 0 & -i\sqrt{k_1^2 + k_2^2 + \omega^2} \end{pmatrix} P_k^{-1}.$$

We therefore expect the oceanic motion to decompose as the sum of a geostrophic motion, corresponding to the non oscillating component, and of Poincaré waves which are either gravity waves (for large wave numbers  $(k_1, k_2)$ ) or rotating waves (for small  $k$ ).

The way waves propagate is then determined by the **geometry of the domain**.

If  $\mathcal{D} = \mathbf{T}^2$ , the spectrum of  $L$  is discrete : waves oscillate endlessly.

If  $\mathcal{D} = \mathbf{R}^2$ , the spectrum of  $L$  is continuous : a nonstationary phase argument shows that waves disperse and that the energy carried by these waves converge locally to zero.

### 2.1.3. The nonlinear coupling

The effect of the nonlinear coupling is observed on longer time scales. Conjugating the Saint-Venant equations with Coriolis force by the group associated to  $L$

$$\partial_t \exp\left(\frac{t}{\varepsilon} L\right) U_\varepsilon + \exp\left(\frac{t}{\varepsilon} L\right) Q(U_\varepsilon) = 0,$$

we indeed see that the time derivative of the filtered unknown  $V_\varepsilon = \exp\left(\frac{t}{\varepsilon} L\right) U_\varepsilon$  is of order 1, meaning that it undergoes non negligible variations only on macroscopic time scales.

In the physical literature, this slow dynamics is usually described by a system of **envelope equations**, which is obtained by a careful study of possible resonances.

## 2.2. Compensated compactness and weak convergence

If we are only interested in describing the mean motion (i.e. the non-oscillating component), a suitable tool is weak convergence, insofar as it does not capture oscillating behaviours

$$\sin\left(\frac{t}{\varepsilon}\right) \rightharpoonup 0.$$

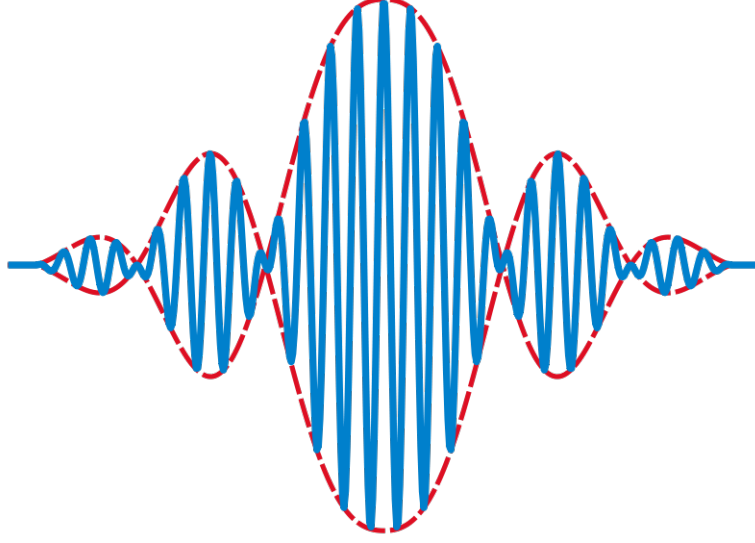


Figure 2.1: Slow dynamics of waves

### 2.2.1. The constraint equation

Provided that we are able to establish convenient bounds on  $U_\varepsilon$ , up to extraction of a subsequence, we have some weak convergence

$$U_\varepsilon \rightharpoonup \bar{U}.$$

We further have

$$LU_\varepsilon = -\varepsilon \partial_t U_\varepsilon - \varepsilon Q(U_\varepsilon) \rightarrow 0$$

in the sense of distributions, so that the limit vector field  $\bar{U}$  satisfies the constraint

$$L\bar{U} = 0.$$

For the viscous Saint-Venant equations with Coriolis force, uniform bounds on  $(\eta_\varepsilon, u_\varepsilon)$  come from the energy estimate

$$\int (1 + \varepsilon \eta_\varepsilon) |u_\varepsilon|^2(t, x) dx + \int |\eta_\varepsilon|^2(t, x) dx + 2\nu \int_0^t \int |\nabla u_\varepsilon|^2(s, x) dx ds \leq 2\mathcal{E}_0.$$

By Sobolev's embeddings, we then have

$$\begin{aligned} \int |u_\varepsilon|^2 dx &= \int (1 + \varepsilon \eta_\varepsilon) |u_\varepsilon|^2 dx + \varepsilon \int \eta_\varepsilon |u_\varepsilon|^2 dx \\ &\leq 2\mathcal{E}_0 + \varepsilon \|\eta_\varepsilon\|_{L^2} \|u_\varepsilon\|_{L^4}^2 \leq 2\mathcal{E}_0 + \varepsilon \|\eta_\varepsilon\|_{L^2} \|u_\varepsilon\|_{L^2} \|\nabla u_\varepsilon\|_{L^2} \\ &\leq 2\mathcal{E}_0 + C_0 \varepsilon (\|u_\varepsilon\|_{L^2}^2 + \|\nabla u_\varepsilon\|_{L^2}^2) \end{aligned}$$

from which we deduce that, up to extraction of a subsequence

$$\begin{aligned} u_\varepsilon &\rightharpoonup \bar{u} \text{ weakly in } L_{loc}^2(H^1) \\ \eta_\varepsilon &\rightharpoonup \bar{\eta} \text{ weakly in } L^\infty(L^2) \\ m_\varepsilon = (1 + \varepsilon \eta_\varepsilon) u_\varepsilon &\rightharpoonup \bar{u} \text{ weakly in } L_{loc}^2(L^p) \end{aligned}$$

for any  $p < 2$ .

Taking limits (in the sense of distributions) in the Saint-Venant equations (2.1), we get the constraint

$$\nabla \cdot \bar{u} = 0, \quad \omega \bar{u}^\perp + \nabla \bar{\eta} = 0,$$

referred to as **geostrophic constraint**, and equivalent to

$$\bar{u} = \frac{1}{\omega} \nabla^\perp \bar{\eta}$$

if  $\omega$  is constant.

### 2.2.2. Regularity of the geostrophic motion

In order to obtain the equations governing the evolution of  $(\bar{\eta}, \bar{u})$ , the idea is to project the Saint-Venant equations on  $\text{Ker} L$ , that is in the space of geostrophic motions.

Let us first **describe this projection**  $\Pi$ . Given  $V = (\eta, u) \in L^2$ ,  $\Pi V = (\bar{\eta}, \bar{u})$  is such that

$$\bar{u} = \frac{1}{\omega} \nabla^\perp \bar{\eta}$$

and for any  $\bar{\rho} \in H^1$ ,

$$\int \left( (\eta - \bar{\eta}) \bar{\rho} + (u - \bar{u}) \cdot \frac{1}{\omega} \nabla^\perp \bar{\rho} \right) dx = 0.$$

Integrating by parts, we get - as there is no boundary -

$$\int \left( (\eta - \bar{\eta}) \bar{\rho} - \frac{1}{\omega} \nabla^\perp \cdot (u - \frac{1}{\omega} \nabla^\perp \bar{\eta}) \bar{\rho} \right) dx = 0,$$

from which we deduce that

$$(I - \frac{1}{\omega^2} \Delta) \bar{\eta} = \eta - \frac{1}{\omega} \nabla^\perp \cdot u.$$

We end up with the following formula

$$\begin{aligned} \bar{\eta} &= (I - \frac{1}{\omega^2} \Delta)^{-1} (\eta - \frac{1}{\omega} \nabla^\perp \cdot u), \\ \bar{u} &= \frac{1}{\omega} \nabla^\perp (I - \frac{1}{\omega^2} \Delta)^{-1} (\eta - \frac{1}{\omega} \nabla^\perp \cdot u). \end{aligned}$$

Using the explicit form of  $\Pi$ , we get some **regularity with respect to space variables** on the geostrophic component of the motion

$$(\bar{\eta}_\varepsilon, \bar{m}_\varepsilon) = \Pi(\eta_\varepsilon, m_\varepsilon).$$

We indeed have  $(\eta_\varepsilon)$  uniformly bounded in  $L_{loc}^2(L^2)$  and

$$m_\varepsilon = u_\varepsilon + \varepsilon \eta_\varepsilon u_\varepsilon$$

with  $(u_\varepsilon)$  uniformly bounded in  $L_{loc}^2(H^1)$  and  $\varepsilon \eta_\varepsilon u_\varepsilon \rightarrow 0$  strongly in  $L_{loc}^2(L^p)$ .

We therefore conclude that, up to a small remainder which converges strongly to zero in  $L_{loc}^2(L^p)$ ,  $(\bar{\eta}_\varepsilon, \bar{m}_\varepsilon)$  is uniformly bounded in  $L_{loc}^2(H^2 \times H^1)$ .

In the analysis of mixed hyperbolic/parabolic systems, such partial regularity and hypoellipticity results are important issues.

Applying  $\Pi$  to the Saint-Venant equations (2.1), we will further obtain some **regularity with respect to time**. Indeed, as  $L$  is skew-symmetric,  $\Pi L = 0$  and

$$\partial_t \Pi \begin{pmatrix} \eta_\varepsilon \\ m_\varepsilon \end{pmatrix} + \Pi \begin{pmatrix} 0 \\ \nabla \cdot (m_\varepsilon \otimes u_\varepsilon) + \nabla \frac{\eta_\varepsilon^2}{2} - \nu \Delta u_\varepsilon \end{pmatrix} = 0. \quad (2.2)$$

The energy inequality provides uniform bounds on the flux terms in  $L^\infty(W^{-1,1})$  and on the dissipation term in  $L^2(H^{-1})$ . Using the continuity of  $\Pi$  in Sobolev spaces, we therefore have

$$\partial_t(\bar{\eta}_\varepsilon, \bar{m}_\varepsilon) \text{ uniformly bounded in } L_{loc}^2(H^{-2}).$$

A standard interpolation argument allows then to conclude that

$$(\bar{\eta}_\varepsilon, \bar{m}_\varepsilon) \text{ is strongly compact in } L_{loc}^2(L^2 + L^p).$$

By continuity of  $\Pi$ , we further have

$$\begin{aligned} (\bar{\eta}_\varepsilon, \bar{m}_\varepsilon) &\rightarrow (\bar{\eta}, \bar{u}) \text{ strongly in } L_{loc}^2(L^2 + L^p), \\ (\eta_\varepsilon - \bar{\eta}_\varepsilon, m_\varepsilon - \bar{m}_\varepsilon) &\rightarrow 0 \text{ weakly in } L_{loc}^2(L^2 + L^p). \end{aligned}$$

### 2.2.3. Compensated compactness

Of course the strong compactness of the geostrophic motion  $(\bar{\eta}_\varepsilon, \bar{m}_\varepsilon)$  is not sufficient a priori to take limits in the nonlinear terms which appear in the geostrophic equation (2.2). The point is that the coupling of oscillating terms could produce a contribution to the mean motion (constructive interferences).

Here the structure of both the linear propagator and the nonlinearity are such that this phenomenon does not occur. More precisely, we can prove that in the sense of distributions,

$$\Pi \begin{pmatrix} 0 \\ \nabla \cdot (m_\varepsilon \otimes u_\varepsilon) + \nabla \frac{\eta_\varepsilon^2}{2} \end{pmatrix} \rightarrow \Pi \begin{pmatrix} 0 \\ \nabla \cdot (\bar{u} \otimes \bar{u}) + \nabla \frac{\bar{\eta}^2}{2} \end{pmatrix}.$$

Let us first rewrite the nonlinear term in convenient form

$$\Pi \begin{pmatrix} 0 \\ \nabla \cdot (m_\varepsilon \otimes u_\varepsilon) + \nabla \frac{\eta_\varepsilon^2}{2} \end{pmatrix} = -(I - \frac{1}{\omega^2} \Delta)^{-1} \begin{pmatrix} 1 \\ \frac{1}{\omega} \nabla^\perp \end{pmatrix} \frac{1}{\omega} \nabla^\perp \otimes \nabla : m_\varepsilon \otimes u_\varepsilon.$$

Note in particular that pressure terms have no contribution.

We further split the motion in its geostrophic and ageostrophic components

$$(\eta_\varepsilon, m_\varepsilon) = (\bar{\eta}_\varepsilon, \bar{m}_\varepsilon) + (\tilde{\eta}_\varepsilon, \tilde{m}_\varepsilon).$$

Because of the strong compactness of  $(\bar{\eta}_\varepsilon, \bar{m}_\varepsilon)$ , we expect that

$$\begin{aligned} \nabla^\perp \otimes \nabla : \tilde{m}_\varepsilon \otimes \tilde{m}_\varepsilon &\rightarrow \nabla^\perp \otimes \nabla : \bar{u} \otimes \bar{u}, \\ \nabla^\perp \otimes \nabla : (\tilde{m}_\varepsilon \otimes \bar{m}_\varepsilon + \bar{m}_\varepsilon \otimes \tilde{m}_\varepsilon) &\rightarrow 0, \end{aligned}$$

as  $\varepsilon \rightarrow 0$ , in the sense of distributions. Note however that the previous quantities are not exactly those appearing in the nonlinear term, and that the products are actually not even defined.

Using the spatial regularity of  $u_\varepsilon$ , we can introduce some spatial regularization

$$m_\varepsilon^\delta = m_\varepsilon * \kappa_\delta, \quad u_\varepsilon^\delta = u_\varepsilon * \kappa_\delta,$$

apply the previous result to the regularized fields, and then prove that the remainder

$$(m_\varepsilon^\delta \otimes m_\varepsilon^\delta - m_\varepsilon \otimes u_\varepsilon)$$

converges to 0 as  $\delta \rightarrow 0$ , uniformly in  $\varepsilon$ .

By definition of  $\Pi$ , the oscillating component  $(\tilde{\eta}_\varepsilon, \tilde{m}_\varepsilon)$  satisfies

$$\begin{aligned}\tilde{\eta}_\varepsilon &= \frac{1}{\omega} \nabla^\perp \cdot \tilde{m}_\varepsilon, \\ \varepsilon \partial_t \tilde{\eta}_\varepsilon + \nabla \cdot \tilde{m}_\varepsilon &= O(\varepsilon), \\ \varepsilon \partial_t \tilde{m}_\varepsilon + \omega \tilde{m}_\varepsilon^\perp + \nabla \tilde{\eta}_\varepsilon &= O(\varepsilon)\end{aligned}\tag{2.3}$$

The **compensated compactness argument** relies then on a simple algebraic identity

$$\begin{aligned}\nabla \cdot (\tilde{m} \otimes \tilde{m}) &= \tilde{m} \nabla \cdot \tilde{m} + \tilde{m}^\perp \nabla^\perp \cdot \tilde{m} + \nabla \frac{|\tilde{m}|^2}{2} \\ &= \tilde{m}(-\varepsilon \partial_t \tilde{\eta} + O(\varepsilon)) + \tilde{\eta}(-\varepsilon \partial_t \tilde{m} - \nabla \tilde{\eta} + O(\varepsilon)) + \nabla \frac{|\tilde{m}|^2}{2} \\ &= -\varepsilon \partial_t (\tilde{\eta} \tilde{m}) + \nabla \frac{|\tilde{m}|^2 - \tilde{\eta}^2}{2} + O(\varepsilon)\end{aligned}$$

and

$$\nabla^\perp \otimes \nabla \cdot (\tilde{m} \otimes \tilde{m}) = -\varepsilon \partial_t \nabla^\perp \cdot (\tilde{\eta} \tilde{m}) + O(\varepsilon).$$

Once again, making this formal computation rigorous requires to introduce regularized quantities

$$\tilde{m}_\varepsilon^\delta = \tilde{m}_\varepsilon * \kappa_\delta, \quad \tilde{\eta}_\varepsilon^\delta = \tilde{\eta}_\varepsilon * \kappa_\delta.$$

Taking limits as  $\delta \rightarrow 0$ , we get finally

$$\partial_t \bar{u} - \frac{1}{\omega} \nabla^\perp (I - \frac{1}{\omega^2} \Delta)^{-1} \nabla^\perp \cdot (\bar{u} \nabla^\perp \cdot \bar{u} - \nu \Delta \bar{u}) = 0.$$

## 2.3. Filtering methods and strong convergence

Describing the departure from geostrophy, namely the evolution of waves, requires a stronger notion of convergence. A natural idea is therefore to build suitable approximate solutions, and then to use some stability result based for instance on the energy inequality to control the accuracy of the approximation.

### 2.3.1. Filtering the oscillations

The slow dynamics of waves (which shall be considered in order to have a good approximation) is obtained from the Saint-Venant equations once the fast time oscillations have been removed.

A classical method to do that is to conjugate the system by the group associated to the singular perturbation

$$\partial_t \exp\left(\frac{t}{\varepsilon} L\right) U_\varepsilon + \exp\left(\frac{t}{\varepsilon} L\right) Q(U_\varepsilon) = 0.$$

Defining

$$V_\varepsilon = \exp\left(\frac{t}{\varepsilon}L\right) U_\varepsilon,$$

we have formally

$$\partial_t V_\varepsilon + \exp\left(\frac{t}{\varepsilon}L\right) Q\left(\exp\left(-\frac{t}{\varepsilon}L\right) V_\varepsilon\right) = 0.$$

Note that, for weak solutions, this formulation does not make sense in general. But this does not matter as far as we only want to describe a formal asymptotics and then to prove that the solution of this formal asymptotics is close to  $U_\varepsilon$ .

Taking (even formal) limits in the filtered equation requires structural assumptions on the nonlinearity  $Q$ , but also a precise description of the spectrum of  $L$ .

- If  $L$  has only continuous spectrum, we expect to be able to prove some dispersion estimate using some non stationary phase argument (Strichartz or Mourre estimate), so that we should have the following convergence

$$\exp\left(\frac{t}{\varepsilon}L\right) Q\left(\exp\left(-\frac{t}{\varepsilon}L\right) V_\varepsilon\right) \rightarrow 0 \text{ strongly on any compact subset of } \mathcal{D}.$$

- If  $L$  has purely discrete spectrum, we introduce the spectral projectors  $\Pi_\lambda$  on  $\text{Ker}(L - i\lambda I)$  and the decomposition

$$\exp\left(\frac{t}{\varepsilon}L\right) Q\left(\exp\left(-\frac{t}{\varepsilon}L\right) V_\varepsilon\right) = \sum_\lambda e^{i\lambda\frac{t}{\varepsilon}} \Pi_\lambda Q\left(\sum_\mu e^{-i\mu\frac{t}{\varepsilon}} \Pi_\mu V_\varepsilon\right).$$

For the sake of simplicity, we only consider the quadratic part  $\mathcal{B}$  of  $Q$  : easy generalizations of the method can be obtained to deal with other nonlinearities. Here, we have then to study the limit of the sum

$$\sum_{\lambda, \mu, \tilde{\mu}} e^{i(\lambda - \mu - \tilde{\mu})\frac{t}{\varepsilon}} \Pi_\lambda \mathcal{B}(\Pi_\mu V_\varepsilon, \Pi_{\tilde{\mu}} V_\varepsilon)$$

where  $V_\varepsilon$  is more or less strongly compact.

- If  $\lambda - \mu - \tilde{\mu} \neq 0$ , it can be proved - integrating by parts - that the corresponding contribution is negligible;
- If  $\lambda - \mu - \tilde{\mu} = 0$ , one has a resonance (constructive interference).

The formal limit is therefore

$$\sum_{\lambda=\mu+\tilde{\mu}} \Pi_\lambda \mathcal{B}(\Pi_\mu V, \Pi_{\tilde{\mu}} V).$$

**Remark 2.1.** *Note that the convergence here is only a weak convergence. In order to get a strong convergence, we have to add a small corrector.*

Let  $V_0$  satisfy

$$\partial_t V_0 + \sum_{\lambda=\mu+\tilde{\mu}} \Pi_\lambda \mathcal{B}(\Pi_\mu V_0, \Pi_{\tilde{\mu}} V_0) = 0 \tag{2.4}$$

and define

$$V_1 = \sum_{\lambda \neq \mu + \tilde{\mu}} \frac{i}{\lambda - \mu - \tilde{\mu}} e^{i(\lambda - \mu - \tilde{\mu})\frac{t}{\varepsilon}} \Pi_\lambda \mathcal{B}(\Pi_\mu V_0, \Pi_{\tilde{\mu}} V_0).$$

Then,

$$\partial_t(V_0 + \varepsilon V_1) + \exp\left(\frac{t}{\varepsilon}L\right) \mathcal{B}\left(\exp\left(-\frac{t}{\varepsilon}L\right)(V_0 + \varepsilon V_1), \exp\left(-\frac{t}{\varepsilon}L\right)(V_0 + \varepsilon V_1)\right) = o(1).$$

The envelope equations (2.4) are well-posed if the set of resonances is not too big, or if the structure of the spectral projectors allows to define a functional framework where the “convolution” makes sense. In these cases, suitable truncations enable us to also define the corrector  $V_1$  and to prove the convergence statement.

### 2.3.2. Poincaré waves

We have seen that the propagation under both the Coriolis force and the gravity can be expressed thanks to the Fourier multiplier

$$L_k = \begin{pmatrix} 0 & ik_1 & ik_2 \\ ik_1 & 0 & -\omega \\ ik_2 & \omega & 0 \end{pmatrix}.$$

Departures from geostrophy are then due to Poincaré waves of symbol

$$\pm i\sqrt{k_1^2 + k_2^2 + \omega^2}.$$

Let us recall that, depending on the relative size of the (non dimensional) parameters  $k$  and  $\omega$ , we actually have three regimes of propagation : a rotating regime if  $k \ll \omega$ , a gravity regime if  $k \gg \omega$  and an intermediate regime :

Frequency $\sigma$	$0.1f$	$f$		$10f$	$N$	$10N$
Period (typical value)	1 week	1 day	6 hr	1hr	10 min	1min
Regime	Quasi-geostrophic	Rotating		Hydrostatic nonrotating		Potential flow
Vertical structure	Evanescent	Evanescent	Wave	Wave	Wave	Evanescent

Figure 2.2: Regimes of propagation

- If  $\mathcal{D} = \mathbf{R}^2$ , solutions to the wave equations

$$\varepsilon^2 \partial_{tt} \psi - \Delta \psi + \omega^2 \psi = 0$$

satisfy the Strichartz estimate

$$\|\psi\|_{L_t^q(L_x^r)} \leq \varepsilon^{1/q} \|\psi_0\|_{H_x^s} \text{ if } \frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2} \text{ and } q, r \geq 2, \quad r \neq \infty.$$

We are therefore able to prove that the ageostrophic component of the motion converges locally strongly to 0.



- If  $\mathcal{D} = \mathbf{T}^2$ , as the eigenvalues of the singular perturbation  $L$  are defined as the roots of

$$\lambda^3 + \lambda(k_1^2 + k_2^2 + \omega^2) = 0, \quad k \in \mathbf{Z}^2,$$

the occurrence of a resonant triad is controlled by the cancellation of some polynomial in  $\omega$ .

As a polynomial has a finite number of zeroes or is identically zero, one can easily prove that for almost all  $\omega$  (more precisely for all  $\omega$  except a countable number) only trivial resonances occur.

Combining this result with the fact that the geostrophic motion decouples from the oscillating component (which is the main result of the previous part), we get a system of envelope equations of the type

$$\partial_t \Pi_0 V + \Pi_0 \mathcal{B}(\Pi_0 V, \Pi_0 V) - \nu \Pi_0 \begin{pmatrix} 0 \\ \Delta \end{pmatrix} \Pi_0 V = 0,$$

$$\partial_t \Pi_\lambda V + 2\Pi_\lambda \mathcal{B}(\Pi_\lambda V, \Pi_0 V) - \nu \Pi_\lambda \begin{pmatrix} 0 \\ \Delta \end{pmatrix} \Pi_\lambda V = 0, \quad \lambda \neq 0,$$

which is globally well-posed.

In both cases, we are then able to build approximate solutions.

### 2.3.3. The modulated energy

It remains then to prove that these approximate solutions remain really close to  $U_\varepsilon = (\eta_\varepsilon, u_\varepsilon)$ .

Classical energy methods are not completely suitable to do that, insofar as they deal with hilbertian norms. We have indeed to introduce some slight modifications on account of the  $(1 + \varepsilon\eta_\varepsilon)$  factor in the kinetic energy. Define the modulated energy

$$\delta \mathcal{E}_\varepsilon = \frac{1}{2} \int \left( (1 + \varepsilon\eta_\varepsilon)(u_\varepsilon - u_{app})^2 + (\eta_\varepsilon - \eta_{app})^2 \right) dx.$$

What we will actually establish is the following inequality :

$$\begin{aligned} \delta \mathcal{E}_\varepsilon(t) &+ \nu \int_0^t \int |\nabla u_\varepsilon - \nabla u_{app}|^2 ds dx \\ &\leq \delta \mathcal{E}_\varepsilon(0) - \frac{1}{2} \int_0^t \int \left( \nabla \cdot u_{app} (\eta_\varepsilon - \eta_{app})^2 + \nabla u_{app} \otimes (1 + \varepsilon\eta_\varepsilon)(u_\varepsilon - u_{app})^{\otimes 2} \right) dx ds \\ &+ \int_0^t \int (1 + \varepsilon\eta_\varepsilon)(u_{app} - u_\varepsilon) \\ &\quad \times \left( \partial_t u_{app} + \frac{1}{\varepsilon} (\omega u_{app}^\perp + \nabla \eta_{app}) - \nu \frac{\Delta u_{app}}{1 + \varepsilon\eta_\varepsilon} + (u_{app} \cdot \nabla) u_{app} \right) dx ds \\ &+ \int (\eta_{app} - \eta_\varepsilon) \left( \partial_t \eta_{app} + \frac{1}{\varepsilon} \nabla \cdot u_{app} + \nabla \cdot (\eta_{app} u_{app}) \right) dx ds \end{aligned} \tag{2.5}$$

If  $(\eta_{app}, u_{app})$  is a smooth approximate solution to the Saint-Venant equations with Coriolis force (2.1) (in the sense that the remainder converges strongly to 0 in  $L_t^2(L_x^p)$  for  $p > 2$ ), then the last two terms in the previous stability inequality converge to 0. We therefore get a Gronwall type inequality which shows that the growth of the modulated energy is controlled by the Lipschitz norm of  $u_{app}$ . With a suitable choice of the initial data, we have then the expected convergence.

Establishing the stability inequality (2.5) relies actually on rather simple (though technical) formal computations. Differentiating the modulated energy with respect to  $t$  and using the global conservation of energy as well as the local conservations of mass and momentum, we get

$$\begin{aligned}
& \frac{d}{dt} \delta \mathcal{E}_\varepsilon(t) + \nu \int |\nabla u_\varepsilon - \nabla u_{app}|^2 dx \\
&= \int (1 + \varepsilon \eta_\varepsilon)(u_{app} - u_\varepsilon) \partial_t u_{app} dx + \int (\eta_{app} - \eta_\varepsilon) \partial_t \eta_{app} dx \\
&+ \int (1 + \varepsilon \eta_\varepsilon) u_{app} \left( \frac{1}{\varepsilon} (\omega u_\varepsilon^\perp + \nabla \eta_\varepsilon) - \frac{\nu}{1 + \varepsilon \eta_\varepsilon} \Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon \right) dx \\
&+ \int \eta_{app} \left( \frac{1}{\varepsilon} \nabla \cdot u_\varepsilon + \nabla \cdot \eta_\varepsilon u_\varepsilon \right) dx - \frac{1}{2} \int (u_{app}^2 - 2u_\varepsilon \cdot u_{app}) (\nabla \cdot u_\varepsilon + \varepsilon \nabla \cdot \eta_\varepsilon u_\varepsilon) dx \\
&+ \nu \int (\nabla u_{app} - 2\nabla u_\varepsilon) \cdot \nabla u_{app} dx
\end{aligned}$$

Note that the blue term above comes from the derivative of the  $(1 + \varepsilon \eta_\varepsilon)$  factor in the kinetic energy. Integrating by parts the linear contributions, we obtain

$$\begin{aligned}
& \frac{d}{dt} \delta \mathcal{E}_\varepsilon(t) + \nu \int |\nabla u_\varepsilon - \nabla u_{app}|^2 dx \\
&= \int (1 + \varepsilon \eta_\varepsilon)(u_{app} - u_\varepsilon) \partial_t u_{app} dx + \int (\eta_{app} - \eta_\varepsilon) \partial_t \eta_{app} dx \\
&- \int \left( \frac{1}{\varepsilon} (1 + \varepsilon \eta_\varepsilon) \omega u_{app}^\perp \cdot u_\varepsilon + \frac{1}{\varepsilon} (\eta_\varepsilon + \frac{\varepsilon}{2} \eta_\varepsilon^2) \nabla \cdot u_{app} + \nu u_{app} \cdot \Delta u_\varepsilon \right) dx \\
&+ \int (1 + \varepsilon \eta_\varepsilon) u_{app} \cdot (u_\varepsilon \cdot \nabla u_\varepsilon) dx \\
&- \int (1 + \varepsilon \eta_\varepsilon) u_\varepsilon \cdot \left( \frac{1}{\varepsilon} \nabla \eta_{app} - \frac{1}{2} \nabla (u_{app}^2 - 2u_\varepsilon \cdot u_{app}) \right) dx \\
&- \nu \int (u_{app} - u_\varepsilon) \cdot \Delta u_{app} dx + \nu \int u_{app} \cdot \Delta u_\varepsilon dx
\end{aligned}$$

Gathering the red terms above together, we get

$$\begin{aligned}
& \frac{d}{dt} \delta \mathcal{E}_\varepsilon(t) + \nu \int |\nabla u_\varepsilon - \nabla u_{app}|^2 dx \\
&= \int (1 + \varepsilon \eta_\varepsilon)(u_{app} - u_\varepsilon) \left( \partial_t u_{app} + \frac{1}{\varepsilon} \omega u_{app}^\perp - \frac{\nu}{1 + \varepsilon \eta_\varepsilon} \Delta u_{app} + u_\varepsilon \cdot \nabla u_{app} \right) dx \\
&+ \int (\eta_{app} - \eta_\varepsilon) \partial_t \eta_{app} dx - \int \left( \frac{1}{\varepsilon} \eta_\varepsilon \nabla \cdot u_{app} + \frac{1}{2} \eta_\varepsilon^2 \nabla \cdot u_{app} \right) dx \\
&- \int (1 + \varepsilon \eta_\varepsilon)(u_\varepsilon - u_{app}) \cdot \frac{1}{\varepsilon} \nabla \eta_{app} dx - \int u_{app} \cdot \frac{1}{\varepsilon} \nabla \eta_{app} dx - \int \eta_\varepsilon u_{app} \cdot \nabla \eta_{app}
\end{aligned}$$

Integrations by parts lead then to

$$\begin{aligned}
& \frac{d}{dt} \delta \mathcal{E}_\varepsilon(t) + \nu \int |\nabla u_\varepsilon - \nabla u_{app}|^2 dx \\
&= \int (1 + \varepsilon \eta_\varepsilon)(u_{app} - u_\varepsilon) \left( \partial_t u_{app} + \frac{1}{\varepsilon} \omega u_{app}^\perp + \frac{1}{\varepsilon} \nabla \eta_{app} - \frac{\nu}{1 + \varepsilon \eta_\varepsilon} \Delta u_{app} + u_\varepsilon \cdot \nabla u_{app} \right) dx \\
&+ \int (\eta_{app} - \eta_\varepsilon) \left( \partial_t \eta_{app} + \frac{1}{\varepsilon} \nabla \cdot u_{app} + \nabla \cdot (\eta_\varepsilon u_{app}) \right) dx
\end{aligned}$$

which is exactly the expected identity. Note that the above calculations can be made rigorous by considering smooth approximate solutions, then taking limits in the final inequality.

## References

- [1] A. Babin, A. Mahalov, and B. Nicolaenko, Global splitting, integrability and regularity of 3D Euler and Navier–Stokes equations for uniformly rotating fluids, *European Journal of Mechanics*, **15** (1996), pages 291–300.
- [2] A. Babin, A. Mahalov, and B. Nicolaenko, Resonances and regularity for Boussinesq equations, *Russian Journal of Mathematical Physics*, **4** (1996), pages 417–428.
- [3] A. Babin, A. Mahalov, and B. Nicolaenko, Global regularity of 3D rotating Navier–Stokes equations for resonant domains, *Indiana University Mathematics Journal*, **48** (1999), pages 1133–1176.
- [4] F. Charve, Global well-posedness and asymptotics for a geophysical fluid system, *Communications in Partial Differential Equations*, **29** (2004), pages 1919–1940.
- [5] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Basics of Mathematical Geophysics*, to appear in *Oxford University Press*, 2006.
- [6] I. Gallagher, Applications of Schochet’s methods to parabolic equations, *Journal de Mathématiques Pures et Appliquées*, **77** (1998), pages 989–1054.
- [7] I. Gallagher & L. Saint-Raymond, On the influence of the Earth’s rotation on geophysical flows, *Handbook of Mathematical Fluid Dynamics*, S. Friedlander and D. Serre Editors Vol 4, Chapter 5, 201–329, 2007.
- [8] H.P. Greenspan, *The theory of rotating fluids*, Cambridge monographs on mechanics and applied mathematics, 1969.
- [9] E. Grenier, Oscillatory perturbations of the Navier–Stokes equations. *Journal de Mathématiques Pures et Appliquées*, **76** (1997), pages 477–498.
- [10] S. Schochet, Fast singular limits of hyperbolic PDEs. *Journal of Differential Equations* **114** (1994), pages 476 – 512.
- [11] W. Thomson (Lord Kelvin), On gravitational oscillations of rotating water. *Proc. Roy. Soc. Edinburgh* **10**, 1879, pages 92–100.

### 3. Boundary layers and Ekman pumping

In this third chapter, we intend to go further in the study of oceanic motions, investigating the influence of boundaries (which has been neglected in the previous chapter).

Such effects are crucial for the understanding of the global oceanic circulation, both at the surface for the transmission of wind energy (*Ekman layers*), and at lateral boundaries for the formation of deep currents (*Munk and Stommel layers*).

#### 3.1. Scalings and asymptotic expansion for the Ekman layers

For the sake of simplicity, we will focus on one type of boundary layers and will further consider a very simple model. However the mathematical methods presented here can be reproduced in many situations.

##### 3.1.1. A simple example

We are interested here in describing the boundary layers near the bottom and near the surface of the oceans, in particular to understand the influence of wind forcing. We have therefore to consider the vertical structure of the flow, and the shallow-water approximation is not relevant anymore :

$$\begin{aligned}\partial_t u + (u \cdot \nabla)u + \nabla p &= \nu_h \Delta_h u + \nu_z \partial_{zz} u + g - \omega \wedge u, \\ \nabla \cdot u &= 0, \\ u|_B &= 0, \quad n \cdot u|_\Sigma = 0 \text{ and } n \cdot (p - \nu(\nabla u + (\nabla u)^T))|_\Sigma = \tau.\end{aligned}$$

On the other hand, we will neglect both the bottom topography and the oscillations of the free surface

$$\begin{aligned}B &= \{z = 0\} \text{ (flat bottom approximation)} \\ \Sigma &= \{z = 1\} \text{ (rigid lid approximation)}\end{aligned}$$

Since we neglect the variations of the water height, no gravity wave can propagate, which corresponds to a purely rotating regime.

The dispersion relation of rotating waves is however modulated by the effect of *PHYSICAL TERMS!vertical stratification*

$$\lambda_k = \pm i\omega \sqrt{\frac{k_3}{|k|}}.$$

A more relevant model should account for both gravity and stratification.

##### 3.1.2. Balance between viscous and rotating terms

Taking formal limits in the Navier-Stokes equations with singular Coriolis force and vanishing vertical viscosity

$$\begin{aligned}\partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla)u_\varepsilon + \nabla p_\varepsilon &= \nu_h \Delta_h u_\varepsilon + \varepsilon \partial_{zz} u_\varepsilon - \frac{1}{\varepsilon} \omega \wedge u_\varepsilon, \\ \nabla \cdot u_\varepsilon &= 0.\end{aligned}$$

we expect the mean motion to be governed by the two-dimensional Navier-Stokes equations

$$\begin{aligned}\partial_t \bar{u}_h + (\bar{u}_h \cdot \nabla) \bar{u}_h + \nabla_h p &= \nu_h \Delta_h \bar{u}_h \\ \nabla_h \cdot \bar{u}_h &= 0 \\ \bar{u}_3 &= 0.\end{aligned}$$

Nevertheless these last equations are not compatible with the **horizontal boundary conditions**

$$u_h|_B = 0 \text{ and } \varepsilon \partial_z u_h|_\Sigma = \tau. \quad (3.1)$$

We therefore expect these boundary conditions to be restored by some boundary layers, the size of which is determined by a balance between the viscous term (compatible with stopping and slipping conditions) and the singular perturbation

$$\frac{1}{\varepsilon}(\omega \wedge w_\varepsilon)_h - \varepsilon \partial_{zz} w_{\varepsilon,h} \sim 0,$$

which can be rewritten in scaled variables  $Z = z/\varepsilon$  or  $Z = (1 - z)/\varepsilon$

$$\partial_{ZZ} w_h - \omega w_h^\perp = 0,$$

with the additional constraint in order that the correction remains localized in the vicinity of boundary that

$$w_h(Z) \rightarrow 0 \text{ as } Z \rightarrow \infty.$$

We finally obtain

$$\begin{aligned}w_1 + iw_2 &= (w_1 + iw_2)|_{Z=0} \exp\left(-\frac{1+i}{\sqrt{2}}\sqrt{\omega}Z\right), \\ w_1 - iw_2 &= (w_1 - iw_2)|_{Z=0} \exp\left(-\frac{1-i}{\sqrt{2}}\sqrt{\omega}Z\right),\end{aligned}$$

where  $w_1|_{Z=0}$  and  $w_2|_{Z=0}$  are defined in terms of  $\tau$  and  $\bar{u}_h$ .

In particular, we have the following boundary estimates for the bottom term

$$\begin{cases} \|w_h^B\|_{L^2([0,1], H^s(\mathcal{D}_h))} \leq C \|\bar{u}_h\|_{H^s(\mathcal{D}_h)} \varepsilon^{1/2}, \\ \|\partial_z w_h^B\|_{L^2([0,1], H^s(\mathcal{D}_h))} \leq C \|\bar{u}_h\|_{H^s(\mathcal{D}_h)} \varepsilon^{-1/2}, \end{cases}$$

and for the surface term

$$\begin{cases} \|w_h^\Sigma\|_{L^2([0,1], H^s(\mathcal{D}_h))} \leq C \|\tau\|_{H^s(\mathcal{D}_h)} \varepsilon^{1/2}, \\ \|\partial_z w_h^\Sigma\|_{L^2([0,1], H^s(\mathcal{D}_h))} \leq C \|\tau\|_{H^s(\mathcal{D}_h)} \varepsilon^{-1/2}, \end{cases}$$

Note that the explicit formula for  $w_h$  gives also  $L^p$  estimates.

### 3.1.3. Resulting vertical motion

By construction, for smooth  $\tau$  and  $\bar{u}$ ,  $\bar{u} + w_\varepsilon^\Sigma + w_\varepsilon^B$  satisfies approximately the evolution equation

$$\partial_t u + (u \cdot \nabla) u + \nabla p = \nu_h \Delta_h u + \varepsilon \partial_{zz} u - \frac{1}{\varepsilon} \omega \wedge u,$$

as well as the boundary conditions

$$u|_B = 0, \quad u_3|_\Sigma = 0 \text{ and } \varepsilon \partial_z u_h|_\Sigma = \tau.$$

However the **incompressibility condition** is not satisfied in general

$$\begin{aligned}\partial_1 w_1 + \partial_2 w_2 &= \frac{1}{2}(\partial_1 - i\partial_2)(w_1 + iw_2)|_{Z=0} \exp\left(-\frac{1+i}{\sqrt{2}}\sqrt{\omega}Z\right) \\ &\quad + \frac{1}{2}(\partial_1 + i\partial_2)(w_1 - iw_2)|_{Z=0} \exp\left(-\frac{1-i}{\sqrt{2}}\sqrt{\omega}Z\right)\end{aligned}$$

We therefore introduce a vertical correction  $w_3$  such that  $\partial_Z w_3 = \pm \varepsilon \partial_z w_3 = \mp \varepsilon (\partial_1 w_1 + \partial_2 w_2)$

$$\begin{aligned}w_3 &= \mp \frac{\varepsilon}{(1-i)\sqrt{2\omega}} (\nabla_h \cdot w_h|_{Z=0} - i\nabla_h^\perp \cdot w_h|_{Z=0}) \exp\left(\sqrt{\frac{\omega}{2}}(i-1)Z\right) \\ &\quad \mp \frac{\varepsilon}{(1+i)\sqrt{2\omega}} (\nabla_h \cdot w_h|_{Z=0} + i\nabla_h^\perp \cdot w_h|_{Z=0}) \exp\left(-\sqrt{\frac{\omega}{2}}(i+1)Z\right).\end{aligned}$$

Now the vector field  $w$  is divergence-free, but it does not satisfy anymore the **zero mass flux condition**.

$$\begin{aligned}w_{3|Z=0}^B &= -\frac{\varepsilon}{\sqrt{2\omega}} \left( \frac{(\partial_1 - i\partial_2)(\bar{u}_1 + i\bar{u}_2)}{1+i} + \frac{(\partial_1 + i\partial_2)(\bar{u}_1 - i\bar{u}_2)}{1-i} \right) \\ w_{3|Z=0}^\Sigma &= \frac{\varepsilon}{2\omega} (i(\partial_1 - i\partial_2)(\tau_1 + i\tau_2) - i(\partial_1 + i\partial_2)(\tau_1 - i\tau_2))\end{aligned}$$

and there are further exponentially small corrections due to  $w_{3|Z=1/\varepsilon}^B$  and  $w_{3|Z=1/\varepsilon}^\Sigma$ . To restore the zero mass flux condition, we shall then add another small correction  $\delta w$  defined by

$$\begin{aligned}\delta w_3 &= -w_{3|z=0}P(1-z) - w_{3|z=1}P(z), \\ \delta w_h &= \nabla_h \Delta_h^{-1} \nabla_h \cdot (-w_{3|z=0}P'(1-z) + w_{3|z=1}P'(z))\end{aligned}$$

for some function  $P$  such that  $P(0) = 0$  and  $P(1) = 1$ .

The small vertical flux  $\delta w_3$  is responsible for global circulation in the whole domain, of small order but not limited to the boundary layer. That process, called *Ekman pumping*, has a very important effect in the energy balance.

Note that  $\bar{u} + w_\varepsilon^\Sigma + w_\varepsilon^B + \delta w_\varepsilon$  does not satisfy exactly the horizontal boundary conditions (3.1), but the error is smaller by one order of magnitude as the one for  $\bar{u}$ .

Our goal now is to show how the presence of Ekman layers modifies the mean flow.

### 3.2. Two-scale analysis and weak convergence

As far as we are only interested in the mean motion, weak compactness methods seem to be the appropriate tool, provided that we are able to establish convenient uniform bounds.

We have seen in the previous chapter that taking limits in nonlinear terms requires then some compensated compactness argument.

The additional difficulty here is to understand how to deal with boundary conditions.

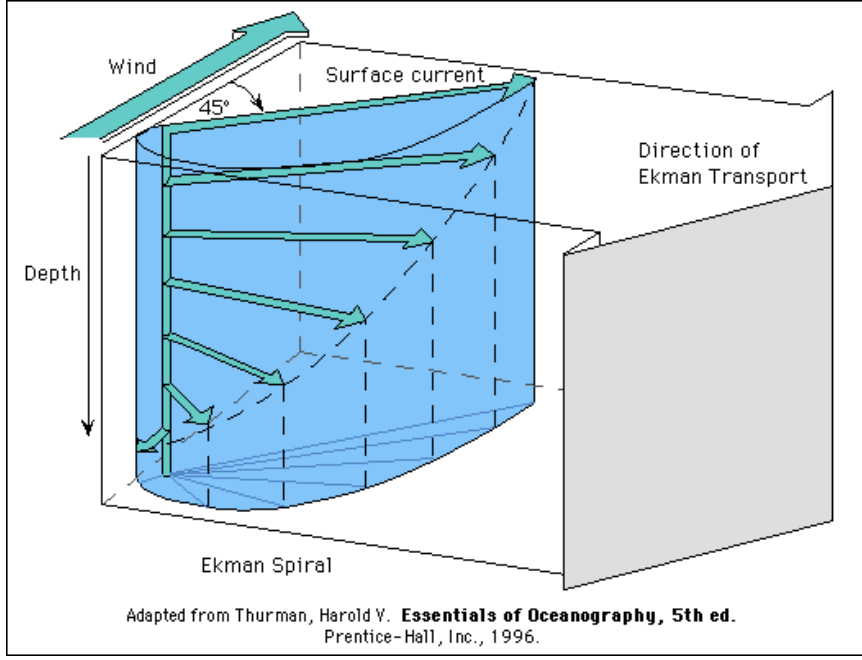


Figure 3.1: Ekman layer

### 3.2.1. Uniform bounds

As usual, the a priori bounds (used in particular to build weak solutions) are inherited from the energy inequality

$$\frac{1}{2} \int |u_\varepsilon|^2 dx + \int_0^t \int (\nu_h |\nabla_h u_\varepsilon|^2 + \varepsilon |\partial_z u_\varepsilon|^2) dx ds \leq \frac{1}{2} \int |u_0|^2 dx + \int_0^t \int u_{\varepsilon,h}|_{z=1} \tau dx_h ds.$$

Together with the trace estimate for divergence free vector fields

$$\varepsilon^{1/2} \|u_{\varepsilon,h}|_{z=1}\|_{L^2(\mathcal{D}_h)}^2 \leq C \varepsilon^{1/2} \|u_{\varepsilon,h}\|_{L^2([0,1] \times \mathcal{D}_h)} \|\partial_z u_{\varepsilon,h}\|_{L^2([0,1] \times \mathcal{D}_h)},$$

it leads to the estimate

$$\begin{aligned} & \frac{1}{2} \int |u_\varepsilon|^2 dx + \int_0^t \int (\nu_h |\nabla_h u_\varepsilon|^2 + \frac{\varepsilon}{2} |\partial_z u_\varepsilon|^2) dx ds \\ & \leq \frac{1}{2} \int |u_0|^2 dx \exp(Ct) + \varepsilon^{-1/4} \int_0^t \|\tau\|_{L^2(\mathcal{D}_h)} \exp(C(t-s)) ds. \end{aligned}$$

In particular, if  $\|\tau\|_{L^2(\mathcal{D}_h)}$  remains finite as  $\varepsilon$  tends to 0, establishing uniform bounds on  $(u_\varepsilon)$  requires to consider a modulated energy inequality which controls typically  $u_\varepsilon - \tilde{w}_\varepsilon$  where  $\tilde{w}_\varepsilon = w_\varepsilon^\Sigma + \delta w_\varepsilon^\Sigma$  is the boundary layer term associated to the surface forcing  $\tau$  and defined in the previous section.

Setting  $v_\varepsilon = u_\varepsilon - \tilde{w}_\varepsilon$ , we get the incompressibility relation

$$\nabla \cdot v_\varepsilon = \nabla \cdot u_\varepsilon - \nabla \cdot (w_\varepsilon^\Sigma + \delta w_\varepsilon^\Sigma) = 0,$$

the identity

$$\begin{aligned} & \partial_t v_\varepsilon + (u_\varepsilon \cdot \nabla) v_\varepsilon + \frac{\omega}{\varepsilon} v_\varepsilon^\perp + \nabla p_\varepsilon - \nu_h \Delta_h v_\varepsilon - \varepsilon \partial_{zz} v_\varepsilon \\ & s = -\partial_t \tilde{w}_\varepsilon - (v_\varepsilon + \tilde{w}_\varepsilon) \cdot \nabla \tilde{w}_\varepsilon + \nu_h \Delta \tilde{w}_\varepsilon + \varepsilon \partial_{zz} \delta w_\varepsilon^\Sigma - \frac{\omega}{\varepsilon} \delta w_\varepsilon^{\Sigma, \perp} \end{aligned}$$

where the red term accounts for the Ekman pumping, and the boundary conditions

$$\begin{aligned} v_{\varepsilon,3}|_{\Sigma} &= v_{\varepsilon,3}|_B = 0, \\ v_{\varepsilon,h}|_B &= 0 \text{ provided that } P'(0) = P'(1) = 0, \\ \varepsilon \partial_z v_{\varepsilon,h}|_{\Sigma} &= \tau \text{ choosing } P \text{ such that } P''(0) = P''(1) = 0. \end{aligned}$$

The energy inequality for  $v_\varepsilon$  then states

$$\begin{aligned} \frac{1}{2} \|v_\varepsilon(t)\|_{L^2}^2 &+ \int_0^t \left( \nu \|\nabla_h v_\varepsilon(s)\|_{L^2}^2 + \varepsilon \|\partial_z v_\varepsilon(s)\|_{L^2}^2 \right) ds \\ &\leq \frac{1}{2} \|v_\varepsilon(0)\|_{L^2}^2 + \int_0^t \|v_\varepsilon(s)\|_{L^2} \left\| \frac{\omega}{\varepsilon} \delta w_\varepsilon^\perp(s) \right\|_{L^2} ds \\ &\quad + \int_0^t \int |v_\varepsilon + \tilde{w}_\varepsilon| \cdot \nabla \tilde{w}_\varepsilon |v_\varepsilon|(s, x) dx ds + o(1) \end{aligned}$$

Note that the blue term above requires a careful treatment involving precised trace estimate. We skip here the arguments providing the suitable trilinear estimates and refer to the convergence proof for very similar computations.

Provided that  $\tau$  is smooth with respect to  $t$  and  $x_h$ , we get uniform a priori bounds on  $v_\varepsilon$  by Gronwall's lemma :

$$\frac{1}{2} \|v_\varepsilon(t)\|_{L^2}^2 + \int_0^t \left( \nu \|\nabla_h v_\varepsilon(s)\|_{L^2}^2 + \varepsilon \|\partial_z v_\varepsilon(s)\|_{L^2}^2 \right) ds \leq C(t). \quad (3.2)$$

### 3.2.2. Weak formulation of the Navier-Stokes equations with boundary conditions

The uniform a priori bounds coming from the previous energy estimate give (up to extraction of a subsequence)

$$\begin{cases} u_\varepsilon \rightharpoonup \bar{u} \text{ weakly in } L^2([0, T] \times \mathcal{D}), \\ \bar{u} \in \text{Ker } L \Leftrightarrow \bar{u} = (\bar{u}_h(x_h), 0) \text{ with } \nabla_h \cdot \bar{u}_h = 0. \end{cases}$$

Our starting point to study the fast rotation limit is then the weak form of the Navier-Stokes-Coriolis equations, and more precisely of their two-dimensional projection :  $\forall \varphi_h \in C_c^\infty(\mathbf{R}^+ \times \omega)$  such that  $\nabla_h \cdot \varphi_h = 0$ ,

$$\begin{aligned} \int u_0 \varphi_h dx &+ \int_0^t \int (u_{\varepsilon,h} \partial_t \varphi_h + u_{\varepsilon,h} \otimes u_{\varepsilon,h} : \nabla_h \varphi_h - \nu_h \nabla_h u_{\varepsilon,h} \cdot \nabla_h \varphi_h) dx ds \\ &= \varepsilon \int_0^t \int (\partial_z u_{\varepsilon,h})|_{z=0} \varphi_h dx_h ds - \int_0^t \int \tau \varphi_h dx_h ds \end{aligned} \quad (3.3)$$

Note that the trace  $(\partial_z u_{\varepsilon,h})|_{z=0}$  does not make sense in general for vector fields  $u_\varepsilon \in L_t^\infty(L_x^2) \cap L_t^2(\dot{H}_x^1)$ . It is actually defined (in a very weak sense) by the equation.

In order to relate the trace  $(\partial_z u_{\varepsilon,h})|_{z=0}$  both to the field  $u_\varepsilon$  and to the trace  $u_{\varepsilon,h}|_{z=0}$ , we will use the more general weak form of the Navier-Stokes-Coriolis equations with



suitable smooth test functions  $\psi$  such that  $\nabla \cdot \psi = 0$  and  $\psi|_B = \psi|_\Sigma = 0$  :

$$\begin{aligned} & \int u_0 \psi dx + \int_0^t \int (u_\varepsilon \cdot \partial_t \psi + u_\varepsilon \otimes u_\varepsilon : \nabla \psi - \nu_h \nabla_h u_\varepsilon \cdot \nabla_h \psi) dx ds \\ & \quad + \int_0^t \int u_\varepsilon \cdot \left( \frac{1}{\varepsilon} \omega \wedge \psi + \varepsilon \partial_{zz} \psi \right) dx ds \\ & = \varepsilon \int_0^t \int (\partial_z u_{\varepsilon,h})|_{z=0} \psi_h|_{z=0} dx_h ds - \int_0^t \int \tau \psi_h|_{z=1} dx_h ds \\ & \quad + \varepsilon \int_0^t \int u_{\varepsilon,h}|_{z=1} \partial_z \psi_h|_{z=1} dx_h ds \end{aligned}$$

### 3.2.3. The trace equation at $z = 0$

To capture the effects of the boundary condition as  $\varepsilon$  tends to 0, we use as test functions a family of solutions to the boundary layer problem, and more precisely to the adjoint boundary layer problem (obtained by changing  $\omega$  in  $-\omega$ , or equivalently  $i$  in  $-i$ ), with  $\varphi_h$  as Dirichlet boundary data. That process is very similar to the two-scale analysis tools introduced by N'Guetseng and Allaire.

By construction of  $w_\varepsilon$  and  $\delta w_\varepsilon$ , we expect all the terms in the weak formulation of the Navier-Stokes-Coriolis equations to be small except

$$\varepsilon \int_0^t \int (\partial_z u_{\varepsilon,h})|_{z=0} w_{\varepsilon,h}|_{z=0} dx_h ds \text{ and } \int_0^t \int u_\varepsilon \cdot \left( \frac{1}{\varepsilon} \omega \wedge \delta w_{\varepsilon,h} \right) dx ds.$$

We should then identify both limits. As  $w_h|_{z=0}$  may be any smooth function of  $x_h$  and  $t$ , this process should allow to take limits in the boundary layer term arising in the mean motion equation

$$\varepsilon \int_0^t \int (\partial_z u_{\varepsilon,h})|_{z=0} w_{\varepsilon,h}|_{z=0} dx_h ds \rightarrow -\sqrt{\frac{\omega}{2}} \int_0^t \int \bar{u}_h \varphi_h dx ds - \int_0^t \int \tau \varphi_h dx_h ds. \quad (3.4)$$

Let us now check that we are indeed able to justify the previous asymptotics. Taking limits in linear terms is rather straightforward

$$\begin{aligned} & \int u_0 \tilde{w}_\varepsilon dx + \int_0^t \int u_\varepsilon \cdot \left( \partial_t \tilde{w}_\varepsilon + \frac{1}{\varepsilon} \omega \delta w_\varepsilon^\perp + \varepsilon \partial_{zz} \delta w_\varepsilon + \nu_h \Delta_h \tilde{w}_\varepsilon \right) dx ds \\ & \sim \int_0^t \int u_\varepsilon \cdot \frac{1}{\varepsilon} (\omega \delta w_\varepsilon^\perp) dx ds \sim - \int_0^t \int \omega \bar{u}_h \cdot \nabla_h^\perp \Delta_h^{-1} w_{3|B} dx ds \\ & \sim \int_0^t \int \sqrt{2\omega} u_h \cdot \varphi_h dx ds \end{aligned}$$

since the  $L^2([0, 1], H^s(\mathcal{D}_h))$  norm of  $\tilde{w}_\varepsilon$  and the  $H^s([0, 1] \times \mathcal{D}_h)$  norm of  $\delta w_\varepsilon$  converge to 0 in  $L^\infty(\mathbf{R}^+)$  as  $\varepsilon \rightarrow 0$  (for any fixed  $s \geq 0$ ).

Dealing with non linear terms is much more complicated. We have

$$\begin{aligned} \int_0^t \int u_\varepsilon \otimes u_\varepsilon : \nabla \tilde{w}_\varepsilon dx ds & = \int_0^t \int u_{\varepsilon,3}^2 \partial_z w_{\varepsilon,3} dx ds + \int_0^t \int u_{\varepsilon,h} \otimes u_{\varepsilon,h} : \nabla_h \tilde{w}_{\varepsilon,h} dx ds \\ & \quad + \int_0^t \int u_{\varepsilon,3} u_{\varepsilon,h} \cdot \partial_z \tilde{w}_{\varepsilon,h} dx ds + \int_0^t \int u_{\varepsilon,3} (u_{\varepsilon,h} \cdot \nabla_h) \tilde{w}_{\varepsilon,3} dx ds \end{aligned}$$

Because of the anisotropic viscosity and the incompressibility constraint, we have some uniform  $L^2$  estimate on  $\nabla_h u_\varepsilon$  and  $\partial_z u_{\varepsilon,3} = -\nabla_h \cdot u_{\varepsilon,h}$ , but not on the vertical derivative of the horizontal velocity  $\partial_z u_{\varepsilon,h}$ . In particular, the blue term above is the

most difficult one to handle : we will therefore concentrate on that term, and admit other convergences which can be proved by similar or easier arguments.

The main idea is to obtain a refined trace estimate on  $u_\varepsilon$  using the Dirichlet condition on  $B$ . We have obviously

$$u_\varepsilon(z) = \int_0^z \partial_z u(z') dz'$$

so that

$$\|u_\varepsilon(z)\|_{L^2(\mathcal{D}_h)} \leq z^{1/2} \|\partial_z u_\varepsilon\|_{L^2([0,1] \times \mathcal{D}_h)}.$$

We then deduce

$$\begin{aligned} \left| \int_0^t \int u_{\varepsilon,3} u_{\varepsilon,h} \cdot \partial_z \tilde{w}_{\varepsilon,h} dx ds \right| &\leq \int_0^t \int_0^1 \|u_{\varepsilon,3}(z)\|_{L^2(\mathcal{D}_h)} \|u_{\varepsilon,h}\|_{L^2(\mathcal{D}_h)} \|\partial_z \tilde{w}_{\varepsilon,h}\|_{L^\infty(\mathcal{D}_h)} dz ds \\ &\leq C \varepsilon^{-1/2} \int_0^1 z \frac{1}{\varepsilon} \exp\left(-\sqrt{\frac{\omega}{2}} \frac{z}{\varepsilon}\right) dz \leq C \varepsilon^{1/2} \end{aligned}$$

Gathering all results together, we obtain the expected convergence (3.4) for the boundary term.

### 3.2.4. The asymptotic motion

The previous study allows to take limits in the boundary term arising in (3.3). In particular, it shows that the influence of the fluid-structure interaction at the bottom on the mean motion, referred to as Ekman pumping, is some kind of dissipation, very similar to friction.

It remains then to describe the asymptotic behaviour of the nonlinear terms, i.e. of the wave coupling

$$\begin{aligned} \int_0^t \int u_{\varepsilon,h} \otimes u_{\varepsilon,h} : \nabla_h \varphi_h dx ds &= \int_0^t \int \bar{u}_{\varepsilon,h} \otimes \bar{u}_{\varepsilon,h} : \nabla_h \varphi_h dx ds + \int_0^t \int \tilde{u}_{\varepsilon,h} \otimes \tilde{u}_{\varepsilon,h} : \nabla_h \varphi_h dx ds \\ &\quad + \int_0^t \int (\tilde{u}_{\varepsilon,h} \otimes \bar{u}_{\varepsilon,h} + \bar{u}_{\varepsilon,h} \otimes \tilde{u}_{\varepsilon,h}) : \nabla_h \varphi_h dx ds \end{aligned}$$

where  $\bar{u}_h = \int_0^1 u_h(x_h, z) dz$  is the projection of  $u$  on the kernel of  $L$ , and  $\tilde{u} = u - \bar{u}$ .

As in the previous chapter, one can prove some regularity on the mean motion (both with respect to  $x$  and  $t$ )

$$\begin{aligned} \bar{u}_{\varepsilon,h} &\rightarrow \bar{u}_h \text{ strongly in } L^2([0, T] \times \mathcal{D}), \\ \tilde{u}_{\varepsilon,h} &\rightharpoonup 0 \text{ weakly in } L^2([0, T] \times \mathcal{D}), \end{aligned}$$

which is enough to take limits in all terms except the blue one.

We then need some compensated compactness argument to prove that

$$\int_0^t \int \tilde{u}_{\varepsilon,h} \otimes \tilde{u}_{\varepsilon,h} : \nabla_h \varphi_h dx ds \rightarrow 0$$

in the sense of distributions. Note that there is an additional difficulty to handle that term due to the lack of regularity of  $\tilde{u}_\varepsilon$  with respect to  $z$

$$\|\partial_z u_{\varepsilon,h}\|_{L^2([0,1] \times \mathcal{D}_h)} = O(\varepsilon^{-1/2}), \quad \|\partial_z u_{\varepsilon,3}\|_{L^2([0,1] \times \mathcal{D}_h)} = O(1).$$

We finally obtain the following damped 2D Navier-Stokes equations :

$$\int u_0 \varphi_h dx + \int_0^t \int (\bar{u}_h \partial_t \varphi_h + \bar{u}_h \otimes \bar{u}_h : \nabla_h \varphi_h - \nu_h \nabla_h \bar{u}_h \cdot \nabla_h \varphi_h) dx ds \\ \sqrt{\frac{\omega}{2}} \int_0^t \int \bar{u}_h \varphi_h dx ds - \int_0^t \int \tau \varphi_h dx_h ds.$$

Up to minor technical difficulties, two-scale analysis is therefore a rather simple method to investigate the influence of boundaries on the mean motion (provided that the forcing has no fast oscillations).

Note that “boundary test functions” are used differently at the surface (Navier condition), and at the bottom (Dirichlet condition).

### 3.3. Oscillating boundary layer terms and strong convergence

In order to describe the oscillating component of the motion, we can extend the energy method presented in the previous chapter.

Note that to each wave (i.e. to each eigenmode of the singular perturbation) is associated a bottom boundary layer - via the Dirichlet condition.

Describing the slow dynamics requires then a careful study of the boundary operator.

#### 3.3.1. The energy method

**Stability results** established in the previous chapter can be extended in presence of boundaries without any difficulty (see for instance the computation of the modulated energy used to get the uniform a priori estimates (3.2)).

**Approximate solutions** are however much more technical to obtain. They are indeed expected to decompose as the sum of

- macroscopic waves
- boundary layer terms corresponding to each wave
- boundary layer terms associated to the surface forcing
- many correctors to ensure that both source terms and boundary terms converge to 0 in appropriate norms.

All the envelope equations are thus expected to involve Ekman pumping terms, which increases the possible couplings.

**Boundary layer terms** allow to transfer a part of the energy via Ekman pumping. Nevertheless they do not provide any approximation of the boundary profiles since they have negligible  $L^2$ -norms. In that respect, strong convergence results are by no mean better than weak convergence results.

#### 3.3.2. Anomalous boundary layers : a systematic approach

In order to match the horizontal boundary conditions, it is natural to seek the boundary term as a sum of oscillating modes, rapidly decaying in  $z$ . Our goal in this paragraph is to characterize these modes, starting from the following Ansatz

$$w(k, \mu) e^{ik_h \cdot x_h} e^{i \frac{t}{\varepsilon} \mu} e^{-\lambda Z} \text{ with } Z = \frac{z}{\varepsilon} \text{ or } \frac{1 - Z}{\varepsilon}.$$

Plugging this Ansatz in the boundary operator  $\varepsilon \partial_t + \mathcal{P}(\omega \wedge \cdot) - \varepsilon \nu_h \Delta_h - \varepsilon^2 \partial_{zz}$  where  $\mathcal{P}$  denotes the Leray projection, we get

$$A_\lambda(k, \mu) w_h(k, \mu) \equiv \begin{pmatrix} i\mu - \lambda^2 + \varepsilon \nu_h k_h^2 - \frac{\varepsilon^2 \omega k_1 k_2}{\lambda^2 - \varepsilon^2 k_h^2} & -\omega + \frac{\varepsilon^2 \omega k_1^2}{\lambda^2 - \varepsilon^2 k_h^2} \\ \omega - \frac{\varepsilon^2 \omega k_2^2}{\lambda^2 - \varepsilon^2 k_h^2} & i\mu - \lambda^2 + \varepsilon \nu_h k_h^2 + \frac{\varepsilon^2 \omega k_1 k_2}{\lambda^2 - \varepsilon^2 k_h^2} \end{pmatrix} w_h(k, \mu) = 0,$$

which expresses the balance between the forcing, the viscosity, the Coriolis force and the pressure.

We have then to determine the kernel of  $A_\lambda(k, \mu)$ , and more precisely the values of  $\lambda$  (with non negative real part) for which this kernel is not reduced to  $\{0\}$ . We have then to distinguish between two cases.

In the **hyperbolic case**, that is if  $|\mu| \neq \omega$

$$\begin{pmatrix} i\mu & -\omega \\ \omega & i\mu \end{pmatrix} \text{ is hyperbolic}$$

in the sense of dynamical systems : eigenvalues have non zero real parts.

Such a property is stable by small perturbation. Boundary layer terms are then obtained as previously by solving the inviscid pressureless equations :

$$\begin{aligned} i \frac{\mu}{\varepsilon} w_h + \frac{1}{\varepsilon} \omega w_h^\perp - \varepsilon \partial_{zz} w_h &= 0 \\ \varepsilon \partial_z w_3 &= -\varepsilon \nabla_h \cdot w_h \end{aligned}$$

In the **degenerate case**, that is if  $|\mu| = \omega$ ,

$$\begin{pmatrix} i\mu & -\omega \\ \omega & i\mu \end{pmatrix} \text{ admits } 0 \text{ as an eigenvalue,}$$

but this property is very sensitive to perturbations.

- If  $k_h \neq 0$ ,  $A_\lambda(\mu, k_h)$  admits actually a non degenerate eigenvalue  $\lambda^2 \sim -2i\mu$ , and another small but non zero eigenvalue  $\lambda^2 = O(\varepsilon)$ . The solution to the boundary problem associated to this second eigenvalue is therefore exponentially decaying, but with a decay rate which is anomalously small : the size of the boundary layer is of order  $\varepsilon^{1/2}$  instead of  $\varepsilon$ .

- If  $k_h = 0$ , the eigenvalues of  $A_\lambda(\mu, k_h)$  are  $\lambda^2 = -2i\mu$  and 0. In other words, resonant wind forcing generates some destabilization process. The precise linear equation governing that process is

$$\begin{cases} \partial_t(w_h e^{-i \frac{t}{\varepsilon} \mu}) - \varepsilon \partial_{zz}(w_h e^{-i \frac{t}{\varepsilon} \mu}) = 0 \\ w_h|_{z=0} = 0, \quad \varepsilon \partial_z(w_h e^{-i \frac{t}{\varepsilon} \mu})|_{z=1} = \tau e^{-i \frac{t}{\varepsilon} \mu} \end{cases}$$

which is nothing else than a heat equation. This implies in particular that, as long as we only consider linear evolution, for finite times the boundary effect remains localized in a thin layer, the size of which depends on  $\varepsilon t$ .

Multiscale analysis fails for long times.

## References

- [1] D. Bresch, B. Desjardins and D. Gérard-Varet, Rotating fluids in a cylinder, *Discrete and Continous Dynamical Systems*. **11** (2004), no. 1, pages 47–82.

- [2] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Basics of Mathematical Geophysics*, to appear in *Oxford University Press*, 2006.
- [3] A.-L. Dalibard, L. Saint-Raymond, Mathematical study of rotating fluids with resonant surface stress. *J. Differential Equations* **246** (2009), 2304–2354.
- [4] A.-L. Dalibard, L. Saint-Raymond, Mathematical study of the beta-plane model for rotating fluids in a thin layer, *J. Math. Pures Appl.* (2010).
- [5] A.-L. Dalibard, L. Saint-Raymond, About degenerate Northern boundary layers, *in preparation*.
- [6] B. Desjardins and E. Grenier, On the Homogeneous Model of Wind-Driven Ocean Circulation, *SIAM Journal on Applied Mathematics* **60** (1999), pages 43–60.
- [7] D. Gérard-Varet, Highly rotating fluids in rough domains, *Journal de Mathématiques Pures et Appliquées* **82** (2003), pages 1453–1498.
- [8] I. Gallagher & L. Saint-Raymond, On the influence of the Earth’s rotation on geophysical flows, *Handbook of Mathematical Fluid Dynamics*, S. Friedlander and D. Serre Editors Vol 4, Chapter 5, 201–329, 2007.
- [9] E. Grenier and N. Masmoudi, Ekman layers of rotating fluids, the case of well prepared initial data, *Communications in Partial Differential Equations* **22** (1997), no. 5-6, pages 953–975.
- [10] N. Masmoudi, Ekman layers of rotating fluids: the case of general initial data, *Communications in Pure and Applied Mathematics*, **53**, (2000), pages 432–483.
- [11] N. Masmoudi, F. Rousset. Stability of oscillating boundary layers in rotating fluids. *Ann. Sci. Éc. Norm. Supér.* **41** (2008), 955–1002.
- [12] F. Rousset. Asymptotic behavior of geophysical fluids in highly rotating balls. *Z. Angew. Math. Phys.* **58** (2007), 53–67.
- [13] Saint-Raymond, Weak compactness methods for singular penalization problems with boundary layers. *SIAM J. Math. Anal.* **41** (2009), 153–177.

## 4. Rossby waves and oceanic eddies

In the previous chapters, we have explained systematic methods to investigate the asymptotic behaviour of singular perturbation problems, when the spectral decomposition of the singular perturbation is known. Classical linear algebra provides such a spectral decomposition for any Fourier multiplier, that is for any singular perturbation with constant coefficients.

In this last chapter, we intend to extend the previous results to the physically more relevant case when the singular perturbation also depends on spatial variables. The key point will be the fact that this dependence holds on larger space scales than the wave oscillations.

### 4.1. Mathematical models for oceanic eddies

As previously, we will specify a (simple) physical situation in which this more elaborate tool is needed. More precisely, our goal here is to get a simple mathematical model to explain the formation of oceanic eddies, as a combination of Rossby waves and zonal flow.

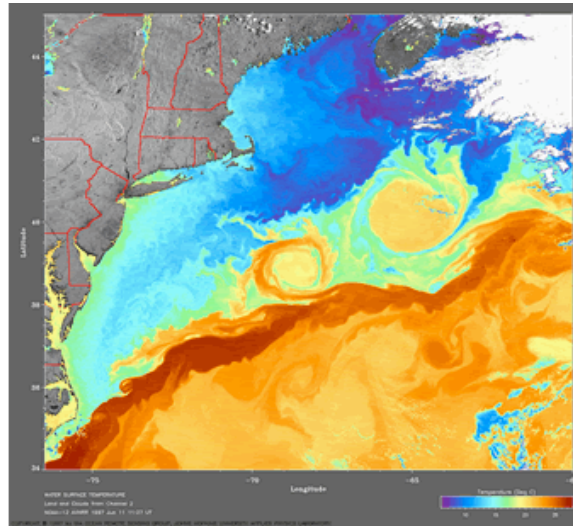


Figure 4.1: West Atlantic eddies

#### 4.1.1. About oceanic eddies, their formation and their propagation

Oceanic eddies are **time-persistent structures of vortex-type** the horizontal extent of which is typically 10 to 100 km, which persist over decades.

The **mechanism of formation** of oceanic eddies is usually described as follows : the wind forcing produces waves with a speed comparable to the bulk velocity, and the convection by zonal flow stops the propagation, creating ventilation zones which are not influenced by external motions.

Orders of magnitude show actually that among waves propagated under hydrostatic pressure and Coriolis force, only **Rossby waves** can generate such structures.

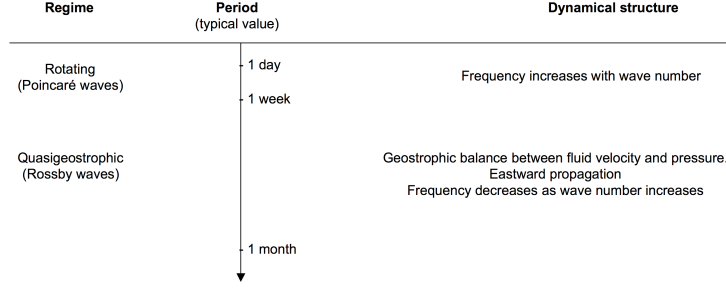


Figure 4.2: Classification of waves

#### 4.1.2. Mathematical description of oceanic motions

As previously, we consider the ocean as an incompressible inviscid fluid with free surface, and further assume that the density of the fluid is homogeneous,. Given the horizontal scale to be considered ( $l_0 \sim 1000km$ ), the aspect ratio is very small and one can use some **shallow-water approximation** : the pressure law is then given by the hydrostatic law, and the motion - essentially horizontal - does not depend on the vertical coordinate.

The evolution of the water height  $h$  and velocity  $v$  is then governed by the Saint-Venant equations with Coriolis force

$$\begin{aligned}\partial_t h + \nabla \cdot (hv) &= 0 \\ \partial_t(hv) + \nabla \cdot (hv \otimes v) + \omega(hv)^\perp + gh\nabla h &= \tau\end{aligned}$$

Macroscopic currents, in particular zonal flows, are stationary solutions. They satisfy the **Sverdrup relation**

$$\nabla \cdot \bar{u} = 0, \quad \nabla \cdot (\bar{u} \otimes \bar{u}) + \omega \bar{u}^\perp = \tau/\underline{h},$$

where  $\underline{h}$  is constant, and  $\tau$  accounts for both the Ekman pumping (related to averages of the wind forcing); and the effects of temperature gradients and topography.

For the sake of simplicity, we will only consider shear flows  $\bar{u}(x) = (\bar{u}_1(x_2), 0)$ .

We will compute the **response to wind fluctuations** assuming that it prescribes the initial data. The pulse at time 0 is a superposition of local waves with very small wave lengths.

#### 4.1.3. Orders of magnitude and scaling

In order to study the propagation of surface waves, we introduce the depth variation  $\rho = \delta h/\underline{h}$  with

$$\underline{h} \sim 1km, \quad \delta h \sim 1m.$$

In order to exhibit structures like eddies, we have further to choose appropriate observation time and length scales

$$t_o \sim 100 \text{ days}, \quad l_o \sim 1000km, \quad v_o \sim 0,1ms^{-1}.$$

to be compared to the typical velocity of macroscopic currents

$$v_c \sim 10ms^{-1}.$$

The **nondimensional equations** then state

$$\begin{aligned} \varepsilon \partial_t \rho + \varepsilon \bar{u} \cdot \nabla \rho + \nabla \cdot u &= -\varepsilon^2 \nabla \cdot (\rho u) \\ \varepsilon \partial_t u + \varepsilon \bar{u} \cdot \nabla u + \frac{1}{\varepsilon} \omega u^\perp + \nabla \rho &= -\varepsilon u \cdot \nabla \bar{u} - \varepsilon^2 u \cdot \nabla u \\ (\rho, u)|_{t=0} &= \sum (\rho_j(x), u_j(x)) \exp\left(\frac{i S_j(x)}{\varepsilon}\right) \end{aligned} \quad (4.1)$$

Given the typical wave numbers in the initial data, we expect spatial derivatives to be of order  $O(1/\varepsilon)$ . The singular perturbation here is then given by

$$\begin{pmatrix} 0 & \varepsilon \partial_1 & \varepsilon \partial_2 \\ \varepsilon \partial_1 & 0 & -\omega(x_2) \\ \varepsilon \partial_2 & \omega(x_2) & 0 \end{pmatrix}.$$

As we are interested in describing the dynamics for very long times (referred to as diffractive times in geometrical optics), we will have to also consider the linear convection  $(\varepsilon \bar{u} \cdot \varepsilon \nabla)$ .

## 4.2. Mathematical study of wave propagation

Because of the scaling of the nonlinear terms, we expect the dynamics to be dominated by the linear propagation. We will thus focus on the linear equation

$$\varepsilon^2 \partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} + A(x, \varepsilon D_x) \begin{pmatrix} \rho \\ u \end{pmatrix} = 0 \quad (4.2)$$

with

$$A(x, \varepsilon D_x) = \begin{pmatrix} (\varepsilon \bar{u} \cdot \varepsilon \nabla) & \varepsilon \partial_1 & \varepsilon \partial_2 \\ \varepsilon \partial_1 & (\varepsilon \bar{u} \cdot \varepsilon \nabla) & -\omega(x_2) + \varepsilon^2 \bar{u}'(x_2) \\ \varepsilon \partial_2 & \omega(x_2) & (\varepsilon \bar{u} \cdot \varepsilon \nabla) \end{pmatrix}.$$

The comparison between linear and nonlinear solutions is postponed to the final section of the chapter. Note that, with the tools we dispose of, we are not able - from the mathematical point of view - to consider stronger couplings.

### 4.2.1. A simple case with explicit spectral resolution

Within the **betaplane approximation**  $\omega(x) = \beta x_2$ , and in the **absence of convection**  $\bar{u} = 0$ , the linear propagator states

$$A_0(x, \varepsilon D_x) = \begin{pmatrix} 0 & \varepsilon \partial_1 & \varepsilon \partial_2 \\ \varepsilon \partial_1 & 0 & -\beta x_2 \\ \varepsilon \partial_2 & \beta x_2 & 0 \end{pmatrix}$$

Introducing a suitable combination of  $\rho$  and  $u$  shows that  $A_0$  can be expressed in terms of  $\varepsilon \partial_1$  and of the creation and annihilation operators  $\varepsilon \partial_2 \pm \beta x_2$ . Therefore it can be diagonalized without any error term using a Fourier basis  $(\exp(\frac{i}{\varepsilon} x_1 \xi_1))$  in  $x_1$  and a Hermite basis  $(\psi_n^\varepsilon(x_2))$  in  $x_2$ .

**Proposition 4.1.** [3.1] *Let  $\tau(n, \xi_1, j)$  ( $j = 0, \pm$ ) be the three roots of*

$$\tau^3 - (\xi_1^2 + \beta \varepsilon (2n + 1)) \tau + \varepsilon \beta \xi_1 = 0, \quad n \in \mathbf{N}, \xi_1 \in \mathbf{R} \quad (4.3)$$

*Then there exists a complete family  $(\Psi_{n, \xi_1, j}^\varepsilon)_{n, \xi_1, j}$  of  $L^2$  such that*

$$A_0(x, \varepsilon D_x) \Psi_{n, \xi_1, j}^\varepsilon = i \tau(n, \xi_1, j) \Psi_{n, \xi_1, j}^\varepsilon$$



The three **scalar propagators** are then obtained from the symbolic equation (4.3) :  $\beta\varepsilon(2n+1)$  is indeed the quantization of the harmonic oscillator  $-\varepsilon^2\partial_2^2 + \beta^2x_2^2$ . We have two Poincaré modes

$$\tau_{\pm} \sim \pm\sqrt{\xi_1^2 + \beta\varepsilon(2n+1)},$$

and one Rossby mode

$$\tau_0 \sim \frac{\varepsilon\beta\xi_1}{\xi_1^2 + \beta\varepsilon(2n+1)}.$$

The propagation of energy associated to Rossby waves holds in diffractive times, i.e. on a time scale for which Poincaré waves do not carry locally any more energy, along the characteristics of  $\tilde{\tau}_0 = \tau_0/\varepsilon$  :

$$\frac{dX_i}{dt} = \frac{\partial\tilde{\tau}_0}{\partial\xi_i}, \quad \frac{d\Xi_i}{dt} = -\frac{\partial\tilde{\tau}_0}{\partial x_i}$$

These characteristics are periodic in  $x_2$ , and linear in  $x_1$  except for

$$\xi_1 \sim \pm\sqrt{\beta^2x_2^2 + \xi_2^2}$$

in which case they are trapped (fixed points).

Note that in this particular case, one can handle even strong nonlinearities, typically

$$\begin{aligned} \varepsilon\partial_t\rho + \nabla \cdot u &= -\varepsilon\nabla \cdot (\rho u) \\ \varepsilon\partial_t u + \frac{\beta}{\varepsilon}x_2u^\perp + \nabla\rho &= -\varepsilon u \cdot \nabla u \end{aligned}$$

Actually nonlinear terms govern the amplitudes according to some envelope equations : using suitable functional spaces (defined with the harmonic oscillator) and a precise characterization of possible resonances, one can indeed prove that the infinite system of ODEs is generically well-posed [3.1].

#### 4.2.2. Tools for asymptotic analysis

For general zonal current  $\bar{u}$  and Coriolis parameter  $\omega$ , there is generally no explicit spectral decomposition for the linear propagator

$$A(x, \varepsilon D_x) = \begin{pmatrix} (\varepsilon\bar{u} \cdot \varepsilon\nabla) & \varepsilon\partial_1 & \varepsilon\partial_2 \\ \varepsilon\partial_1 & (\varepsilon\bar{u} \cdot \varepsilon\nabla) & -\omega(x_2) + \varepsilon^2\bar{u}'(x_2) \\ \varepsilon\partial_2 & \omega(x_2) & (\varepsilon\bar{u} \cdot \varepsilon\nabla) \end{pmatrix}$$

Since the Rossby and Poincaré part are expected to exhibit very different behaviours, we have even no theoretical result giving a qualitative description of the spectrum. (Even in the particular case considered in the previous paragraph, i.e. when  $\bar{u} = 0$  and  $\omega(x_2) = \beta x_2$ ,  $A$  is neither compact, nor with compact resolvent.

Nevertheless, as  $\varepsilon \rightarrow 0$ , **semiclassical analysis** provides good approximations of the dynamics. In that framework, (pseudo-)differential calculus is replaced by symbolic computations in the phase space

$$(x_2, \varepsilon\partial_2) \rightarrow (x_2, \xi_2).$$

Commutators are indeed of higher order in  $\varepsilon$  :

$$[x_2, \varepsilon\partial_2] = -\varepsilon.$$

We can then proceed by successive approximations, computing first of all the principal symbols, then subsymbols at any order recursively.

#### 4.2.3. Main difficulties

In the present situation, additional difficulties come from the **matricial structure** of the propagator  $A$ , and from the **difference of scaling** between Rossby and Poincaré modes. Note that because of this last feature, polarization cannot be obtained directly by the theory of normal forms.

Our strategy here is to come down to a scalar equation (of higher order) and then to compute the three elementary propagators using a kind of implicit function theorem.

### 4.3. Semiclassical analysis of the linear propagation

For the sake of simplicity, we will not detail here all the technical steps of the study. We will just mention the main difficulties and refer to [3.1] for completely rigorous arguments.

#### 4.3.1. Polarization of Poincaré and Rossby waves

We first compute the “characteristic polynomial”. Substitutions and linear combinations leads for instance to the subsystem

$$\begin{pmatrix} (\varepsilon \bar{u}_1 \xi_1 - \tau) & \xi_1 \\ \xi_1 & (\varepsilon \bar{u}_1 \xi_1 - \tau) \end{pmatrix} \begin{pmatrix} \rho \\ u_1 \end{pmatrix} = \begin{pmatrix} i\varepsilon \partial_2 u_2 \\ -i\omega(x_2)u_2 \end{pmatrix}$$

and to the scalar equation

$$\varepsilon \partial_2 \left( \frac{i\omega \xi_1 u_2 + i(\varepsilon \bar{u}_1 \xi_1 - \tau)\varepsilon \partial_2 u_2}{(\varepsilon \bar{u}_1 \xi_1 - \tau)^2 - \xi_1^2} \right) - \omega \left( \frac{i\xi_1 \varepsilon \partial_2 u_2 + (\varepsilon \bar{u}_1 \xi_1 - \tau)i\omega u_2}{(\varepsilon \bar{u}_1 \xi_1 - \tau)^2 - \xi_1^2} \right) + i(\varepsilon \bar{u}_1 \xi_1 - \tau)u_2 = 0$$

Note that we have to deal with another subsystem when the previous one is not invertible, i.e. when  $\tau \sim \xi_1$  at principal order. But the final form of the characteristic polynomial is of course independent of the way of computing the determinant. Such a method has to be compared to the usual “pivot de Gauss” in classical linear algebra, the only difference being the fact that the field is non commutative!

Using pseudo-differential functional calculus, we get the **three scalar propagators**  $T_{\pm}$ ,  $T_0$

- by solving the symbolic equation

$$(\tau - \varepsilon \bar{u}_1 \xi_1)^3 - (\tau - \varepsilon \bar{u}_1 \xi_1)(\xi_2^2 + \xi_1^2 + \omega^2(x_2)) + \varepsilon \omega'(x_2)\xi_1 = O(\varepsilon^2), \quad (4.4)$$

- defining some associate operator by any quantization,
- then by computing recursively the expansions of the symbol.

We further prove that any initial condition can be decomposed microlocally on the eigenmodes of the scalar propagators  $T_{\pm}$ ,  $T_0$ .

**Proposition 4.2.** [3.1] *Let  $U_{\varepsilon,0}$  be  $\varepsilon$ microlocalized in a compact set  $\mathcal{C}$  satisfying*

$$\mathcal{C} \cap \{\xi_1 = 0\} = \emptyset \text{ and } \mathcal{C} \cap \Sigma = \emptyset,$$

*where  $\Sigma$  is a codimension 1 subset of  $\mathbf{R}^4$  consisting of pathological trajectories.*

*For any parameter  $\varepsilon > 0$ , denote by  $V_\varepsilon$  the associate solution to (4.2).*

*Then for all  $t \geq 0$  one can write  $V_\varepsilon(t)$  as the sum of a "Rossby" vector field and a "Poincaré" vector field:*

$$V_\varepsilon(t) = V_\varepsilon^R(t) + V_\varepsilon^P(t)$$

*where  $V_\varepsilon^R$  evolves according to some scalar operator  $T_0$  of principal symbol*

$$\tau_0 = \varepsilon \bar{u}(x_2) \xi_1 + \frac{\varepsilon \omega'(x_2) \xi_1}{\xi_1^2 + \xi_2^2 + \omega^2(x_2)}$$

*while  $V_\varepsilon^P$  is the sum of two Poincaré waves propagated by  $T_\pm$  of principal symbols*

$$i\tau_\pm = \pm i \sqrt{\xi_1^2 + \xi_2^2 + \omega^2(x_2)}.$$

Note that the subsymbols depend strongly on the choice of quantization, which is not the case of principal symbols. Furthermore, in that approach, the Rossby Hamiltonian apperas as the principal symbol associated to the root of order  $\varepsilon$ , and not as a subsymbol. In particular, this implies that Rossby modes are intrinsic to the physical system, and tell us something about the propagation of a part of the energy.

Another important remark is that  $T_\pm$  and  $T_0$  are defined up to a proper microlocalization. The proof of Proposition 4.2 thus requires additional informations about the evolution of the microlocalization under the dynamics. There is therefore some bootstrap argument involving the following results about the propagation of energy by Poincaré and Rossby modes.

#### 4.3.2. Dispersion of Poincaré waves

We propose to establish the dispersion of Poincaré waves using some spectral argument which relies on rather explicit computations. Here the assumption that coefficients depend only on  $x_2$  is crucial.

If the initial data is microlocalized away from  $\xi_2^2 + \omega^2(x_2) + \xi_1^2 = 0$ , we can find pseudo-differential operators  $H_{2\pm}(\xi_1)$  of principal symbols  $\xi_2^2 + \omega^2(x_2)$  such that

$$\hat{T}_\pm(\xi_1) = \pm \sqrt{H_{2\pm}(\xi_1) + \xi_1^2}.$$

If  $H_{2\pm}$  has no hyperbolic fixed point, the **Bohr-Sommerfeld quantization condition** (with subsymbol) gives that the eigenvalues of  $H_{2\pm}(\xi_1)$  are of the form:

$$\lambda_\pm^k(\xi_1) = \lambda_\pm \left( \left( k + \frac{1}{2} \right) \varepsilon \right) + \varepsilon \mu_\pm^k(\xi_1) + O(\varepsilon^2),$$

where  $\lambda_\pm$  is the energy  $\xi_2^2 + \omega^2(x_2)$  defined on action variable, and  $\varepsilon \mu_\pm^k(\xi_1) \in C^\infty$  is the correction due to the subsymbol.

The Poincaré component is therefore a superposition of elementary waves, indexed by  $(q, p)$  (coherent state in  $x_1$ ) and  $k$  (quantization in  $x_2$ )

$$\int \exp \left( i(x_1 - q)\xi_1 - (\xi_1 - p)^2 \right) \exp \left( \pm i \frac{(\lambda_\pm^k(\xi_1) + \xi_1^2)^{\frac{1}{2}} t}{\varepsilon} \right) \Psi_{k, \xi_1, \pm}^\varepsilon(x_2) d\xi_1.$$

Such integrals are  $O(\epsilon^\infty)$  except if there exists a stationary point for the phase , given by the conditions:

$$\xi_1 = p \text{ and } \epsilon(x_1 - q) \pm \frac{(2\xi_1 + \epsilon\partial_{\xi_1}\mu_\pm^k)}{2\sqrt{\lambda_\pm^k + \xi_1^2}}t = 0.$$

We finally conclude that for **diffractive times**  $t \sim 1$ , as there is no critical point for  $x_1$  in compact sets, the energy carried by Poincaré waves exit from any compact set.

**Proposition 4.3.** [3.1] *Suppose that  $\omega^2$  has only one non degenerate critical value (meaning that  $(\omega^2)'$  only vanishes at one point, where  $(\omega^2)''$  does not vanish). Then for any compact set  $K$  in  $\mathbf{R}^2$ , one has*

$$\forall t > 0, \quad \|V_\epsilon^P(t)\|_{L^2(K)} = O(\epsilon^\infty).$$

Another method would be to use some Mourre estimate. This would require to check that the bootstrap argument concerning the microlocalization still holds true and to get a uniform bound from below on  $[x_1, \mathbf{T}_\pm]$ . The positive point is that some assumptions on  $\omega$  should be relaxed in that way.

#### 4.3.3. Trapping of Rossby waves

As the Rossby Hamiltonian  $\tau_0$  is smaller by one order of magnitude, the propagation on diffractive times is nothing else than the semiclassical propagation for the rescaled Hamiltonian

$$\tilde{\tau}_0(x, \xi) = \bar{u}(x_2)\xi_1 + \frac{\omega'(x_2)\xi_1}{\xi_1^2 + \xi_2^2 + \omega^2(x_2)},$$

that is the transport along the bicharacteristics

$$\begin{aligned} \frac{dx_1^t}{dt} &= \frac{\partial \tilde{\tau}_0}{\partial \xi_1}(x^t, \xi^t), & \frac{dx_2^t}{dt} &= \frac{\partial \tilde{\tau}_0}{\partial \xi_2}(x^t, \xi^t), \\ \frac{d\xi_1^t}{dt} &= -\frac{\partial \tilde{\tau}_0}{\partial x_1}(x^t, \xi^t), & \frac{d\xi_2^t}{dt} &= -\frac{\partial \tilde{\tau}_0}{\partial x_2}(x^t, \xi^t). \end{aligned}$$

In particular, trajectories are submanifolds of the energy surfaces  $\tilde{\tau}_0(x, \xi) = \tau$ , i.e.

$$\xi_2^2 = \frac{\omega'(x_2)\xi_1}{\tau - \bar{u}(x_2)\xi_1} - \xi_1^2 - \omega^2(x_2) \equiv V_\tau(x_2).$$

The motion along  $x_2$  can be of two types depending on the possible existence of a singularity of  $V_\tau$  between two roots of this same function. More precisely define

$$\begin{aligned} x_{min} &= \max\{x_2 \leq x_2^0 / V_\tau(x_2) = 0 \text{ or } \tau - \bar{u}(x_2)\xi_1 = 0\} \\ x_{min} &= \min\{x_2 \geq x_2^0 / V_\tau(x_2) = 0 \text{ or } \tau - \bar{u}(x_2)\xi_1 = 0\}. \end{aligned}$$

Then, generically,

- if  $x_{min}$  and  $x_{max}$  are turning points (zeros of  $V_\tau$ ), the trajectory is periodic (see the first picture in Fig. 4.3);
- if  $x_{min}$  or  $x_{max}$  is a singular point (singularity of  $V_\tau$ ), the trajectory is called asymptotic (see the second picture in Fig. 4.3).

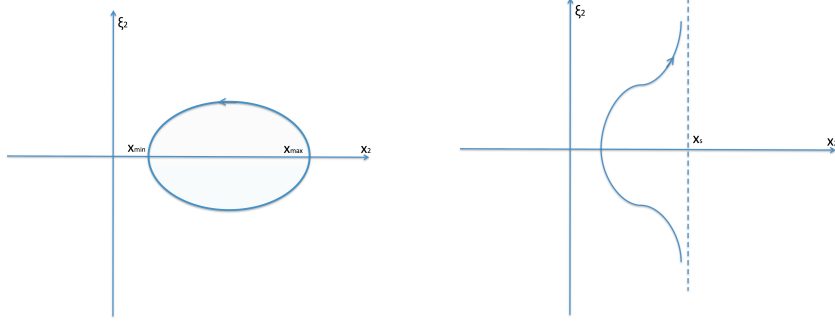


Figure 4.3: Trajectories of the Rossby Hamiltonian

We then obtain qualitative informations on the energy propagation by integrating

$$\frac{d}{dt}x_1^t = \bar{u}(x_2^t) + \frac{\omega'(x_2^t)(-(\xi_1^t)^2 + (\xi_2^t)^2 + \omega^2(x_2^t))}{(\xi_1^t)^2 + (\xi_2^t)^2 + \omega^2(x_2^t)^2}$$

Performing a change of variable, we can express the **trapping condition** in terms of the initial parameters  $(\xi_1, x_2^0, \xi_2^0)$ .

**Proposition 4.4.** [3.1] *There is a submanifold  $\Lambda$  of  $\mathbf{R}^4$ , invariant under translations in the  $x_1$ -direction, such that the following property holds :*

$$\exists K \subset \subset \mathbf{R}^2, \quad \forall t \geq 0, \quad \|V_\varepsilon^R(t)\|_{L^2(K)} \neq O(\varepsilon^\infty).$$

*is equivalent to*

$$\text{the } \varepsilon\text{-frequency set of } V_\varepsilon^R(0) \text{ intersects } \Lambda.$$

If  $\omega(x_2) = \beta x_2$  and  $\bar{u}$  is not identically positive, there is a submanifold of codimension 1 of initial data giving rise to trapped Rossby waves, **spatially concentrated** on lines  $x_2 = x_s$

$$X_2(t) \rightarrow x_s; \quad X_1(t) \rightarrow x_1^\infty$$

and **strongly oscillating** with respect to  $x_2$

$$|\Xi_2(t)| \rightarrow \infty.$$

In particular, the associated vorticity concentrates on zero measure sets. These features are reminiscent of oceanic eddies, even though not completely realistic : the translation invariance with respect to  $x_1$  (both in the zonal flow  $\bar{u}_1 \equiv \bar{u}_1(x_2)$  and as regards the domain  $\mathcal{D}$ )) prevents from localizing eddies in the longitudinal direction.

#### 4.4. Case of a weak coupling

We would like now to transcribe the previous results about the linear propagation (4.2) on the nonlinear system (4.1). Weak compactness methods are obviously not suitable to the situation insofar as we are interested in the propagation of waves (oscillations) and in the formation of eddies (vorticity concentrations). Filtering methods introduced in the second chapter can neither be applied since precise informations on the spectral decomposition are missing.

We will thus restrict our attention to the case when the nonlinear coupling is weak enough in order that the linear equation provides a good approximation. Actually

we will even impose stronger restrictions on the nonlinearity in order to be able to get a uniform lifespan for the solutions to the scaled Saint-Venant equations. This restriction is due to the bad  $L^\infty$  control on multiscaled functions such as WKB profiles.

#### 4.4.1. A symmetrizable hyperbolic system

Let us first recall that, setting  $U = (\frac{2}{\varepsilon^2}(\sqrt{1 + \varepsilon^2 \rho} - 1), u)$ , system  $(SW_\varepsilon)$  states

$$\varepsilon^2 \partial_t U + A(x, \varepsilon D_x) U + \varepsilon^3 \sum_{j=1,2} S_j(U) \varepsilon \partial_j U = 0$$

where  $S_j(U)$  are symmetric matrices.

For such symmetrizable hyperbolic systems, the Cauchy problem is locally well-posed in  $H^s$  for  $s > \frac{d}{2} + 1$ . Here, as  $\partial_2$  does not commute with the singular perturbation, the life span depends a priori on  $\varepsilon$  (no uniform lower bound).

To get uniform regularity estimates and thus a uniform lifespan, one has to define specific norms :

- suited to the semiclassical framework (in order that the norm of the initial data is of order  $O(1)$ );
- having good commutation properties with  $A(x, \varepsilon D_x)$  (to be propagated uniformly in  $\varepsilon$ ).

#### 4.4.2. Propagation of regularity

The idea is to extend a result of regularity propagation [3.1] which is very specific to the betaplane approximation, taking advantage of the semiclassical framework, namely of the fact that commutators are of higher order with respect to  $\varepsilon$ .

An easy change of variables show that  $A(x, \varepsilon D_x)$  is equivalent to the propagator

$$\tilde{A}(x, \varepsilon D_x) =$$

Define then

$$D_\varepsilon \sim \begin{pmatrix} \varepsilon^2 \partial_2^2 - \omega^2 + 2\varepsilon\omega' & 0 & 0 \\ 0 & \varepsilon^2 \partial_2^2 - \omega^2 - 2\varepsilon\omega' & 0 \\ 0 & 0 & \varepsilon^2 \partial_2^2 - \omega^2 \end{pmatrix}.$$

Because of the fundamental identity

$$[\varepsilon^2 \partial_2^2 - \omega^2, \varepsilon \partial_2 \pm \omega] = \pm 2\varepsilon\omega'(\varepsilon \partial_2 \pm \omega) \pm \varepsilon^2 \omega''$$

$D_\varepsilon$  **almost commutes with**  $\tilde{A}(x, \varepsilon D_x)$

$$[D_\varepsilon, \tilde{A}(x, \varepsilon D_x)] = O(\varepsilon^2 (Id - D_\varepsilon)).$$

We therefore introduce weighted Sobolev spaces  $W_\varepsilon^{s_1, s_2}$  using powers of  $\varepsilon \partial_1$  and of  $D_\varepsilon$ . Using some Gronwall's inequality, we can then prove that  $W_\varepsilon^{s_1, s_2}$ -norms of the solutions to the linear equation (4.2) are uniformly controlled locally in time.

Propagating such estimates under the nonlinear dynamics (4.1) requires to further establish product laws in the weighted spaces  $W_\varepsilon^{s_1, s_2}$ . Note that  $D_\varepsilon$  controls two  $\varepsilon$ -derivatives with respect to  $x_2$ . Moreover, as the principal symbol of  $D_\varepsilon$  is scalar, the symmetry of the higher order nonlinear term is conserved.

We therefore get the following **trilinear estimate**

$$\left| \langle U | S_j(U) \varepsilon \partial_j U \rangle_{W_\varepsilon^{3,2}} \right| \leq \|\varepsilon D_x U\|_{L^\infty} \|U\|_{W_\varepsilon^{3,2}}^2.$$

Note that, because of the semiclassical scaling, we lose one power of  $\varepsilon$  in the embedding

$$\|\varepsilon D_x U\|_{L^\infty} \leq \frac{1}{\varepsilon} \|U\|_{W_\varepsilon^{3,2}},$$

which is obviously not optimal considering for instance oscillating functions such as  $x \mapsto \exp(\frac{i}{\varepsilon} k \cdot x)$ .

Gathering all the previous properties leads to a precised energy estimate in  $W_\varepsilon^{3,2}$ . The same arguments as in the theorem by Fujita and Kato give then the existence of a solution  $U_\varepsilon$  on a uniform time interval  $[0, T^*[$ .

#### 4.4.3. Linear approximation

Within the scalings considered to get solutions to the nonlinear equations on a uniform time interval, we expect the coupling to be asymptotically negligible. What can be proved is the following. Denote by  $U_\varepsilon^\eta$  the solutions of the weakly nonlinear system

$$\varepsilon^2 \partial_t U + A(x, \varepsilon D_x) U + \varepsilon^{3+\eta} S_j(U) \varepsilon \partial_j U = 0, \quad \eta \geq 0.$$

and by  $V_\varepsilon$  the solution to the linear system.

- If  $\eta > 0$ , for any  $T > 0$

$$\|U_\varepsilon^\eta - V_\varepsilon\|_{L^2} \rightarrow 0 \text{ on } [0, T] \text{ as } \varepsilon \rightarrow 0.$$

In particular, the  $L^2$ -norm of  $U_\varepsilon^\eta$  on any given compact remains bounded from below if there are trapped Rossby waves, i.e. if the initial wavefront set does intersect  $\Lambda$ .

- If  $\eta = 0$  (which is the scaling introduced at the beginning the chapter) and if we further assume that the solution  $V_\varepsilon$  to the linear system (4.2) satisfies

$$\|\varepsilon V_\varepsilon\|_{L^\infty} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \tag{4.5}$$

then

$$\|U_\varepsilon^0 - V_\varepsilon\|_{L^2} \rightarrow 0 \text{ on } [0, T^*[ \text{ as } \varepsilon \rightarrow 0.$$

In particular, for any  $t \in ]0, T^*[$ , the energy of  $U_\varepsilon^0$  on any compact is carried only by Rossby waves. Note that we hope to be able to prove the estimate (4.5)

## References

- [3.1] C. Cheverry, I. Gallagher, T. Paul & L. Saint-Raymond, Trapping Rossby waves, *C. R. Math. Acad. Sci. Paris* **347** (2009), 879–884.
- [3.1] C. Cheverry, I. Gallagher, T. Paul & L. Saint-Raymond. Semiclassical and spectral analysis of oceanic waves. Submitted (2010).
- [3.1] M. Dimassi & S. Sjöstrand, *Spectral Asymptotics in the Semi-Classical Limit*; Cambridge University Press, London Mathematical Society Lecture Note Series **268**, 1999.
- [3.1] A. Dutrifoy & A. J. Majda, Fast Wave Averaging for the Equatorial Shallow Water Equations, *Comm. PDE*, **32** (2007), 1617–1642.
- [3.1] A. Dutrifoy, A. J. Majda & S. Schochet, A Simple Justification of the Singular Limit for Equatorial Shallow-Water Dynamics, in Communications on Pure and Applied Math. LXI (2008) 0002-0012.
- [3.1] I. Gallagher and L. Saint-Raymond, Weak convergence results for inhomogeneous rotating fluid equations, *Journal d'Analyse Mathématique*, **99** (2006), 1-34.
- [3.1] I. Gallagher & L. Saint-Raymond, Mathematical study of the betaplane model: equatorial waves and convergence results. Mém. Soc. Math. Fr. (N.S.). 107 (2006), v+116 pp.
- [3.1] I. Gallagher & L. Saint-Raymond, On the influence of the Earth's rotation on geophysical flows, *Handbook of Mathematical Fluid Dynamics*, S. Friedlander and D. Serre Editors Vol 4, Chapter 5, 201-329, 2007.
- [3.1] A. E. Gill, Atmosphere-Ocean Dynamics, *International Geophysics Series*, **Vol. 30**, 1982.
- [3.1] A. E. Gill & M. S. Longuet-Higgins, Resonant interactions between planetary waves, *Proc. Roy. Soc. London*, **A 299** (1967), 120–140.
- [3.1] J.-L. Joly, G. Métivier & J. Rauch, Generic Rigorous Asymptotic Expansions for Weakly Nonlinear Multidimensional Oscillatory Waves, *Duke Mathematical Journal* (1993), **70**, 373-404.
- [3.1] J.-L. Joly, G. Métivier & J. Rauch, Coherent nonlinear waves and the Wiener algebra, *Ann. Inst. Fourier (Grenoble)* **44** (1994), no. 1, 167–196.
- [3.1] M. Majdoub, M. Paicu. Uniform local existence for inhomogeneous rotating fluid equations. *J. Dynam. Differential Equations* **21** (2009), 21–44.
- [3.1] A. Martinez, *An introduction to semiclassical and microlocal analysis*, Springer (2002)
- [3.1] J. Pedlosky, *Geophysical fluid dynamics*, Springer (1979).
- [3.1] J. Pedlosky, *Ocean Circulation Theory*, Springer (1996).



- [3.1] S. Vũ Ngoc, Systèmes intégrables semi-classiques : du local au global, *Panoramas et Synthèses* **22**, 2006.

UNIVERSITÉ PARIS VI AND DMA, ECOLE NORMALE SUPÉRIEURE, 45 RUE D'ULM,  
75230 PARIS CEDEX 05, FRANCE  
Laure.Saint-Raymond@ens.fr

# Index

## MATHEMATICAL NOTIONS

- anomalous boundary layers, 34
- Bohr-Sommerfeld quantization, 42
- boundary condition, 7
  - Dirichlet boundary condition, 7
  - kinematic condition, 7
  - Navier boundary condition, 7
- boundary layers, 27, 34
- compensated compactness, 20
- constraint equation, 18
- diffractive time, 39
- dispersion, 42
- Ekman pumping, 29
- energy estimate, 11
- envelope equations, 23
- filtering methods, 21
- free surface, 7, 9
- mean motion, 19
- modulated energy, 24, 34
- Navier-Stokes equations, 6, 10
- resonance, 22
- Saint-Venant equations, 13, 16, 38
- scaling invariance, 12
- semiclassical analysis, 40
- singular perturbation, 16
- stationary phase, 43
- Strichartz estimate, 23
- symmetrizable hyperbolic system, 14, 45
- trapping, 43
- two-scale analysis, 29
- viscous dissipation, 7
- weak Leray solution, 11

## PHYSICAL TERMS

- approximation
  - betaplane approximation, 39
  - geostrophic approximation, 9
  - rigid lid approximation, 9
  - shallow-water approximation, 8
- bottom topography, 3
- boundary layers
  - Ekman layers, 27
  - Munk layers, 27
  - Stommel layers, 27
- Coriolis force, 6
- Ekman pumping, 5, 9, 29
- envelope equations, 17
- geostrophic motion, 19
- hydrostatic law, 6
- nondimensional parameters
  - aspect ratio, 8
  - Froude number, 16

- Rossby number, 9, 16
- oceanic currents, 5
- oceanic eddies, 37
- pycnocline, 3
- ripples, 4
- storm waves, 5
- Sverdrup relation, 38
- swell, 4
- thermohaline circulation, 6
- tides, 5
- tsunamis, 5
- waves
  - gravity waves, 17
  - Poincaré waves, 41
  - Poincaré waves, 10, 17, 23
  - Rossby waves, 10, 37, 41
  - rotating waves, 17, 27
- wind forcing, 3, 6
- zonal flow, 37
- stratification, 27