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Un théorème d'existence en théorie non linéaire des coques minces

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A minimization problem arising in nonlinear thin shell theory

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Résumé

Les équations bidimensionnelles d'une coque non linéairement élastique "en flexion" ont été récemment justifiées par V. Lods et B. Miara par la méthode des développements asymptotiques formels appliquée aux équations de l'élasticité non linéaire tridimensionnelle. Ces équations se mettent sous la forme d'un problème de point critique d'une fonctionnelle dont l'intégrande est une expression quadratique en termes de la différence exacte entre les tenseurs de courbure des surfaces déformée et non déformée, sur un ensemble de déformations admissibles qui préservent le tenseur métrique de la surface moyenne et satisfont des conditions aux limites ad hoc.

Nous montrons ici comment l'existence d'un minimiseur peut être établie

1. The two-dimensional equations of a nonlinearly elastic "flexural" shell.

Greek indices and exponents take their values in the set $\{1,2\}$, Latin indices and exponents in the set $\{1,2,3\}$, and the summation convention with respect to repeated indices and exponents is used. The scalar product, the exterior product, and the Euclidean norm of $\mathbf{a}, \mathbf{b} \in \mathbf{R}^3$ are denoted $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \wedge \mathbf{b}$, and $|\,\mathbf{a}\,|$.

Let ω be a bounded open connected subset of $\mathbf{R}^2$, with a Lipschitz-continuous boundary γ, the set ω being locally on one side of γ. We denote by $y = (y_\alpha)$ a generic point of $\bar{\omega}$ and we let $\partial_\alpha = \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} = \partial^2/\partial y_\alpha \partial y_\beta$.

Let $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{R}^3)$ be an injective mapping such that the two vectors $\mathbf{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are linearly independent at each point of $\bar{\omega}$. We define a normal vector at each point of the *surface* $S = \boldsymbol{\theta}(\bar{\omega})$ by

$$\mathbf{a}_3 = \frac{\mathbf{a}_1 \wedge \mathbf{a}_2}{|\,\mathbf{a}_1 \wedge \mathbf{a}_2\,|} .$$

The vectors $\mathbf{a}_i$ and the vectors $\mathbf{a}^i$ defined by $\mathbf{a}_j \cdot \mathbf{a}^i = \delta^i_j$ respectively constitute the covariant and the contravariant basis at each point of the surface $S = \boldsymbol{\theta}(\bar{\omega})$.

We denote by $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ and $a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta$ the covariant and contravariant components of the *metric tensor* of S, and we let $a = \det(a_{\alpha\beta})$. The covariant components of the *curvature tensor* of S are defined by $b_{\alpha\beta} = \mathbf{a}^3 \cdot \partial_\beta \mathbf{a}_\alpha$.

Using a *formal asymptotic analysis* of the three-dimensional equations of nonlinear elasticity with the thickness as the "small" parameter, Lods & Miara [4] have identified the *two-dimensional equations of a nonlinearly elastic "flexural" shell with middle surface S and thickness 2ε*, constituted by a *Saint Venant- Kirchhoff* material with *Lamé constants* $\lambda > 0$ and $\mu > 0$, and *clamped over a portion of its lateral surface.*

These two-dimensional equations take the form of variational equations, whose solutions are the critical points of a functional, the *two-dimensional energy of the shell,* over a manifold of *admissible displacements,* defined in the next theorem.

The unknown is the *deformation* $\varphi : \bar{\omega} \to \mathbf{R}^3$ of the middle surface of the shell: This means that, for each $y \in \bar{\omega}$, $\varphi(y)$ is the position taken by the point $\boldsymbol{\theta}(y)$ of the undeformed middle surface under the action of the applied forces.

2. The existence theorem.

We show here how the existence of at least one minimizer of the energy of the shell can be established; for a detailed proof, see Ciarlet & Coutand [3]. Given any field $\psi \in \mathrm{H}^2(\omega; \mathbf{R}^3)$, the vector fields $\mathbf{a}_\alpha(\psi) = \partial_\alpha \psi$ and the functions $a_{\alpha\beta}(\psi) = \mathbf{a}_\alpha(\psi) \cdot \mathbf{a}_\beta(\psi)$ are defined almost everywhere in ω. If the two vectors $a_\alpha(\psi)$ are linearly independent almost everywhere in ω, the vector field

$$\mathbf{a}_3(\psi) = \frac{\mathbf{a}_1(\psi) \wedge \mathbf{a}_2(\psi)}{|\, \mathbf{a}_1(\psi) \wedge \mathbf{a}_2(\psi)\, |}$$

the vector fields $\mathbf{a}^i(\psi)$ defined by the relations $\mathbf{a}_j(\psi) \cdot \mathbf{a}^i(\psi) = \delta^i_j$ and the functions $b_{\alpha\beta}(\psi) = \mathbf{a}_3(\psi) \cdot \partial_\beta \mathbf{a}_\alpha(\psi)$ are also defined almost everywhere in ω.

Theorem 1 *Let $\boldsymbol{\theta} \in W^{2,p}(\omega; \mathbf{R}^3)$ $(p > 2)$ be such that the two vectors $\mathbf{a}_\alpha(\boldsymbol{\theta}) = \partial_\alpha \boldsymbol{\theta}$ are linearly independent at each point of $\bar{\omega}$. Let there also be given a portion γ_o of γ with length $\gamma_o > 0$ and a mapping $\boldsymbol{\Phi}_o : \gamma_o \to \mathbf{R}^3$ such that the manifold of inextensional admissible deformations, defined by:*

$$\boldsymbol{\Phi}_F(\omega) = \{\psi \in H^2(\omega; \mathbf{R}^3);\ a_{\alpha\beta}(\psi) = a_{\alpha\beta} \text{ in } \omega;\ \psi = \boldsymbol{\Phi}_o \text{ on } \gamma_o\}$$

is not empty.

Then, if $\psi \in \boldsymbol{\Phi}_F(\omega)$, the vectors $\mathbf{a}_\alpha(\psi) = \partial_\alpha \psi$ are linearly independent almost everywhere in ω and the functions $b_{\alpha\beta}(\psi)$ belong to $L^2(\omega)$.

Let there be given a continuous linear form L on $H^2(\omega; \mathbf{R}^3)$ and let the energy $I_F : \boldsymbol{\Phi}_F(\omega) \to \mathbf{R}$ be defined by

$$I_F(\psi) = \frac{\varepsilon^3}{6} \int_\omega a^{\alpha\beta\sigma\tau}(b_{\sigma\tau}(\psi) - b_{\sigma\tau})(b_{\alpha\beta}(\psi) - b_{\alpha\beta})\sqrt{a}\, d\omega - L(\psi)$$

for all $\psi \in \boldsymbol{\Phi}_F(\omega)$, where

$$a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma})$$

denote the contravariant components of the two-dimensional elasticity tensor of the shell.

Then there exists at least one $\varphi \in \mathbf{\Phi}_F(\omega)$ *such that*

$$I_F(\varphi) = \inf_{\psi \in \mathbf{\Phi}_F} (\omega) I_F(\psi).$$

3. Sketch of the proof of the existence theorem.

The proof is "naturally" broken into several steps, of which only the statements are given (we refer to Ciarlet & Coutand [3] for complete proofs) . The first four steps deal with properties of isometric surfaces, while the remaining two deal with properties of the functional I_F over the manifold $\mathbf{\Phi}_F(\omega)$.

In the following, the usual norm of $\mathrm{L}^2(\omega)$ is denoted $||\ .\ ||_{0,\omega}$ and the usual norm of the Sobolev space $\mathrm{W}^{m,p}(\omega;\mathbf{R}^3)$ $(m > 0,\ p > 0)$ is denoted $||\ .\ ||_{m,p,\omega}$.

Step 1. For each $\psi \in \mathbf{\Phi}_F(\omega)$, the vectors $a_\alpha(\psi)$ are linearly independent a.e. in ω. Consequently, the functions $b_{\alpha\beta}(\psi)$ are well defined and belong to the space $\mathrm{L}^2(\omega)$ and thus, the functional I_F is well defined over the manifold $\mathbf{\Phi}_F(\omega)$.

Step 2. There exists a constant $C_1 > 0$ such that:

$$||\ \psi\ ||_{1,\infty,\omega} \leq C_1, \quad \text{for all } \psi \in \mathbf{\Phi}_F(\omega)\ .$$

In other words, the manifold $\mathbf{\Phi}_F(\omega)$ is included in a bounded subset of $\mathrm{W}^{1,\infty}(\omega;\mathbf{R}^3)$.

Step 3. There exists a constant C_2 such that:

$$\sum_{\alpha,\beta} ||\ b_{\alpha\beta}(\psi)\ ||_{0,\omega}^2 \ \geq\ ||\ \psi\ ||_{2,2,\omega}^2 \ +\ C_2, \quad \text{for all } \psi \in \mathbf{\Phi}_F(\omega).$$

This means that over $\mathbf{\Phi}_F(\omega)$ the $\mathrm{L}^2(\omega)$-norm of the curvature tensor (which is well defined by step 1) "uniformly controls" the $\mathrm{H}^2(\omega;\mathbf{R}^3)$-norms of the fields in the manifold $\mathbf{\Phi}_F(\omega)$.

The next step can be easily derived from the compactness of the Sobolev imbedding of $\mathrm{H}^2(\omega;\mathbf{R}^3)$ into $\mathrm{H}^1(\omega;\mathbf{R}^3)$ and the compactness of the trace operator:

Step 4. The manifold $\mathbf{\Phi}_F(\omega)$ is sequentially weakly closed in $\mathrm{H}^2(\omega;\mathbf{R}^3)$ (we let $\rightharpoonup$ denote weak convergence):

$$\psi^l \in \mathbf{\Phi}_F(\omega), l \geq 1, \text{ and } \psi^l \rightharpoonup \psi \text{ in } \mathrm{H}^2(\omega;\mathbf{R}^3) \Rightarrow \psi \in \mathbf{\Phi}_F(\omega).$$

The next steps establish that, while the functional I_F is not defined in general "outside" the manifold $\mathbf{\Phi}_F(\omega)$, it nevertheless possesses the usual properties required to apply the fundamental theorem of the calculus of variations.

Step 5. The functional I_F is sequentially weakly lower semi-continuous over the manifold $\mathbf{\Phi}_F(\omega)$:

$$\psi^l \in \mathbf{\Phi}_F(\omega), l \geq 1, \text{ and } \psi^l \rightharpoonup \psi \text{ in } \mathrm{H}^2(\omega;\mathbf{R}^3) \Rightarrow I_F(\psi) \leq \liminf_{l\to\infty} I_F(\psi^l).$$

Step 6. There exists constants $C_3 > 0$ and C_4 such that:

$$I_F(\psi) \geq C_3\ ||\ \psi\ ||_{2,2,\omega}^2 + C_4, \qquad \text{for all } \psi \in \mathbf{\Phi}_F(\omega).$$

The proof is then concluded by combining steps 4, 5 and 6.

4. Commentary.

(1) While Lods & Miara [4] assumed $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbf{R}^3)$ in order to carry out the asymptotic analysis that leads to the two-dimensional equations of a "flexural" shell, the assumption $\boldsymbol{\theta} \in \mathrm{W}^{2,p}(\omega; \mathbf{R}^3)$ for some $p > 2$ is enough to ensure the existence of a minimizer of the energy over the manifold of admissible deformations.

(2) The existence result immediately extends to the case of a flexural shell submitted to the boundary conditions "of clamping" $\boldsymbol{\varphi} = \boldsymbol{\Phi}_o$ and $\partial_\nu \boldsymbol{\varphi} = \partial_\nu \boldsymbol{\Phi}_o$ along γ_o.

(3) The "interesting" cases covered by this existence theorem are those where $\boldsymbol{\Phi}_F(\omega)$ does not reduce to a discrete set, *i.e.* $\boldsymbol{\Phi}_F(\omega)$ is a "genuine" manifold. This is for instance the case if the middle surface of the shell is a portion of a cone (excluding its vertex), or a portion of a cylinder clamped along a portion of one of its generatrices (if S is a non-flat cylinder, it may be clamped along portions of two generatrices).

(4) A comprehensive presentation and analysis of the different plate and shell problems is given in Ciarlet [1, 2].

Références

[1] Ciarlet P.G., 1997. *Mathematical Elasticity, Volume II: Theory of Plates*, North Holland, Amsterdam.

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