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Johannes Sjöstrand

**Lecture notes : Spectral properties of non-self-adjoint operators**

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# LECTURE NOTES : Spectral properties of non-self-adjoint operators

Johannes Sjöstrand

**Résumé**

Ce texte contient une version légèrement completée de mon cours de 6 heures au colloque d'équations aux dérivées partielles à Évian-les-Bains en juin 2009. Dans la première partie on expose quelques résultats anciens et récents sur les opérateurs non-autoadjoints. La deuxième partie est consacrée aux résultats récents sur la distribution de Weyl des valeurs propres des opérateurs elliptiques avec des petites perturbations aléatoires. La partie III, en collaboration avec B. Helffer, donne des bornes explicites dans le théorème de Gearhardt-Prüss pour des semi-groupes.

**Abstract**

This text contains a slightly expanded version of my 6 hour mini-course at the PDE-meeting in Évian-les-Bains in June 2009. The first part gives some old and recent results on non-self-adjoint differential operators. The second part is devoted to recent results about Weyl distribution of eigenvalues of elliptic operators with small random perturbations. Part III, in collaboration with B. Helffer, gives explicit estimates in the Gearhardt-Prüss theorem for semi-groups.

## CONTENTS



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## 1. Introduction

For self-adjoint and more generally normal operators on some complex Hilbert space $\mathcal{H}$ we have a nice theory, including the spectral theorem and the wellknown and important resolvent estimate,

$$\|(z-P)^{-1}\| \leq (\mathrm{dist}\,(z, \sigma(P)))^{-1}, \tag{1.0.1}$$

where $\sigma(P)$ denotes the spectrum of $P$. The spectral theorem also gives very nice control over functions of self-adjoint operators, so for instance if $P$ is self-adjoint with spectrum contained in the half interval $[\lambda_0, +\infty[$, then

$$\|e^{-tP}\| \leq e^{-\lambda_0 t},\ t \geq 0. \tag{1.0.2}$$

However, non-normal operators appear frequently in different problems; Scattering poles, Convection-diffusion problems, Kramers-Fokker-Planck equation, damped wave equations, linearized operators in fluid dynamics. Then typically, $\|(z-P)^{-1}\|$ may be very large even when $z$ is far from the spectrum and this implies mathematical difficulties:

– When studying the distribution of eigenvalues,

– When studying functions of the operator, like $e^{-tP}$ and its norm.

The largeness of the norm of the resolvent far away from the spectrum also makes the eigenvalues very unstable under small perturbations of the operator and this is a source of mathematical and numerical difficulties.

There are two natural reactions to this problem:

- Change the Hilbert space norm to make the operators look more normal. This is quite natural to do when there is no clear unique choice of the ambient Hilbert space, like in problems for scattering poles (resonances).

- Recognize that the region of the $z$-plane where $\|(z-P)^{-1}\|$ is large, has its own interest. One can then introduce the $\epsilon$-pseudospectrum which is the set of points $z \in \mathbf{C}$, which are either in the spectrum or such that $\|(z-P)^{-1}\| > \epsilon$. and study this set in its own right.

The first point of view will be illustrated in the first part of these notes, and is at the basis of a whole range of methods from that of analytic ditations in the study of resonances to more microlocal methods.

The second point of view has been promoted by numerical analysts like L.N. Trefethen and then made its way into analysis through contributions by E.B. Davies, M. Zworski and others. That spectral instability is not only a nuisance, but can be at the origin of nice and previously unexpected results, will hopefully be clear from the second and main part of these lecture notes, where we shall describe some results about Weyl asymptotics of the distribution of eigenvalues of elliptic operators with small random perturbations. These results have been obtained in recent works by M. Hager [36, 37, 38], Hager and the author [39], W. Bordeaux Montrieux [10], the author [97, 98] and Bordeaux Montriex and the author [11].

The results in the second part of the lectures rely on microlocal analysis, combined with quite classical methods for non-self-adjoint operators and holomorphic functions of one variable and some elementary probability theory. It therefore was natural to include a first part that treats some older and newer results and methods about non-self-adjoint operators. Many of these methods combined with microlocal analysis have been used with great succes in resonance theory, but that is beyond the scope of these lectures.. Let us nevertheless mention results by T. Christiansen [16, 17] and Christiansen–P. Hislop [18] that establish Weyl type lower bounds for the number of scattering poles in large discs in generic situations.

During the meeting we started a discussion with Bernard Helffer about the important theorem of Gearhardt, Prüss et al about how to go from resolvent bounds to semi-group bounds. There seemed to be a lack of explicit estimates in the literature that can be applied in parameter dependent situations. As a result we derived such bounds in a joint work which is included as Part III. It is is related to many questions discussed in Part I and also to Helffer's contribution [40] to these proceedings.

Here is the plan of the notes:

Part I is devoted to some old and recent general results for non-self-adjoint differential operators.

In Section 2 we describe some classical results for the distribution of eigenvalues of elliptic operators, starting with a result of T. Carleman about Weyl asymptotics of the eigenvalues for operators with real principal symbol. We also give a result by S. Agmon about the completeness of the set of generalized eigenvalues as a well as further results by M.S. Agranovich, A.S. Markus, V.I. Matseev.

In Section 3, we start by giving some basic definitions and facts about the $\epsilon$-pseudospectrum, then go on by describing the Davies–Hörmander quasi-mode construction under the assumption that a certain Poisson bracket is non-zero and of suitable sign.

In Section 4, we describe some estimates on the size of norm of the resolvent when the spectral parameter is close to the range of the semi-classical principal symbol. These estimates are closely related to subellipticity estimates for operators of principal type and the results here are due to Dencker–Sjöstrand–Zworski with some recent partial improvement of the author.

In Section 5, we discuss some recent results from two different areas; the Kramers-Fokker-Planck operator and spectral asymptotics for analytic operators in two dimensions. The first topic (based on joint works with F. Hérau–C. Stolk, F. Hérau–M. Hitrik and inspired by works of Hérau–F. Nier, B. Helffer–F. Nier) was chosen because it has a very concrete importance, while the second topic (based on joint works with M. Hitrik and S. Vũ Ngọc) is of interest as an example of how to get precise information about individual eigenvalues. In both situations the methods exploit suitable changes of the Hilbert space norms.

Part II is devoted to Weyl asymptotics for the eigenvalues of elliptic operators with small random perturbations.

In Section 6, we give a result, that generalizes and improves earlier results by Hager and Hager–Sjöstrand about the number of zeros of holomorphic functions of exponential growth, which is clearly close to classical results for entire functions but that we have not found in the literature. This result is used in an essential way in the sequel.

In Section 7 we treat the one-dimensional semi-classical case very much in the spirit of Hager. This case is easier than the general case, and it has some special features that permit to have more precise results, and is most likely the first testing case for more refined questions about statistics and correlation of eigenvalues.

In Section 8 we establish Weyl asymptotics in the multi-dimensional semi-classical case, by combination of complex analysis (Section 6), microlocal analysis spectral theory and probabilistic arguments.

In Section 9, we consider the large eigenvalues of elliptic operators. In the semi-classical case the results say that we have Weyl asymptotics with a probability tending to 1 very fast when Planck's constant tends to zero. The study of large eigenvalues of elliptic operators can often be reduced to a semi-classical study, and by performing such a reduction and applying the Borel-Cantelli lemma, we show that Weyl asymptotics holds almost surely. Such a result was obtained by W. Bordeaux Montrieux for elliptic operators on $S^1$ using the results and the approach of Hager. Here we mainly describe a corresponding multi-dimensional result, obtained jointly with Bordeaux Montrieux, where the semi-classical part is the one described in Section 8.

In Section 10 we formulate some open problems.

As for Part III, we refer to the introduction of that work below.

PART I

# Some general results

## 2. Elliptic non-self-adjoint operators, some classical results

This is a classical area. T. Carleman [14] considered the Dirichlet realization $P$ of a second order elliptic operator in a bounded domain $\Omega \Subset \mathbf{R}^3$, assuming enough smoothnes on the coefficients and on the boundary, and he also assumed that the principal symbol is strictly positive so that the non-self-adjointness can come only from the lower order symbols. In this case we see easily that the spectrum consists of isolated eigenvalues of finite algebraic multiplicity and we will allways count the eigenvalues with their multiplicity. Carleman showed that the eigenvalues are contained in a parabolic neighborhood of the positive real axis and that the real parts are distributed according to the Weyl asymptotics, i.e.

$$\#\{\mu \in \sigma(P);\, \Re\mu \leq \lambda\} = \frac{1}{2\pi}\lambda^{3/2}(\mathrm{vol}\,\{(x,\xi) \in T^*\Omega;\, p(x,\xi) \leq 1\} + o(1)),\ \ \lambda \to \infty \tag{2.0.1}$$

His method of proof consisted in studying the trace of $(P+\kappa^2)^{-1} - (P+\kappa_0^2)^{-1}$ for a fixed $\kappa_0$ in the limit when $\kappa \to \infty$ and to apply a Tauberian argument.

After Carleman there have been important results of Keldysch which have inspired later workers in the field, like Agmon, Agranovich, Markus and Matseev. In a spirit similar to that of Carleman, M.S. Agranovich and A.S. Markus [3] considered a non-self-adjoint elliptic classical pseudodifferential operator on a compact manifold $\Omega$ of dimension $n$, of order $m > 0$. Let $p$ denote the principal symbol. Assume that the range of $p$ is in a sector $\{z \in \mathbf{C};\, |\arg p| < \theta\}$ where $\theta < \pi$. If the quantity

$$d := \frac{1}{(2\pi)^n n}\int_\Omega \int_{|\xi|=1} p(x,\xi)^{-n/m} S(d\xi)dx \tag{2.0.2}$$

is $\neq 0$ then the authors show that

$$N(\lambda) \asymp \lambda^{n/m},\ \ \lambda \to \infty \tag{2.0.3}$$

where $N(\lambda) = \#(\sigma(P) \cap D(0,\lambda))$ and $D(0,\lambda)$ denotes the open disc of radius $\lambda$ centered at 0. They also obtain some more precise inequalities for the quantities $\limsup_{\lambda\to\infty} \lambda^{-n/m}N(\lambda)$ and $\liminf_{\lambda\to\infty} \lambda^{-n/m}N(\lambda)$. Especially when these two limits are equal, then the corresponding quantity is in the interval $[|d|, \Delta]$, where

$$\Delta := \frac{1}{(2\pi)^n n}\int_\Omega \int_{|\xi|=1} |p(x,\xi)|^{-n/m} d\xi dx. \tag{2.0.4}$$

Here it is interesting to notice that $d \neq 0$ when the angle $2\theta$ is smaller than $\pi m/n$. To prove these results, the authors first establish the asymptotic trace formula

$$\mathrm{tr}\,((\mu + A)^{-\ell}) \sim \mathrm{Const.}\mu^{n/m-\ell},\ \ \mu \to +\infty \tag{2.0.5}$$

for $\ell m > n$, where the constant can be expressed with the help of $d$, and then apply a Tauberian argument. Here $(\mu + A)^{-\ell}$ is of trace class when $\ell m > n$.

The operator $P = f(x)D_x + g(x)$ on $S^1$ is elliptic when $f \neq 0$, and its spectral behaviour can be studied explicitly. When the range of $f$ is in a sector of angle less than $\pi$, then we have a nice spectrum sitting on a line, while for larger values, strange things may happen ([10], [92]) and the spectrum may be either equal to $\mathbf{C}$ or empty.

Another interesting and potentially important question is that of the completeness of the set of generalized eigenfunctions. This question was studied by Keldysh and Agmon. Agmon studied elliptic boundary value problems, let us here formulate his result in the case of elliptic differential operators on manifolds without boundary [1] (see also [2]). Assume that $P$ is such an operator of even order $m > 0$ and assume that the symbol $p$ takes its values away from a finite union of closed half-rays $e^{i\theta_j}[0, +\infty[$, where $\theta_1 < \theta_2 < ... < \theta_N$ belong to $[0, 2\pi[$ and assume that the angles $\theta_{j+1} - \theta_j$ are all strictly smaller than $m\pi/n$ (with the convention that $\theta_{N+1} = \theta_1 + 2\pi$). Then the generalized eigenfunctions of $P$ are complete in $L^2$, in the sense that they span a dense subspace.

The proof uses the following ingredients:

- First we show that the resolvent $(z-P)^{-1}$ is well-defined and of norm $\mathcal{O}(|z|^{-1})$ when $z$ tends to infinity along one of the half rays given by $\arg z = \theta_j$. In particular the resolvent exists for at least one value of $z$ and by analytic Fredholm theory this implies that the spectrum of $P$ consists of isolated eigenvalues and each such value is of finite algebraic multiplicity.

- Using a suitable functional determinant which is an entire function of fractional exponential grow and whose zeros are the eigenvalues, we get a polynomial control over the number $N(\lambda)$ when $\lambda \to \infty$ and a corresponding exponential control over the resolvent on a family of circles $\partial D(0, r_j)$, $r_j \to \infty$, when $j \to \infty$:
$$\|(z-P)^{-1}\| \leq \mathcal{O}(1)\exp(|z|^{\frac{n}{m}+\epsilon}),\ |z| = \rho_j, \tag{2.0.6}$$
where $\epsilon > 0$ can be chosen arbitrarily small.

- Of course, $(z-P)^{-1}$ will have poles at the eigenvalues, but if $f_0 \in L^2$ is orthogonal to all the generalized eigenfunctions, then $F_f(z) := ((z-P)^{-1}f|f_0)_{L^2}$ turns out to be an entire function for every $f \in L^2$. Thanks to the condition on the angles, and the exponential control of the resolvent, we can apply the Phragmén Lindelöf theorem to conclude that $F_f(z)$ is actually constant. By restricting to a ray of minimal growth we see that the constant has to be zero, and varying $f$ it is not hard to see from this that $f_0$ has to be zero.

Let us mention that rays of minimal growth of the resolvent have also been used by R. Seeley [91]. We could also point out that under the same ellipticity condition and the assumption that the range of $p$ is not equal to all of $\mathbf{C}$, we have the classical upper bound
$$N(\lambda) = \mathcal{O}(\lambda^{n/m}),\ \lambda \to +\infty. \tag{2.0.7}$$
This can be proved in a similar manner using a relative determinant and the Jensen formula. A more elementary proof is the following, that was pointed out to me by M.S. Agranovich [4]: After replacing $P$ by $P - \lambda_0$ for a sufficiently large $\lambda_0$ on a ray of minimal growth, we may assume that $P$ is bijective. Let $s_1(P^{-1}) \geq s_2(P^{-1}) \geq ...$

be the singular values of the inverse $P^{-1}$ (i.e. the decreasing sequence of eigenvalues of $(P^*P)^{-1/2}$ which is bounded from $L^2$ to $H^m$). If $C > 0$ is large enough we have $(P^*P)^{-1} \leq C(1-\Delta)^{-m}$ where $\Delta$ is the Laplace Beltrami operator on $X$ for some smooth Riemannian metric and this implies corresponding inequalities for the eigenvalues. Applying the Weyl asymptotics for $-\Delta$, we conclude that $s_j(P^{-1}) \leq \mathcal{O}(j^{-m/n})$. Then Corollary 3.2 in Chapter III of [33] implies that $\lambda_j(P^{-1}) = \mathcal{O}(j^{-m/n})$, where $\lambda_j(P^{-1})$ denote the eigenvalues of $P^{-1}$ arranged so that $j \mapsto |\lambda_j(P^{-1})|$ is decreasing. Then (2.0.7) follows.

Markus and V.I. Matseev [72] have established interesting estimates on the difference of the counting function for a self-adjoint (or more generally normal) operator and the counting function for the real parts of the eigenvalues of a small perturbation of that operator. The proofs are based on the use of relative determinants. We will not state the general results, but simply mention a corollary of the general result: Let $P$ be an elliptic differential operator of order $m$ on a compact manifold with positive principal symbol $p(x,\xi)$. Then the eigenvalues are contained in a thin neighborhood of the positive real axis and if $N(\lambda)$ denotes the number of such eigenvalues with real part $\leq \lambda$, then $N(\lambda) = (2\pi)^{-n}\mathrm{Vol}\, p^{-1}([0,1])\lambda^{n/m} + \mathcal{O}(\lambda^{(n-1)/m})$. Notice that the remainder estimate is the same as in the general result of Avakumović [7], Levitan [69] ($m = 2$) and Hörmander [56] (general $m$) for self-adjoint elliptic operators which in turn depend on trace formulas, not for resolvents as in Carleman's approach but for hyperbolic evolution problems.

## 3. Pseudospectrum, quasi-modes and spectral instability

Let $\mathcal{H}$ be a complex Hilbert space and let $P : \mathcal{H} \to \mathcal{H}$ be a closed densely defined operator. Recall that the resolvent set is defined as

$$\rho(P) = \{z \in \mathbf{C};\ P - z : \mathcal{D}(P) \to \mathcal{H} \text{ has a bounded 2-sided inverse}\}.$$

It is an open set, if $z \in \rho(P)$ and $\|(z-P)^{-1}\| = 1/\epsilon$, then the open disc $D(z,\epsilon)$ is contained in $\rho(P)$. The spectrum of $P$ is the closed set

$$\sigma(P) = \mathbf{C} \setminus \rho(P).$$

Following Trefethen–M. Embree [106] we define, for $\epsilon > 0$, the $\epsilon$-pseudospectrum as the open set

$$\sigma_\epsilon(P) = \sigma(P) \cup \{z \in \rho(P);\ \|(z-P)^{-1}\| > 1/\epsilon\}. \tag{3.0.1}$$

Unlike the spectrum, the $\epsilon$-pseudospectrum will change if we replace the given norm on $\mathcal{H}$ by an equivalent one.

$\sigma_\epsilon(P)$ can be characterized as a set of spectral instability, by the following simplified version of a theorem of Roch and Silberman:

$$\sigma_\epsilon(P) = \bigcup_{\substack{Q \in \mathcal{L}(\mathcal{H},\mathcal{H}) \\ \|Q\| < \epsilon}} \sigma(P+Q). \tag{3.0.2}$$

The result becomes more sublte if we use the more traditional definition with a non-strict inequality in (3.0.1).

*Proof.* Let $\widetilde{\sigma}_\epsilon(P)$ denote the right hand side in (3.0.2). If $z \in \mathbf{C} \setminus \sigma_\epsilon(P)$, then by a perturbation argument, we see that $z \in \mathbf{C} \setminus \widetilde{\sigma}_\epsilon(P)$.

Let $z \in \sigma_\epsilon(P)$. If $z \in \sigma(P)$ we also have $z \in \widetilde{\sigma}_\epsilon(P)$, so we may assume that $z \in \rho(P)$. Then $\exists\ u \in \mathcal{D}(P)$, $v \in \mathcal{H}$ such that $\|u\| = 1$, $\|v\| < \epsilon$, $(P-z)u = v$. Let $Q$ be the rank one operator from $\mathcal{H}$ to $\mathcal{H}$, given by $Q\phi = -(\phi|u)v$. Then $\|Q\| = \|u\|\|v\| < \epsilon$ and $(P+Q-z)u = v + Qu = v - v = 0$, so $z \in \sigma(P+Q)$, and $z \in \widetilde{\sigma}_\epsilon(P)$. □

Using the subharmonicity of the function $z \mapsto \|(z-P)^{-1}\|$ we notice that every bounded connected component of $\sigma_\epsilon(P)$ contains an element of $\sigma(P)$.

We next discuss the construction of *quasimodes* for non-normal differential operators which shows that very often we get large $\epsilon$-pseudospectra. The background and starting point is a result by E.B. Davies [21] for non-selfadjoint Schrödinger operators in dimension 1. M. Zworski [113] observed that this is essentially an old result of Hörmander [54, 55](1960), and that we have the following generalization, with $\{a, b\} = a'_\xi \cdot b'_x - a'_x \cdot b'_\xi = H_a(b)$ denoting the Poisson bracket of $a = a(x, \xi)$, $b(x, \xi)$.

**Theorem 3.1.** *Let*

$$P(x, hD_x) = \sum_{|\alpha|\le m} a_\alpha(x)(hD_x)^\alpha,\ D_x = \frac{1}{i}\frac{\partial}{\partial x} \tag{3.0.3}$$

*have smooth coefficients in the open set* $\Omega \subset \mathbf{R}^n$. *Put* $p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$. *Assume* $z = p(x_0, \xi_0)$ *with* $\frac{1}{i}\{p, \overline{p}\}(x_0, \xi_0) > 0$. *Then* $\exists\ u = u_h$, *with* $\|u\| = 1$, $\|(P-z)u\| = \mathcal{O}(h^\infty)$, *when* $h \to 0$.

In the case when the coefficients are all analytic we can replace "$h^\infty$" by "$e^{-1/Ch}$ for some $C > 0$".

Notice that this implies that if the resolvent $(P-z)^{-1}$ exists then its norm is greater than any negative power of $h$ when $h \to 0$ (and even exponentially large in the analytic case).

In the case $n \ge 2$, we noticed with A. Melin in [77] that if $z = p(\rho)$ and $\Re p$, $\Im p$ are independent at $\rho$, then $\frac{1}{i}\{p, \overline{p}\}$ times the natural Liouville measure is equal to a constant times the restriction to $p^{-1}(z)$ of $\sigma^{n-1}$ which is a closed form. It follows that if $\Gamma$ is a compact connected component of $p^{-1}(z)$ on which $d\Re p$ and $d\Im p$ are pointwise independent, then the average of $\frac{1}{i}\{p, \overline{p}\}$ over $\Gamma$ with respect to the Liouville measure has to vanish. Hence if there is a point on $\Gamma$ where the Poisson bracket is $\ne 0$ then there is also point where it is positive. In the case $n = 1$ we have a similar phenomenon: If (for instance thanks to suitable ellipticity assumption) we know that $p^{-1}(z)$ is finite and that $\frac{1}{i}\{p, \overline{p}\}$ is $\ne 0$ everywhere on that set, then this set is finite and if it is contained in the interior of a connected bounded set $\Omega$ in phase space with smooth boundary such that the variation of $\arg(p - z)$ along that boundary is equal to zero, then we have to have an equal number of points in $p^{-1}(z)$ where $\frac{1}{i}\{p, \overline{p}\}$ is positive and where it is negative. This follows from the observation that the argument variation of $p - z$ along a small positively oriented circle around a point in $p^{-1}(z)$ is $\mp 2\pi$ when $\pm\frac{1}{i}\{p, \overline{p}\} > 0$ at that point.

**Example 3.2.** $P = -h^2\Delta + V(x)$, $p(x, \xi) = \xi^2 + V(x)$, $\frac{1}{i}\{p, \overline{p}\} = -4\xi \cdot \Im V'(x)$.

More recently K. Pravda-Starov [82] improved this result by adapting a more refined quasi-mode construction of R. Moyer (in 2 dimensions) and Hörmander [58] for adjoints of operators that do not satisfy the Nirenberg-Trèves condition $(\Psi)$ for local solvability.

The proof in the $C^\infty$-case in [113] is by a standard reduction of semi-classical results to classical results in ordinary microlocal analysis. In [26] we gave a direct proof and also treated the case of analytic coefficients, which is also essentially quite old. Here is a brief outline of a

*Proof.* In the following we use the notation neigh $(a, A)$ for "some neighborhood of $a$ in $A$". If $\phi \in C^\infty(\text{neigh}\,(x_0, \mathbf{R}^n))$ satisfies $\phi'(_0) : \xi_0 \in \mathbf{R}^n$, and

$$\Im\phi''(x_0) > 0, \tag{3.0.4}$$

then we can define the complex Lagrangian manifold

$$\Lambda_\phi := \{(x, \phi'(x));\ x \in \text{neigh}\,(x_0, \mathbf{C}^n)\} \tag{3.0.5}$$

where we extend $\phi$ to a complex neighborhood by taking an almost holomorphic extension, i.e. a smooth extension such that $\overline{\partial}\phi = \mathcal{O}((\Im x)^\infty)$. In this case we can content ourselves with working with formal Taylor expansions at $x_0$, and then $\Lambda_\phi$ can be viewed as an equivalence class of real submanifolds of the complexified phase space $\mathbf{C}^{2n}$ where two submanifolds are equivalent if they agree to infinite order at $(x_0, \xi_0)$. As observed by Hörmander [57] and developped a lot by the author with A. Melin in [76] and in other works, the positivity assumption (3.0.4) can be formulated equivalently by saying that

$$\frac{1}{i}\sigma(t, \overline{t}) > 0,\ 0 \neq t \in T_{(x_0,\xi_0)}(\Lambda_\phi), \tag{3.0.6}$$

where $\sigma$ denotes the symplectic 2-form, here viewed as a bilinear form on the complexified tangentspace of the cotangent space at $(x_0, \xi_0)$.

Let $z, p, (x_0, \xi_0)$ be as in the theorem. Then we observe that $\frac{1}{i}\sigma(H_p, \overline{H_p}) = \frac{1}{i}\{p, \overline{p}\} > 0$. Moreover, the real set $\Sigma := p^{-1}(z)$ is a smooth symplectic manifold near $(x_0, \xi_0)$ and using the Darboux theorem, we can identify it with $\mathbf{R}^{2(n-1)}$ and hence find a Lagrangian submanifold $\Lambda'$ in its compexification passing through $(x_0, \xi_0)$ that satisfies the positivity condition (3.0.6). Viewing the complexification of $\Sigma$ as a submanifold of $\mathbf{C}^{2n}$, we can take $\Lambda = \{\exp sH_p(\rho);\ s \in \text{neigh}\,(0, \mathbf{C}),\ \rho \in \text{neigh}\,((x_0, \xi_0), \mathbf{C}^{2n})\}$. Using that $H_p$ is symplectically orthogonal to the tangent space of $\Sigma$ it is then quite easy to verify that $\Lambda$ is a complex Lagrangian manifold to $\infty$ order at $(x_0, \xi_0)$ contained (to infinite order) in the complex characteristic hypersurface $\{\rho \in \text{neigh}\,((x_0, \xi_0), \mathbf{C}^{2n});\ p(\rho) = 0\}$ and satisfying the positivity condition (3.0.6). Hence to infinite order, $\Lambda$ is of the form $\Lambda_\phi$ for a function $\phi$ as in (3.0.4), (3.0.5), which also fulfills the eiconal equation

$$p(x, \phi'(x)) = \mathcal{O}(|x - x_0|^\infty). \tag{3.0.7}$$

We normalize $\phi$ by requiring that $\phi(x_0) = 0$. Then the function $e^{i\phi(x)/h}$ is rapidly decreasing with all its derivatives away from any neighborhood of $x_0$, and by a complex version of the standard WKB-construction we can construct an elliptic symbol $a(x; h) \asymp a_0(x) + ha_1(x) + ...$, by solving the suitable transport equations to infinite order at $x_0$, such that if $\chi \in C_0^\infty(\text{neigh}\,(x_0, \mathbf{R}^n))$ is equal to 1 near $x_0$, then $u(x; h) = \chi(x)h^{-n/4}a(x; h)e^{i\phi(x)/h}$ has the required properties. ☐

# 4. Boundary estimates of the resolvent

## 4.1. Introduction

In this section we are interested in bounds on the resolvent of an $h$-pseudodifferential operator when $z$ is close to the boundary of the range of $p$. As with the quasi-mode construction this question is closely related to classical results in the general theory of linear PDE, and with N. Dencker and Zworski ([26]) we were able to find quite general results closely related to the classical topic of subellipticity for pseudodifferential operators of principal type, studied by Egorov, Hörmander and others. See [58].

In [26] we obtained resolvent estimates at certain boundary points,
(A) under a non-trapping condition,
and
(B) under a stronger "subellipticity condition".

In case (A) we could apply quite general and simple arguments related to the propagation of regularity and in case (B) we were able to adapt general Weyl-Hörmander calculus and Hörmander's treatment of subellipticity for operators of principal type ([58]). In the first case we obtained that the resolvent extends and has temperate growth in $1/h$ in discs of radius $\mathcal{O}(h\ln 1/h)$ centered at the appropriate boundary points, while in case (B) we got the corresponding extension up to distance $\mathcal{O}(h^{k/(k+1)})$, where the integer $k\geq 2$ is determined by a condition of "subellipticity type".

Using a method based on semi-groups led to a strengthened result in case (B): The resolvent can be extended to a disc of radius $\mathcal{O}((h\ln 1/h)^{k/(k+1)})$ around the appropriate boundary points.

Let $X$ be equal to $\mathbf{R}^n$ or equal to a compact smooth manifold of dimension $n$.

In the first case, let $m\in C^\infty(\mathbf{R}^{2n};[1,+\infty[)$ be an order function (see [27] for more details about the pseudodifferential calculus) in the sense that for some $C_0,N_0>0$,

$$m(\rho)\leq C_0\langle\rho-\mu\rangle^{N_0}m(\mu),\ \ \rho,\mu\in\mathbf{R}^{2n}, \tag{4.1.1}$$

where $\langle\rho-\mu\rangle=(1+|\rho-\mu|^2)^{1/2}$. Let $P=P(x,\xi;h)\in S(m)$, meaning that $P$ is smooth in $x,\xi$ and satisfies

$$|\partial^\alpha_{x,\xi}P(x,\xi;h)|\leq C_\alpha m(x,\xi),\ \ (x,\xi)\in\mathbf{R}^{2n},\ \alpha\in\mathbf{N}^{2n}, \tag{4.1.2}$$

where $C_\alpha$ is independent of $h$. We also assume that

$$P(x,\xi;h)\sim p_0(x,\xi)+hp_1(x,\xi)+...,\ \text{ in } S(m), \tag{4.1.3}$$

and write $p=p_0$ for the principal symbol. We impose the ellipticity assumption

$$\exists w\in\mathbf{C},\ C>0,\ \text{such that } |p(\rho)-w|\geq m(\rho)/C,\ \forall\rho\in\mathbf{R}^{2n},\ |\rho|\geq C. \tag{4.1.4}$$

In this case we let

$$P=P^w(x,hD_x;h)=\mathrm{Op}(P(x,h\xi;h)) \tag{4.1.5}$$

be the Weyl quantization of the symbol $P(x,h\xi;h)$ that we can view as a closed unbounded operator on $L^2(\mathbf{R}^n)$.

In the second case when $X$ is compact manifold, we let $P\in S^m_{1,0}(T^*X)$ (the classical Hörmander symbol space ) of order $m\geq 0$, meaning that

$$|\partial^\alpha_x\partial^\beta_\xi P(x,\xi;h)|\leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|},\ \ (x,\xi)\in T^*X, \tag{4.1.6}$$

where $C_{\alpha,\beta}$ are independent of $h$. We also assume that we have an expansion of the type (4.1.3), now in the sense that

$$P(x,\xi;h)-\sum_0^{N-1} h^j p_j(x,\xi)\in h^N S_{1,0}^{m-N}(T^*X),\ N=1,2,... \tag{4.1.7}$$

and we quantize the symbol $P(x,h\xi;h)$ in the standard (non-unique) way, by doing it for various local coordinates and paste the quantizations together by means of a partition of unity. In the case $m>0$ we impose the ellipticity condition

$$\exists C>0,\ \text{such that}\ |p(x,\xi)|\geq \frac{\langle\xi\rangle^m}{C},\ |\xi|\geq C. \tag{4.1.8}$$

Let $\Sigma(p)=\overline{p^*(T^*X)}$ and let $\Sigma_\infty(p)$ be the set of accumulation points of $p(\rho_j)$ for all sequences $\rho_j\in T^*X$, $j=1,2,3,..$ that tend to infinity. By pseudodifferential calculus, if $K\subset \mathbf{C}\setminus\Sigma(p)$ is compact, then $(z-P)^{-1}$ exists and has uniformly bounded operator norm for $z\in K$, $0<h\ll 1$.

The following theorem ([99]) is a partial improvement of corresponding results in [26].

**Theorem 4.1.** *We adopt the general assumptions above. Let $z_0\in\partial\Sigma(p)\setminus\Sigma_\infty(p)$ and assume that $dp\neq 0$ at every point of $p^{-1}(z_0)$. Then for every such point $\rho$ there exists $\theta\in\mathbf{R}$ (unique up to a multiple of $\pi$) such that $d(e^{-i\theta}(p-z_0))$ is real at $\rho$. We write $\theta=\theta(\rho)$. Consider the following two cases:*

- *(A) For every $\rho\in p^{-1}(z_0)$, the maximal integral curve of $H_{\Re(e^{-i\theta(\rho)}p)}$ through the point $\rho$ is not contained in $p^{-1}(z_0)$.*
- *(B) There exists an integer $k\geq 1$ such that for every $\rho\in p^{-1}(z_0)$, there exists $j\in\{1,2,..,k\}$ such that*

  $$p^*(\exp tH_p(\rho))=at^j+\mathcal{O}(t^{j+1}),\ t\to 0,$$

  *where $a=a(\rho)\neq 0$. Here $p$ also denotes an almost holomorphic extension to a complex neighborhood of $\rho$ and we put $p^*(\mu)=\overline{p(\overline{\mu})}$. Equivalently, $H_p^j(\overline{p})(\rho)/(j!)=a\neq 0$.*

*Then, in case (A), there exists a constant $C_0>0$ such that for every constant $C_1>0$ there is a constant $C_2>0$ such that the resolvent $(z-P)^{-1}$ is well-defined for $|z-z_0|<C_1h\ln\frac{1}{h}$, $h<\frac{1}{C_2}$, and satisfies the estimate*

$$\|(z-P)^{-1}\|\leq\frac{C_0}{h}\exp(\frac{C_0}{h}|z-z_0|). \tag{4.1.9}$$

*In case (B), there exists a constant $C_0>0$ such that for every constant $C_1>0$ there is a constant $C_2>0$ such that the resolvent $(z-P)^{-1}$ is well-defined for $|z-z_0|<C_1(h\ln\frac{1}{h})^{k/(k+1)}$, $h<\frac{1}{C_2}$ and satisfies the estimate*

$$\|(z-P)^{-1}\|\leq\frac{C_0}{h^{\frac{k}{k+1}}}\exp(\frac{C_0}{h}|z-z_0|^{\frac{k+1}{k}}). \tag{4.1.10}$$

In [26] we obtained (4.1.10) for $z=z_0$, implying that the resolvent exists and satisfies the same bound for $|z-z_0|\leq h^{k/(k+1)}/\mathcal{O}(1)$ in case (B) and with $k/(k+1)$ replaced by 1 in case (A). In case (A) we also showed that the resolvent exists with norm bounded by a negative power of $h$ in any disc $D(z_0,C_1h\ln(1/h))$. (The condition in case (B) was formulated a little differently in [26], but the two conditions lead to the same microlocal models and hence they are equivalent.) The case (A) of the theorem is basically identical with the corresponding result in [26] and was proved using weighted estimates with weights that have at most polynomial growth in $h$. We will not disuss that in detail here and instead we concentrate on the partially new result in the case (B).

When $k = 2$ more direct methods are available and more precise bounds can be given, at least in special cases. Such results have been obtained by J. Martinet [73], Y. Almog, B. Helffer, X. Pan, see [40] and W. Bordeaux Montrieux [10].

Let us now consider the special situation of potential interest for evolution equations, namely the case when

$$z_0 \in i\mathbf{R}, \tag{4.1.11}$$

$$\Re p(\rho) \geq 0 \text{ in neigh}\,(p^{-1}(z_0), T^*X). \tag{4.1.12}$$

**Theorem 4.2.** *We adopt the general assumptions above. Let $z_0 \in \partial\Sigma(p) \setminus \Sigma_\infty(p)$ and assume (4.1.11), (4.1.12). Also assume that $dp \neq 0$ on $p^{-1}(z_0)$, so that $d\Im p \neq 0$, $d\Re p = 0$ on that set. Consider the two cases of Theorem 4.1:*

- *(A) For every $\rho \in p^{-1}(z_0)$, the maximal integral curve of $H_{\Im p}$ through the point $\rho$ contains a point where $\Re p > 0$.*
- *(B) There exists an integer $k \geq 1$ such that for every $\rho \in p^{-1}(z_0)$, we have $H_{\Im p}^j \Re p(\rho) \neq 0$ for some $j \in \{1, 2, ..., k\}$.*

*Then, in case (A), there exists a constant $C_0 > 0$ such that for every constant $C_1 > 0$ there is a constant $C_2 > 0$ such that the resolvent $(z - P)^{-1}$ is well-defined for*

$$|\Im(z - z_0)| < \frac{1}{C_0},\ \frac{-1}{C_0} < \Re z < C_1 h \ln \frac{1}{h},\ h < \frac{1}{C_2},$$

*and satisfies the estimate*

$$\|(z - P)^{-1}\| \leq \begin{cases} \frac{C_0}{|\Re z|}, & \Re z \leq -h, \\ \frac{C_0}{h} \exp(\frac{C_0}{h} \Re z), & \Re z \geq -h. \end{cases} \tag{4.1.13}$$

*In case (B), there exists a constant $C_0 > 0$ such that for every constant $C_1 > 0$ there is a constant $C_2 > 0$ such that the resolvent $(z - P)^{-1}$ is well-defined for*

$$|\Im(z - z_0)| < \frac{1}{C_0},\ \frac{-1}{C_0} < \Re z < C_1 (h \ln \frac{1}{h})^{\frac{k}{k+1}},\ h < \frac{1}{C_2}, \tag{4.1.14}$$

*and satisfies the estimate*

$$\|(z - P)^{-1}\| \leq \begin{cases} \frac{C_0}{|\Re z|}, & \Re z \leq -h^{\frac{k}{k+1}}, \\ \frac{C_0}{h^{\frac{k}{k+1}}} \exp(\frac{C_0}{h} (\Re z)_+^{\frac{k+1}{k}}), & \Re z \geq -h^{\frac{k}{k+1}}. \end{cases} \tag{4.1.15}$$

## 4.2. Outline of the proof in case (B)

Away from the set $p^{-1}(z_0)$ we can use ellipticity, so the problem is to obtain microlocal estimates near a point $\rho_0 \in p^{-1}(z_0)$. After a standard factorization of $P - z$ in such a region, we can further reduce the proof of the first theorem to that of the second one.

The main (quite standard) idea of the proof of Theorem 4.2 is to study $\exp(-tP/h)$ (microlocally) for $0 \le t \ll 1$ and to show that in this case

$$\| \exp -\frac{tP}{h} \| \le C \exp(-\frac{t^{k+1}}{Ch}), \tag{4.2.1}$$

for some constant $C > 0$. Noting that that implies that $\| \exp -\frac{tP}{h} \| = \mathcal{O}(h^\infty)$ for $t \ge h^\delta$ when $\delta(k+1) < 1$, and using the formula

$$(z-P)^{-1} = -\frac{1}{h} \int_0^\infty \exp(\frac{t(z-P)}{h}) dt, \tag{4.2.2}$$

we get to (4.1.15).

The most direct way of studying $\exp(-tP/h)$, or rather a microlocal version of that operator, is to view it as a Fourier integral operator with complex phase ([74, 63, 76, 75]) of the form

$$U(t)u(x) = \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}(\phi(t,x,\eta) - y\cdot\eta)} a(t,x,\eta;h) u(y) dy d\eta, \tag{4.2.3}$$

where the phase $\phi$ should have a non-negative imaginary part and satisfy the Hamilton-Jacobi equation:

$$i\partial_t \phi + p(x, \partial_x \phi) = \mathcal{O}((\Im \phi)^\infty), \text{ locally uniformly}, \tag{4.2.4}$$

with the initial condition

$$\phi(0,x,\eta) = x \cdot \eta. \tag{4.2.5}$$

The amplitude $a$ will be bounded with all its derivatives and has an asymptotic expansion where the terms are determined by transport equations. This can indeed be carried out in a classical manner for instance by adapting the method of [76] to the case of non-homogeneous symbols following a reduction used in [79, 75]. It is based on making estimates on the fonction

$$S_\gamma(t) = \Im (\int_0^t \xi(s) \cdot dx(s)) - \Re \xi(t) \cdot \Im x(t) + \Re \xi(0) \cdot \Im x(0)$$

along the complex integral curves $\gamma : [0,T] \ni s \mapsto (x(s), \xi(s))$ of the Hamilton field of $p$. Notice that here and already in (4.2.4), we need to take an almost holomorphic extension of $p$. Using the property (B) one can show that $\Im \phi(t,x,\eta) \ge C^{-1} t^{k+1}$ and from that we can obtain (a microlocalized version of) (4.2.1) quite easily.

Finally, we prefered a variant: Let

$$Tu(x) = Ch^{-\frac{3n}{4}} \int e^{\frac{i}{h}\phi(x,y)} u(y) dy,$$

be an FBI – or (generalized) Bargmann-Segal transform that we treat in the spirit of Fourier integral operators with complex phase as in [94]. Here $\phi$ is holomorphic in a neighborhood of $(x_0, y_0) \in \mathbf{C}^n \times \mathbf{R}^n$, and $-\phi'_y(x_0,y_0) = \eta_0 \in \mathbf{R}^n$, $\Im \phi''_{y,y}(x_0,y_0) > 0$, $\det \phi''_{x,y}(x_0,y_0) \neq 0$. Let $\kappa_t : (y, -\phi'_y(x,y)) \mapsto (x, \phi'_x(x,y))$ be the associated canonical transformation. Then microlocally, $T$ is bounded $L^2 \to H_{\Phi_0} := \mathrm{Hol}\,(\Omega) \cap L^2(\Omega, e^{-2\Phi_0/h} L(dx))$ and has (microlocally) a bounded inverse, where $\Omega$ is a small complex neighborhood of $x_0$ in $\mathbf{C}^n$. Here the weight $\Phi_0$ is smooth and strictly plurisubharmonic. If $\Lambda_{\Phi_0} := \{(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}); x \in \mathrm{neigh}\,(x_0)\}$, then (in the sense of germs) $\Lambda_{\Phi_0} = \kappa_T(T^*X)$. The conjugated operator $\widetilde{P} = TPT^{-1}$ can be defined locally modulo $\mathcal{O}(h^\infty)$ (see also [64]) as a bounded operator from $H_\Phi \to H_\Phi$ provided that the weight $\Phi$ is smooth and satisfies $\Phi' - \Phi'_0 = \mathcal{O}(h^\delta)$ for some $\delta > 0$. (In the analytic frame work this condition can be relaxed.) Egorov's theorem applies in this situation, so the leading symbol $\widetilde{p}$ of $\widetilde{P}$ is given by $\widetilde{p}\circ\kappa_T = p$. Thus (under the assumptions of Theorem 4.2) we have $\Re\widetilde{p}_{|\Lambda_{\Phi_0}} \geq 0$, which in turn can be used to see that for $0 \leq t \leq h^\delta$, we have $e^{-t\widetilde{P}/h} = \mathcal{O}(1)$: $H_{\Phi_0} \to H_{\Phi_t}$, where $\Phi_t \leq \Phi_0$ is determined by the real Hamilton-Jacobi problem

$$\frac{\partial\Phi_t}{\partial t} + \Re\widetilde{p}(x, \frac{2}{i}\frac{\partial\Phi_t}{\partial x}) = 0,\ \Phi_{t=0} = \Phi_0. \tag{4.2.6}$$

Here is a somewhat formal derivation of (4.2.6):

Consider formally:

$$(e^{-t\widetilde{P}/h}u|e^{-t\widetilde{P}/h}u)_{H_{\Phi_t}} = (u_t|u_t)_{H_{\Phi_t}},\ u\in H_{\Phi_0},$$

and try to choose $\Phi_t$ so that the time derivative of this expression vanishes to leading order. We get

$$\begin{aligned} 0 &\approx h\partial_t\int u_t\overline{u}_t e^{-2\Phi_t/h}L(dx) \\ &= -\left((\widetilde{P}u_t|u_t)_{H_{\Phi_t}} + (u_t|\widetilde{P}u_t)_{H_{\Phi_t}} + \int 2\frac{\partial\Phi_t}{\partial t}(x)|u|^2 e^{-2\Phi_t/h}L(dx)\right). \end{aligned}$$

Here

$$(\widetilde{P}u_t|u_t)_{H_{\Phi_t}} = \int(\widetilde{p}_{|\Lambda_{\Phi_t}} + \mathcal{O}(h))|u_t|^2 e^{-2\Phi_t/h}L(dx),$$

and similarly for $(u_t|\widetilde{P}u_t)_{H_{\Phi_t}}$, so we would like to have

$$0 \approx \int(2\frac{\partial\Phi_t}{\partial t} + 2\Re\widetilde{p}_{|\Lambda_{\Phi_t}} + \mathcal{O}(h))|u_t|^2 e^{-2\Phi_t/h}L(dx).$$

We choose $\Phi_t$ to be the solution of (4.2.6). Then the preceding discussion again shows that $e^{-t\widetilde{P}/h} = \mathcal{O}(1) : H_{\Phi_0} \to H_{\Phi_t}$.

To get (4.2.1), it suffices to show that $\Phi_t \leq \Phi_0 - t^{k+1}/C$ for $0 \leq t \ll 1$. By a geometric discussion, this follows from

$$G_t(\rho) \leq -t^{k+1}/C, \tag{4.2.7}$$

where $G_t$ is a smooth function in a real neighborhood of $\rho_0$, given by

$$\frac{\partial G_t(\rho)}{\partial t} + \Re p(\rho + iH_{G_t}(\rho)) = 0,\ G_0 = 0. \tag{4.2.8}$$

The behaviour of $G_t$ is easy to understand by means of Taylor expansion of the first equation in (4.2.8) at the point $\rho$.

## 4.3. Examples

Consider

$$P = -h^2\Delta + iV(x),\ V\in C^\infty(X;\mathbf{R}), \tag{4.3.1}$$

where either $X$ is a smooth compact manifold of dimension $n$ or $X = \mathbf{R}^n$. In the second case we assume that $p = \xi^2 + iV(x)$ belongs to a symbol space $S(m)$ where $m \geq 1$ is an order function. It is easy to give quite general sufficient condition for this to happen, let us just mention that if $V \in C_b^\infty(\mathbf{R}^2)$ then we can take $m = 1 + \xi^2$ and if $\partial^\alpha V(x) = \mathcal{O}((1+|x|)^2)$ for all $\alpha\in\mathbf{N}^n$ and satisfies the ellipticity condition $|V(x)| \geq C^{-1}|x|^2$ for $|x| \geq C$, for some constant $C > 0$, then we can take $m = 1 + \xi^2 + x^2$.

We have $\Sigma(p) = [0, \infty[+i\overline{V(X)}$. When $X$ is compact then $\Sigma_\infty(p)$ is empty and when $X = \mathbf{R}^n$, we have $\Sigma_\infty(p) = [0, \infty[+i\Sigma_\infty(V)$, where $\Sigma_\infty(V)$ is the set of accumulation points at infinity of $V$.

Let $z_0 = x_0 + iy_0 \in \partial\Sigma(p) \setminus \Sigma_\infty(p)$.

- In the case $x_0 = 0$ we see that Theorem 4.2 (B) is applicable with $k = 2$, provided that $y_0$ is not a critical value of $V$. This is close to problems from fluid dynamics, studied by I. Gallagher, T. Gallay, F. Nier [32].

- Now assume that $x_0 > 0$ and that $y_0$ is either the maximum or the minimum of $V$. In both cases, assume that $V^{-1}(y_0)$ is finite and that each element of that set is a non-degenerate maximum or minimum. Then Theorem 4.2 (B) is applicable to $\pm iP$. By allowing a more complicated behaviour of $V$ near its extreme points, we can produce examples where 4.2 (B) applies with $k > 2$.

Now, consider the non-self-adjoint harmonic oscillator

$$Q = -\frac{d^2}{dy^2} + iy^2 \tag{4.3.2}$$

on the real line, studied by Boulton [12] and Davies [22], K. Pravda-Starov [83]. Consider a large spectral parameter $E = i\lambda + \mu$ where $\lambda \gg 1$ and $|\mu| \ll \lambda$. The change of variables $y = \sqrt{\lambda}x$ permits us to identify $Q$ with $Q = \lambda P$, where $P = -h^2 \frac{d^2}{dx^2} + ix^2$ and $h = 1/\lambda \to 0$. Hence $Q - E = \lambda(P - (i + \frac{\mu}{\lambda}))$ and Theorem 4.2 (B) is applicable with $k = 2$. We conclude that $(Q - E)^{-1}$ is well-defined and of polynomial growth in $\lambda$ (which can be specified further) when

$$\frac{\mu}{\lambda} \leq C_1(\lambda^{-1} \ln \lambda)^{\frac{2}{3}},$$

for any fixed $C_1 > 0$, i.e. when

$$\mu \leq C_1 \lambda^{\frac{1}{3}} (\ln \lambda)^{\frac{2}{3}}. \tag{4.3.3}$$

M. Hitrik and K. Pravda-Starov [51] have obtained interesting results on the characterization of the exponential decay of the the semi-groups generated by differential operators with quadratic symbols.

# 5. Survey of some recent results

## 5.1. Introduction

In this section we survey some recent results about Kramers-Fokker-Planck type operators and about Bohr-Sommerfeld quantization conditions in dimension 2. In both cases our approach uses quite essentially the possibility of modifying the Hilbert space structure by means of exponential weights.

In recent years there has been many works that apply the commutator methods developed by Kohn for subelliptic operators to equations of Fokker-Planck type and non-equilibriaum stat physics models. Especially there is the work by F. Hérau–F. Nier [47] devoted to the Kramers-Fokker-Planck operator

$$P = y \cdot h\partial_x - V'(x) \cdot h\partial_y + \frac{1}{2}(y - h\partial_y) \cdot (y + h\partial_y),\ (x, y) \in \mathbf{R}^{2n} = \mathbf{R}^n_x \times \mathbf{R}^n_y. \quad (5.1.1)$$

Symbol: $p = ip_2 + p_1$, where

$$p_1(x, y, \xi, \eta) = \frac{1}{2}(y^2 + \eta^2),\ p_2(x, y, \xi, \eta) = y \cdot \xi - V'(x) \cdot \eta \quad (5.1.2)$$

Here we have suppressed some physical parameters and only kept $h$ which is proportional to the temperature. Using commutator techniques Hérau and Nier establish an interesting link to a Witten Laplacian and under various general symbol type assumtions on $V$ they show:

- global subellipticity ($P$ is not elliptic even locally), and $m$-accretivity,
- absence of large eigenvalues and corresponding power-decay of the resolvent in certain "parabolic" neighborhoods of $i\mathbf{R}$,
- Estimates on the smallest non-vanishing eigenvalue (relating it to the corresponding quantity for the Witten laplacian), and especially estimates on this quantity in the high and low temperature limit.
- Precise estimates on the return to equilibrium (when 0 is an eigenvalue) or simply decay when 0 is not an eigenvalue, including estimates on the exponential rate of convergence.

With F. Hérau and C. Stolk [48] we applied microlocal methods in order to study the semi-classical (low temperature) limit, the results are fewer than the ones in [47] and for me easier to describe.

Assume that $V$ is a Morse function, such that

$$|V'(x)| \geq 1/C \text{ when } |x| > C, \quad \partial^\alpha V''(x) = \mathcal{O}(1), \forall \alpha \in \mathbf{N}^n. \quad (5.1.3)$$

We showed that $\exists c > 0$ such that for any $C \geq 1$ we have have for $h > 0$ small enough:

- The eigenvalues in the disc $D(0, Ch)$ are of the form $\mu = \mu(\lambda) = \lambda h + o(h)$, where $\lambda$ are the eigenvalues of the quadratic approximation of $P_{|h=1}$ at the critical points of $V$.
- For $|z| \geq Ch$ and $\Re z \leq c|z|^{1/3}h^{2/3}$ the resolvent exists and satisfies the estimate
$$\|(z - P)^{-1}\| \leq \frac{C}{|z|^{1/3}h^{2/3}}.$$

This is similar to Theorem 4.2 (B) in the case $k = 2$. with an essential difference: $p$ has no values in $i\mathbf{R} \setminus \{0\}$, so $\partial\Sigma \subset \Sigma_\infty$! Nevertheless we can compute $H_{p_2}^2 p_1 = V'(x)^2 + \xi^2 - (V''(x)y \cdot y + V''(x)\eta \cdot \eta)$ and see that when $p_1$ is small, then $H_{p_2}^2 p_1 > 0$ except near points where $\xi = V'(x) = 0$. This means that where $p_1$ is very small, the short time averages are not small. This can be exploited by conjugating the operator by an operator which is bounded and has a bounded inverse and for which the conjugated operator has a symbol with a larger real part.

Below we shall discuss a more general supersymmetric situation and also give detailed results on the return to equilibrium for the associated heat equation, following two joint works with F. Hérau and M. Hitrik.

In the case of the KFP equation we make a moderate change of the Hilbert space norm in order to increase the real part of the operator away from certain critical points of the symbol. When we have analyticity assumptions such changes of the norm can be much larger and sometime allow us to study all eigenvalues also at fixed distance inside the range of $p$. Here is one such result by M. Hitrik [50]:

Let $P = P(x, hD_x; h)$ on $\mathbf{R}$ satisfying the general conditions of section 4. Also assume that $P$ has a holomorphic extension to a tubular neighborhood of $\mathbf{R}^2$ in $\mathbf{C}^2$ which is still $\mathcal{O}(m(\Re(x,\xi))$ Let $z_0 \in \partial\Sigma\backslash\Sigma_\infty$ be a point such that $p^{-1}(z_0) = \{(x_0, \xi_0)\}$ and such that $|p(x,\xi)-z_0| \asymp |(x-x_0, \xi-\xi_0)|^2$ for $(x,\xi)$ in a neighborhood of $(x_0, \xi_0)$. Also assume that there is a truncated sector $z_0+]0,\epsilon_0]e^{i[\theta_0-\epsilon_0,\theta_0+\epsilon_0]}$ which is disjoint from $\Sigma(p)$. Then in a small but fixed neighborhood of $z_0$, the spectrum is given by the set of values $z_0 + G((k+\frac{1}{2})h; h) + \mathcal{O}(h^\infty)$, where $G(\cdot; h)$ is holomorphic and $\sim \sum_0^\infty h^j G_j(\cdot; h)$, $k = 0, 1, 2, \ldots$, where $G_0(0) = 0$, $G_0'(0) \neq 0$.

The main idea is to construct an IR-manifold $\Lambda$ (a complex deformation of real phase space), containing $(x_0, \xi_0)$, such that on $\Lambda$, $p - z_0$ is elliptic outside an arbitrarily small neighborhood of $(x_0, \xi_0)$ and such that near that point, the restriction of the quadratic part of $p$ to $\Lambda$ takes its values along a ray in the complex plane. To implement this picture, we make a Bargmann transform, mapping $L^2$ into an exponentially weighted space of holomorphic functions, then deform the weight.

In the second part of this section we discuss precise Bohr-Sommerfeld rules in dimension 2 for non-self-adjoint operators in the semi-classical limit. What is remarkable here is that thanks to the non-self-adjointness we get better results than what would be possible in the self-adjoint case. Again analyticity assumptions and the use of exponential weights on the Bargmann transform side is essential. We will follow recent works with M. Hitrik and S. Vũ Ngọc [53, 52]

## 5.2. Kramers-Fokker-Planck type operators, spectrum and return to equilibrium

### 5.2.1. Introduction

There has been a renewed interest in the problem of "return to equilibrium" for various 2nd order operators. One example is the Kramers-Fokker-Planck operator:

$$P = y \cdot h\partial_x - V'(x) \cdot h\partial_y + \frac{\gamma}{2}(-h\partial_y + y) \cdot (h\partial_y + y), \tag{5.2.1}$$

where $x, y \in \mathbf{R}^n$ correspond respectively to position and speed of the particles and $h > 0$ corresponds to temperature. The constant $\gamma > 0$ is the friction. (Since we will only discuss $L^2$ aspects we here present right away an adapted version of the operator, obtained after conjugation by a Maxwellian factor.)

The associated evolution equation is:

$$(h\partial_t + P)u(t, x, y) = 0.$$

*Problem of return to equilibrium:* Study the rate of convergence of $u(t, x, y)$ to a multiple of the "ground state" $u_0(x, y) = e^{-(y^2/2+V(x))/h}$ when $t \to +\infty$, assuming that $V(x) \to +\infty$ sufficiently fast when $x \to \infty$ so that $u_0 \in L^2(\mathbf{R}^{2n})$. Notice here that $P(u_0) = 0$ and that the vector field part of $P$ is $h$ times the Hamilton field of $y^2/2 + V(x)$, when we identify $\mathbf{R}^{2n}_{x,y}$ with the cotangent space of $\mathbf{R}^n_x$.

A closely related problem is to study the difference between the first eigenvalue (0) and the next one, $\mu(h)$. (Since our operator is non-self-adjoint, this is only a very approximate formulation however.)

Some contributions: L. Desvillettes–C. Villani [28], J.P. Eckmann–M. Hairer [29], F. Hérau–F. Nier [47], B. Helffer–F. Nier [41], Villani [107]. In the work [47] precise estimates on the exponential rates of return to equilibrium were obtained with methods close to those used in hypoellipticity studies and this work was our starting point. With Hérau and C.Stolk [48] we made a study in the semi-classical limit and studied small eigenvalues modulo $\mathcal{O}(h^\infty)$. More recently with Hérau and M. Hitrik [45] we have made a precise study of the exponential decay of $\mu(h)$ when $V$ has two local minima (and in that case $\mu(h)$ turns out to be real). This involves tunneling, i.e. the study of the exponential decay of eigenfunctions. As an application we have a precise result on the return to equilibrium [46]. This has many similarities with older work on the tunnel effect for Schrödinger operators in the semi-classical limit by B. Helffer–Sjöstrand [42, 43] and B. Simon [93] but for the Kramers-Fokker-Planck operator the problem is richer and more difficult since $P$ is neither elliptic nor self-adjoint. We have used a supersymmetry observation of J.M. Bismut [8] and J. Tailleur–S. Tanase-Nicola–J. Kurchan [104], allowing arguments similar to those for the standard Witten complex [43].

### 5.2.2. Statement of the main results

Let $P$ be given by (5.2.1) where $V \in C^\infty(\mathbf{R}^n;\mathbf{R})$, and

$$\partial^\alpha V(x) = \mathcal{O}(1),\ |\alpha| \ge 2, \tag{5.2.2}$$

$$|\nabla V(x)| \ge 1/C,\ |x| \ge C, \tag{5.2.3}$$

$$V \text{ is a Morse function.} \tag{5.2.4}$$

We also let $P$ denote the graph closure of $P$ from $\mathcal{S}(\mathbf{R}^{2n})$ which coincides with the maximal extension of $P$ in $L^2$ (see [47, 41, 45]). We have $\Re P \ge 0$ and the spectrum of $P$ is contained in the right half plane. In [48] the spectrum in any strip $0 \le \Re z \le Ch$ (and actually in a larger parabolic neighborhood of the imaginary axis, in the spirit of [47]) was determined asymptotically mod $(\mathcal{O}(h^\infty))$. It is discrete and contained in a sector $|\Im z| \le C\Re z + \mathcal{O}(h^\infty)$:

**Theorem 5.1.** *The eigenvalues in the strip* $0 \le \Re z \le Ch$ *are of the form*

$$\lambda_{j,k}(h) \sim h(\mu_{j,k} + h^{1/N_{j,k}}\mu_{j,k,1} + h^{2/N_{j,k}}\mu_{j,k,2} + ..) \tag{5.2.5}$$

*where* $\mu_{j,k}$ *are the eigenvalues of the quadratic approximation ("non-selfadjoint oscillator")*

$$y\cdot\partial_x - V''(x_j)x\cdot\partial_y + \frac{\gamma}{2}(-\partial_y + y)\cdot(\partial_y + y),$$

*at the points* $(x_j, 0)$*, where* $x_j$ *are the critical points of* $V$*.*

The $\mu_{j,k}$ are known explicitly and it follows that when $x_j$ is not a local minimum, then $\Re\lambda_{j,k} \ge h/C$ for some $C > 0$. When $x_j$ is a local minimum, then precisely one of the $\lambda_{j,k}$ is $\mathcal{O}(h^\infty)$ while the others have real part $\ge h/C$. Furthermore, when

$V \to +\infty$ as $x \to \infty$, then 0 is a simple eigenvalue. *In particular, if* $V$ *has only one local minimum, then*

$$\inf \Re(\sigma(P) \setminus \{0\}) \sim h(\mu_1 + h\mu_2 + \dots), \quad \mu_1 > 0.$$

(or possibly an expansion in fractional powers) and we obtained a corresponding result for the problem of return to equilibrium. It should be added that when $\mu_{j,k}$ is a simple eigenvalue of the quadratic approximation then $N_{j,k} = 1$ so there are no fractional powers of $h$ in (5.2.5).

The following is the main new result that we obtained with F. Hérau and M. Hitrik in [45]:

**Theorem 5.2.** *Assume that* $V$ *has precisely 3 critical points; 2 local minima,* $x_{\pm 1}$ *and one "saddle point",* $x_0$ *of index 1. Then for* $C > 0$ *sufficiently large and* $h$ *sufficiently small,* $P$ *has precisely 2 eigenvalues in the strip* $0 \le \Re z \le h/C$*, namely* $0$ *and* $\mu(h)$*, where* $\mu(h)$ *is real and of the form*

$$\mu(h) = h(a_1(h)e^{-2S_1/h} + a_{-1}(h)e^{-2S_{-1}/h}), \tag{5.2.6}$$

*where* $a_j$ *are real,*

$$a_j(h) \sim a_{j,0} + ha_{j,1} + \dots, \; h \to 0, \quad a_{j,0} > 0,$$

$$S_j = V(x_0) - V(x_j).$$

As for the problem of return to equilibrium, we obtained the following result with F. Hérau and M. Hitrik in [46]:

**Theorem 5.3.** *We make the same assumptions as in Theorem 5.2 and let* $\Pi_j$ *be the spectral projection associated with the eigenvalue* $\mu_j$*,* $j = 0, 1$*, where* $\mu_0 = 0$*,* $\mu_1 = \mu(h)$*. Then we have*

$$\Pi_j = \mathcal{O}(1) : L^2 \to L^2, \quad h \to 0. \tag{5.2.7}$$

*We have furthermore, uniformly as* $t \ge 0$ *and* $h \to 0$*,*

$$e^{-tP/h} = \Pi_0 + e^{-t\mu_1/h}\Pi_1 + \mathcal{O}(1)e^{-t/C}, \text{ in } \mathcal{L}(L^2, L^2), \tag{5.2.8}$$

*where* $C > 0$ *is a constant.*

Actually, as we shall see in the outline of the proofs, these results (as well as (5.2.5)) hold for more general classes of supersymmetric operators.

Very recently we observed with F. Hérau and M. Hitrik that we can actually treat the case of any (finite) number of local minima. The basic observation here is that there is a Hermitian product which becomes a scalar product on the space spanned by the $N_0$ lowest eigenvalues, where $N_0$ denotes the number of local minima, and for which our operator is formally self-adjoint. This makes it possible to apply very much the same methods as for the standard Witten complex.

#### 5.2.3. A partial generalization of [48]

Consider on $\mathbf{R}^n$ ($2n$ is now replaced by $n$):

$$\begin{aligned} P &= \sum_{j,k} hD_{x_j} b_{j,k}(x) hD_{x_k} + \\ &\quad \frac{1}{2}\sum_j (c_j(x) h\partial_{x_j} + h\partial_{x_j} \circ c_j(x)) + p_0(x) \\ &= P_2 + iP_1 + P_0, \end{aligned}$$

where $b_{j,k}, c_j, p_0$ are real and smooth. The associated symbols are:

$$\begin{aligned} p(x,\xi) &= p_2(x,\xi) + ip_1(x,\xi) + p_0(x), \\ p_2 &= \sum b_{j,k}\xi_j\xi_k, \ \ p_1 = \sum c_j\xi_j. \end{aligned}$$

Assume,

$$p_2 \geq 0, \ \ p_0 \geq 0,$$

$$\begin{aligned} \partial_x^\alpha b_{j,k} &= \mathcal{O}(1), \ \ |\alpha| \geq 0, \\ \partial_x^\alpha c_j &= \mathcal{O}(1), \ \ |\alpha| \geq 1, \\ \partial_x^\alpha p_0 &= \mathcal{O}(1), \ \ |\alpha| \geq 2. \end{aligned}$$

Assume that

$$\{x;\, p_0(x) = c_1(x) = .. = c_n(x) = 0\}$$

is finite $= \{x_1, ..., x_N\}$ and put $\mathcal{C} = \{\rho_1, ..., \rho_n\}$, $\rho_j = (x_j, 0)$. Put

$$\widetilde{p}(x,\xi) = \langle\xi\rangle^{-2} p_2(x,\xi) + p_0(x), \ \ \langle\xi\rangle = \sqrt{1+|\xi|^2}$$

$$\langle\widetilde{p}\rangle_{T_0} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \widetilde{p} \circ \exp(tH_{p_1})dt, \ \ T_0 > 0 \text{ fixed}.$$

Here in general we let $H_a = a'_\xi \cdot \frac{\partial}{\partial x} - a'_x \cdot \frac{\partial}{\partial \xi}$ denote the Hamilton field of the $C^1$-function $a = a(x,\xi)$.

Dynamical assumptions: Near each $\rho_j$ we have $\langle\widetilde{p}\rangle_{T_0} \sim |\rho - \rho_j|^2$ and in any compact set disjoint from $\mathcal{C}$ we have $\langle\widetilde{p}\rangle_{T_0} \geq 1/C$. (Near infinity this last assumption has to be modified slightly and we refer to [45] for the details.) The following result from [45] is very close to the main result of [48] and generalizes Theorem 5.1:

**Theorem 5.4.** *Under the above assumptions, the spectrum of $P$ is discrete in any band $0 \leq \Re z \leq Ch$ and the eigenvalues have asymptotic expansions as in (5.2.5).*

Put

$$q(x,\xi) = -p(x, i\xi) = p_2(x,\xi) + p_1(x,\xi) - p_0(x).$$

The linearization of the Hamilton field $H_q$ at $\rho_j$ (for any fixed $j$) has eigenvalues $\pm\alpha_k$, $k = 1, .., n$ with real part $\neq 0$. Let $\Lambda_+ = \Lambda_{+,j}$ be the unstable manifold through $\rho_j$ for the $H_q$-flow. Then $\Lambda_+$ is Lagrangian and of the form $\xi = \phi'_+(x)$ near $x_j$ ($\phi_+ = \phi_{+,j}$), where

$$\phi_+(x_j) = 0, \ \ \phi'_+(x_j) = 0, \ \ \phi''_+(x_j) > 0.$$

The next result is from [45]:

**Theorem 5.5.** *Let $\lambda_{j,k}(h)$ be a simple eigenvalue as in (5.2.5) and assume there is no other eigenvalue in a disc $D(\lambda_{j,k}, h/C)$ for some $C > 0$. Then, in the $L^2$ sense, the corresponding eigenfunction is of the form $e^{-\phi_+(x)/h}(a(x;h) + \mathcal{O}(h^\infty))$ near $x_j$, where $a(x;h)$ is smooth in $x$ with an asymptotic expansion in powers of $h$. Away from a small neighborhood of $x_j$ it is exponentially decreasing.*

The proof of the first theorem uses microlocal weak exponential estimates, while the one of the last theorem also uses local exponential estimates.

### 5.2.4. Averaging and exponential weights.

The basic idea of the proof of Theorem 5.4 is taken from [48], but we reworked it in order to allow for non-hypoelliptic operators. We will introduce a weight on $T^*\mathbf{R}^n$ of the form

$$\psi_\epsilon = -\int J(\frac{t}{T_0})\widetilde{p}_\epsilon \circ \exp(tH_{p_1})dt, \tag{5.2.9}$$

for $0 < \epsilon \ll 1$. Here $J(t)$ is the odd function given by

$$J(t) = \begin{cases} 0,\ |t| \geq \frac{1}{2}, \\ \frac{1}{2} - t,\ 0 < t \leq \frac{1}{2}, \end{cases} \tag{5.2.10}$$

and we choose $\widetilde{p}_\epsilon(\rho)$ to be equal to $\widetilde{p}(\rho)$ when $\mathrm{dist}\,(\rho, \mathcal{C}) \leq \epsilon$, and flatten out to $\epsilon\widetilde{p}$ away from a fixed neighborhood of $\mathcal{C}$ in such a way that $\widetilde{p}_\epsilon = \mathcal{O}(\epsilon)$. Then

$$H_{p_1}\psi_\epsilon = \langle \widetilde{p}_\epsilon \rangle_{T_0} - \widetilde{p}_\epsilon. \tag{5.2.11}$$

We let $\epsilon = Ah$ where $A \gg 1$ is independent of $h$. Then the weight $\exp(\psi_\epsilon/h)$ is uniformly bounded when $h \to 0$. Indeed, $\psi_\epsilon = \mathcal{O}(h)$.

Using Fourier integral operators with complex phase, we can define a Hilbert space of functions that are "microlocally $\mathcal{O}(\exp(\psi_\epsilon/h))$ in the $L^2$ sense". The norm is uniformly equivalent to the one of $L^2$, but the natural leading symbol of $P$, acting in the new space, becomes

$$p(\exp(iH_{\psi_\epsilon})(\rho)),\ \rho \in T^*\mathbf{R}^n \tag{5.2.12}$$

which by Taylor expansion has real part $\approx p_2(\rho) + p_0(\rho) + \langle \widetilde{p}_\epsilon \rangle - \widetilde{p}_\epsilon$.

Very roughly, the real part of the new symbol is $\geq \epsilon$ away from $\mathcal{C}$ and behaves like $\mathrm{dist}\,(\rho, \mathcal{C})^2$ in a $\sqrt{\epsilon}$-neighborhood of $\mathcal{C}$. This can be used to show that the spectrum of $P$ (viewed as an operator on the weighted space) in a band $0 \leq \Re z < \epsilon/C$ comes from an $\sqrt{\epsilon}$-neighborhood of $\mathcal{C}$. In such a neighborhood, we can treat $P$ as an elliptic operator and the spectrum is to leading order determined by the quadratic approximation of the dilated symbol (5.2.12). This gives Theorem 5.4.

We next turn to the proof of Theorem 5.5, and we work near a point $\rho_j = (x_j, \xi_j) \in \mathcal{C}$. Recall that $\Lambda_+ : \xi = \phi'_+(x)$ is the unstable manifold for the $H_q$-flow, where $q(x,\xi) = -p(x, i\xi)$. We have $q(x, \phi'_+(x)) = 0$.

In general, if $\psi \in C^\infty$ is real, then $P_\psi := e^{\psi/h} \circ P \circ e^{-\psi/h}$ has the symbol

$$p_\psi(x,\xi) = p_2(x,\xi) - q(x, \psi'(x)) + i(q'_\xi(x, \psi'(x)) \cdot \xi \tag{5.2.13}$$

- As long as $q(x, \psi'(x)) \leq 0$, we have $\Re p_\psi \geq 0$ and we may hope to establish good apriori estimates for $P_\psi$.

- This is the case for $\psi = 0$ and for $\psi = \phi_+$. Using the convexity of $q(x, \cdot)$, we get suitable weights $\psi$ with $q(x, \psi'(x)) \leq 0$, equal to $\phi_+(x)$ near $x_j$, strictly positive away from $x_j$ and constant outside a neighborhood of that point.

- It follows that the eigenfunction in Theorem 5.5 is (roughly) $\mathcal{O}(e^{-\phi_+(x)/h})$ near $x_j$ in the $L^2$ sense.

- On the other hand, we have quasi-modes of the form $a(x; h)e^{-\phi_+(x)/h}$ as in [42].

- Applying the exponentially weighted estimates, indicated above, to the difference of the eigenfunction and the quasi-mode, we then get Theorem 5.5.

### 5.2.5. Supersymmetry and the proof of Theorem 5.2

We review the supersymmetry from [8], [104], see also G. Lebeau [67]. Let $A(x) : T_x^*\mathbf{R}^n \to T_x\mathbf{R}^n$ be linear, invertible and smooth in $x$. Then we have the nondegenerate bilinear form

$$\langle u|v\rangle_{A(x)} = \langle \wedge^k A(x)u|v\rangle,\ \ u, v \in \wedge^k T_x^*\mathbf{R}^n,$$

and we also write $(u|v)_{A(x)} = \langle u|\overline{v}\rangle_{A(x)}$.

If $u, v$ are smooth $k$-forms with compact support, put

$$(u|v)_A = \int (u(x)|v(x))_{A(x)}dx.$$

The formal "adjoint" $Q^{A,*}$ of an operator $Q$ is then given by

$$(Qu|v)_A = (u|Q^{A,*}v)_A.$$

Let $\phi : \mathbf{R}^n \to \mathbf{R}$ be a smooth Morse function with $\partial^\alpha\phi$ bounded for $|\alpha| \geq 2$ and with $|\nabla\phi| \geq 1/C$ for $|x| \geq C$. Introduce the Witten-De Rham complex:

$$d_\phi = e^{-\frac{\phi}{h}} \circ hd \circ e^{\frac{\phi}{h}} = \sum_j (h\partial_{x_j} + \partial_{x_j}\phi) \circ dx_j^\wedge,$$

where $d$ denotes exterior differentiation and $dx_j^\wedge$ left exterior multiplication with $dx_j$. The corresponding Laplacian is then: $-\Delta_A = d_\phi^{A,*}d_\phi + d_\phi d_\phi^{A,*}$. Its restriction to $q$-forms will be denoted by $-\Delta_A^{(q)}$. Notice that:

$$-\Delta_A^{(0)}(e^{-\phi/h}) = 0.$$

Write $A = B + C$ with $B^t = B$, $C^t = -C$. $-\Delta_A$ is a second order differential operator with scalar principal symbol in the semi-classical sense ($\frac{h}{i}\frac{\partial}{\partial x_j} \mapsto \xi_j$) of the form:

$$p(x, \xi) = \sum_{j,k} b_{j,k}(\xi_j\xi_k + \partial_{x_j}\phi\partial_{x_k}\phi) + 2i\sum_{j,k} c_{j,k}\partial_{x_k}\phi\,\xi_j.$$

**Example.** Replace $n$ by $2n$, $x$ by $(x, y)$, let

$$A = \frac{1}{2}\begin{pmatrix} 0 & I \\ -I & \gamma \end{pmatrix}.$$

Then

$$\begin{aligned} -\Delta_A^{(0)} &= h(\phi'_y \cdot \partial_x - \phi'_x \cdot \partial_y) \\ &+\frac{\gamma}{2}\sum_j (-h\partial_{y_j} + \partial_{y_j}\phi)(h\partial_{y_j} + \partial_{y_j}\phi). \end{aligned}$$

When $\phi = y^2/2 + V(x)$ we recover the KFP operator (5.2.1)

The results of Subsubsection 5.2.3 apply, if we make the additional dynamical assumptions there; $-\Delta_A^{(q)}$ has an asymptotic eigenvalue $= o(h)$ associated to the critical point $x_j$ precisely when the index of $x_j$ is equal to $q$ (as for the Witten complex and analogous complexes in several complex variables). In order to cover the cases $q > 0$ we also assume that

$$A = \text{Const.} \tag{5.2.14}$$

*The Double well case.* Keep the assumption (5.2.14). Assume that $\phi$ is a Morse function with $|\nabla\phi| \geq 1/C$ for $|x| \geq C$ such that $-\Delta_A$ satisfies the extra dynamical conditions of Subsubsection 5.2.3 and having precisely three critical points, two local minima $U_{\pm 1}$ and a saddle point $U_0$ of index 1.

Then $-\Delta_A^{(0)}$ has precisely 2 eigenvalues: 0, $\mu$ that are $o(h)$ while $-\Delta_A^{(1)}$ has precisely one such eigenvalue: $\mu$. (Here we use as in the study of the Witten complex, that $d_\phi$ and $d_\phi^{A,*}$ intertwine our Laplacians in degeree 0 and 1. The detailed justification is more complicated however.) $e^{-\phi/h}$ is the eigenfunction of $\Delta_A^{(0)}$ corresponding to the eigenvalue 0. Let $S_j = \phi(U_0) - \phi(U_j)$, $j = \pm 1$, and let $D_j$ be the connected component of $\{x \in \mathbf{R}^n;\ \phi(x) < \phi(U_0)\}$ containing $U_j$ in its interior.

Let $E^{(q)}$ be the corresponding spectral subspaces so that $\dim E^{(0)} = 2$, $\dim E^{(1)} = 1$. Truncated versions of the function $e^{-\phi(x)/h}$ can be used as approximate eigenfunctions, and we can show:

**Proposition 5.6.** *$E^{(0)}$ has a basis $e_1, e_{-1}$, where*

$$e_j = \chi_j(x)e^{-\frac{1}{h}(\phi(x)-\phi(U_j))} + \mathcal{O}(e^{-\frac{1}{h}(S_j-\epsilon)}),\ \textit{in the } L^2\textit{-sense.}$$

*Here, we let $\chi_j \in C_0^\infty(D_j)$ be equal to 1 on $\{x \in D_j;\ \phi(x) \leq \phi(U_0) - \epsilon\}$.*

The theorems 5.4, 5.5 can be adapted to $-\Delta_A^{(1)}$ and lead to:

**Proposition 5.7.** *$E^{(1)} = \mathbf{C}e_0$, where*

$$e_0(x) = \chi_0(x)a_0(x;h)e^{-\frac{1}{h}\phi_+(x)} + \mathcal{O}(e^{-\epsilon_0/h}),$$

*$\phi_+(x) \sim (x - U_0)^2$, $\epsilon_0 > 0$ is small enough, $a_0$ is an elliptic symbol, $\chi_0 \in C_0^\infty(\mathbf{R}^n)$, $\chi_0 = 1$ near $U_0$.*

Let the matrices of $d_\phi : E^{(0)} \to E^{(1)}$ and $d_\phi^{A,*} : E^{(1)} \to E^{(0)}$ with respect to the bases $\{e_{-1}, e_1\}$ and $\{e_0\}$ be

$$\begin{pmatrix} \lambda_{-1} & \lambda_1 \end{pmatrix} \text{ and } \begin{pmatrix} \lambda_{-1}^* \\ \lambda_1^* \end{pmatrix} \text{ respectively.}$$

Using the preceding two results in the spirit of tunneling estimates and computations of Helffer–Sjöstrand ([42, 43]) we can show:

**Proposition 5.8.** *Put $S_j = \phi(U_0) - \phi(U_j)$, $j = \pm 1$. Then we have*

$$\begin{pmatrix} \lambda_{-1} \\ \lambda_1 \end{pmatrix} = h^{\frac{1}{2}}(I + \mathcal{O}(e^{-\frac{1}{Ch}})) \begin{pmatrix} \ell_{-1}(h)e^{-S_{-1}/h} \\ \ell_1(h)e^{-S_1/h} \end{pmatrix},$$

$$\begin{pmatrix} \lambda_{-1}^* \\ \lambda_1^* \end{pmatrix} = h^{\frac{1}{2}}(I + \mathcal{O}(e^{-\frac{1}{Ch}})) \begin{pmatrix} \ell_{-1}^*(h)e^{-S_{-1}/h} \\ \ell_1^*(h)e^{-S_1/h} \end{pmatrix},$$

*where $\ell_{\pm 1}$, $\ell_{\pm 1}^*$ are real elliptic symbols of order* $0$ *such that $\ell_j \ell_j^* > 0$, $j = \pm 1$.*

From this we get Theorem 5.2.3, since $\mu = \lambda_{-1}^* \lambda_{-1} + \lambda_1^* \lambda_1$. □

Thanks to the fact that we have only two local minima, certain simplifications were possible in the proof. In particular it was sufficent to control the exponential decay of general eigenfunctions in some small neighborhood of the critical points.

### 5.2.6. Return to equilibrium, ideas of the proof of Theorem 5.3

Keeping the same assumptions, let $\Pi_0$, $\Pi_1$ be the rank 1 spectral projections corresponding to the eigenvalues $\mu_0 := 0$, $\mu_1 := \mu$ of $-\Delta_A^{(0)}$ and put $\Pi = \Pi_0 + \Pi_1$. Then $e_{-1,}, e_1$ is a basis for $\mathcal{R}(\Pi)$ and the restriction of $P$ to this range, has the matrix

$$\begin{pmatrix} \lambda_{-1}^* \\ \lambda_1^* \end{pmatrix} \begin{pmatrix} \lambda_{-1} & \lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_{-1}^*\lambda_{-1} & \lambda_{-1}^*\lambda_1 \\ \lambda_1^*\lambda_{-1} & \lambda_1^*\lambda_1 \end{pmatrix} \tag{5.2.15}$$

with the eigenvalues 0 and $\mu = \lambda_{-1}^*\lambda_{-1} + \lambda_1^*\lambda_1$. A corresponding basis of eigenvectors is given by

$$\begin{aligned} v_0 &= \frac{1}{\sqrt{\mu_1}}(\lambda_1 e_{-1} - \lambda_{-1} e_{-1}) \\ v_1 &= \frac{1}{\sqrt{\mu_1}}(\lambda_{-1}^* e_{-1} + \lambda_1^* e_{-1}). \end{aligned} \tag{5.2.16}$$

The corresponding dual basis of eigenfunctions of $P^*$ is given by

$$\begin{aligned} v_0^* &= \frac{1}{\sqrt{\mu}}(\lambda_1^* e_{-1}^* - \lambda_{-1}^* e_{-1}^*) \\ v_1^* &= \frac{1}{\sqrt{\mu}}(\lambda_{-1} e_{-1}^* + \lambda_1 e_1^*), \end{aligned} \tag{5.2.17}$$

where $e_{-1}^*, e_1^* \in \mathcal{R}(\Pi^*)$ is the basis that is dual to $e_{-1}, e_1$. It follows that $v_j, v_j^* = \mathcal{O}(1)$ in $L^2$, when $h \to 0$.

From this discussion we conclude that $\Pi_j = (\cdot|v_j^*)v_j$, *are uniformly bounded when* $h \to 0$. A non-trivial fact, based on the analysis described in Subsubsections 5.2.3, 5.2.4, is that after replacing the standard norm and scalar product on $L^2$ by certain uniformly equivalent ones, we have

$$\Re(Pu|u) \geq \frac{h}{C}\|u\|^2, \quad \forall u \in \mathcal{R}(1 - \widetilde{\Pi}), \tag{5.2.18}$$

where $\widetilde{\Pi}$ is the spectral projection corresponding to the spectrum of $P$ in $D(0, Bh)$ for some $B \gg 1$.

This can be applied to the study of $u(t) := e^{-tP/h}u(0)$, where the initial state $u(0) \in L^2$ is arbitrary: Write

$$u(0) = \Pi_0 u(0) + \Pi_1 u(0) + (1 - \Pi)u(0) =: u^0 + u^1 + u^\perp. \tag{5.2.19}$$

Then

$$\begin{aligned} \|u^0\|, \|u^1\|, \|u^\perp\| &\le \mathcal{O}(1)\|u(0)\| && (5.2.20) \\ \|e^{-tP/h}u^\perp\| &\le Ce^{-t/C}\|u(0)\| && (5.2.21) \\ e^{-tP/h}u_j &= e^{-t\mu_j/h}u_j,\ j = 0, 1. && (5.2.22) \end{aligned}$$

Here (5.2.21) follows if we write $u^\perp = (1 - \widetilde{\Pi})u + (\Pi - \widetilde{\Pi})u$, apply (5.2.18) to the evolution of the first term, and use that the last term is the (bounded) spectral projection of $u$ to a finite dimensional spectral subspace of $P$, for which the corresponding eigenvalues all have real part $\ge h/C$. □

## 5.3. Spectral asymptotics in 2 dimensions

### 5.3.1. Introduction

This subsection is mainly based on recent joint works with S. Vũ Ngọc and M. Hitrik [52] [53], but we shall start by recalling some earlier results that we obtained with A. Melin [78] where we discovered that in the two dimensional case one often can have Bohr-Sommerfeld conditions to determine all the individual eigenvalues in some region of the spectral plane, provided that we have analyticity. In the self-adjoint case such results are known (to the author) only in 1 dimension and in very special cases for higher dimensions.

Subsequently, with M. Hitrik we have studied small perturbations of self-adjoint operators. First we studied the case when the classical flow of the unperturbed operator is periodic, then also with S. Vũ Ngọc we looked at the more general case when it is completely integrable, or just when the energy surface contains some invariant Diophantine Lagrangian tori.

### 5.3.2. Bohr-Sommerfeld rules in two dimensions

For (pseudo-)differential operators in dimension 1, we often have a Bohr-Sommerfeld rule to determine the asymptotic behaviour of the eigenvalues. Consider for instance the semi-classical Schrödinger operator

$$P = -h^2\frac{d^2}{dx^2} + V(x), \text{ with symbol } p(x,\xi) = \xi^2 + V(x),$$

where we assume that $V \in C^\infty(\mathbf{R};\mathbf{R})$ and $V(x) \to +\infty$, $|x| \to \infty$. Let $E_0 \in \mathbf{R}$ be a non-critical value of $V$ such that (for simplicity) $\{x \in \mathbf{R}; V(x) \le E_0\}$ is an interval. Then in some small fixed neighborhood of $E_0$ and for $h > 0$ small enough, the eigenvalues of $P$ are of the form $E = E_k$, $k \in \mathbf{Z}$, where

$$\frac{I(E)}{2\pi h} = k - \theta(E;h), \quad I(E) = \int_{p^{-1}(E)} \xi \cdot dx,\ \theta(E;h) \sim \theta_0(E) + \theta_1(E)h + ...$$

In the non-self-adjoint case we get the same results, provided that $\Im V$ is small and $V$ *is analytic*. The eigenvalues will then be on a curve close to the real axis.

For self-adjoint operators in dimension $\ge 2$ it is generally admitted that Bohr-Sommerfeld rules do not give all eigenvalues in any fixed domain except in certain (completely integrable) cases. Using the KAM theorem one can sometimes describe some fraction of the eigenvalues.

With A. Melin [78]: we considered an $h$-pseudodifferential operator with leading symbol $p(x,\xi)$ that is bounded and holomorphic in a tubular neighborhood of $\mathbf{R}^4$ in $\mathbf{C}^4 = \mathbf{C}^2_x \times \mathbf{C}^2_\xi$. Assume that

$$\mathbf{R}^4 \cap p^{-1}(0) \neq \emptyset \text{ is connected.} \tag{5.3.1}$$

$$\text{On } \mathbf{R}^4 \text{ we have } |p(x,\xi)| \geq 1/C, \text{ for } |(x,\xi)| \geq C, \tag{5.3.2}$$

for some $C > 0$,

$$d\Re p(x,\xi), d\Im p(x,\xi) \text{ are linearly independent for all } (x,\xi) \in p^{-1}(0) \cap \mathbf{R}^4. \tag{5.3.3}$$

(Here the boundedness assumption near $\infty$ and (5.3.2) can be replaced by a suitable ellipticity assumption.) It follows that $p^{-1}(0) \cap \mathbf{R}^4$ is a compact (2-dimensional) surface.

Also assume that

$$|\{\Re p, \Im p\}| \text{ is sufficiently small on } p^{-1}(0) \cap \mathbf{R}^4. \tag{5.3.4}$$

"Sufficiently small" here refers to some positive bound that can be defined whenever the the other conditions are satisfied uniformly.

When the Poisson bracket vanishes on $p^{-1}(0)$, this set becomes a Lagrangian torus, and more generally it is a torus. The following is a complex version of the KAM theorem without small divisors (cf T.W. Cherry [15], J. Moser [80]),

**Theorem 5.9.** *([78]) There exists a smooth 2-dimensional torus $\Gamma \subset p^{-1}(0) \cap \mathbf{C}^4$, close to $p^{-1}(0) \cap \mathbf{R}^4$ such that $\sigma_{|\Gamma} = 0$ and $I_j(\Gamma) \in \mathbf{R}$, $j = 1, 2$. Here $I_j(\Gamma) := \int_{\gamma_j} \xi \cdot dx$ are the actions along the two fundamental cycles $\gamma_1, \gamma_2 \subset \Gamma$, and $\sigma = \sum_1^2 d\xi_j \wedge dx_j$ is the complex symplectic (2,0)-form.*

Replacing $p$ by $p - z$ for $z$ in a neighborhood of $0 \in \mathbf{C}$, we get tori $\Gamma(z)$ depending smoothly on $z$ and a corresponding smooth action function $I(z) = (I_1(\Gamma(z)), I_2(\Gamma(z)))$, which are important in the Bohr-Sommerfeld rule for the eigenvalues near 0 in the semi-classical limit $h \to 0$:

**Theorem 5.10.** *([78]) Under the above assumptions, there exists $\theta_0 \in (\frac{1}{2}\mathbf{Z})^2$ and $\theta(z;h) \sim \theta_0 + \theta_1(z)h + \theta_2(z)h^2 + ..$ in $C^\infty(\mathrm{neigh}\,(0,\mathbf{C}))$, such that for $z$ in an $h$-independent neighborhood of 0 and for $h > 0$ sufficiently small, we have that $z$ is an eigenvalue of $P = p(x, hD_x)$ iff*

$$\frac{I(z)}{2\pi h} = k - \theta(z;h), \text{ for some } k \in \mathbf{Z}^2. \quad (BS)$$

Recently, a similar result was obtained by S. Graffi, C. Villegas Bas [34].

An application of this result is that we get all resonances (scattering poles) in a fixed neighborhood of $0 \in \mathbf{C}$ for $-h^2\Delta + V(x)$ if V is an analytic real potential on $\mathbf{R}^2$ with a nondegenerate saddle point at $x = 0$, satisfying $V(0) = 0$ and having $\{(x,\xi) = (0,0)\}$ as its classically trapped set in the energy surface $\{p(x,\xi) = 0\}$.

### 5.3.3. Diophantine case

In this and the next subsubsection we describe a result from [53] and the main result of [52] about individual eigenvalues for small perturbations of a self-adjoint operator with a completely integrable leading symbol. We start with the case when only Diophantine tori play a role.

Let $P_\epsilon(x, hD; h)$ on $\mathbf{R}^2$ have the leading symbol $p_\epsilon(x, \xi) = p(x, \xi) + i\epsilon q(x, \xi)$ where $p$, $q$ are real and extend to bounded holomorphic functions on a tubular neighborhood of $\mathbf{R}^4$. Assume that $p$ fulfills the ellipticity condition (5.3.2) near infinity and that

$$P_{\epsilon=0} = P(x, hD) \tag{5.3.5}$$

is self-adjoint. (The conditions near infinity can be modified and we can also replace $\mathbf{R}^2_x$ by a compact 2-dimensional analytic manifold.)

Also, assume that $P_\epsilon(x, \xi; h)$ depends smoothly on $0 \le \epsilon \le \epsilon_0$ with values in the space of bounded holomorphic functions in a tubular neighborhood of $\mathbf{R}^4$, and $P_\epsilon \sim p_\epsilon + hp_{1,\epsilon} + h^2 p_{2,\epsilon} + ...$, when $h \to 0$.

Assume

$$p^{-1}(0) \text{ is connected and } dp \neq 0 \text{ on that set.} \tag{5.3.6}$$

Assume complete integrability for $p$: There exists an analytic real valued function $f$ on $T^*\mathbf{R}^2$ such that $H_p f = 0$, with the differentials $df$ and $dp$ being linearly independent almost everywhere on $p^{-1}(0)$. ($H_p = p'_\xi \cdot \frac{\partial}{\partial x} - p'_x \cdot \frac{\partial}{\partial \xi}$ is the Hamilton field.)

Then we have a disjoint union decomposition

$$p^{-1}(0) \cap T^*\mathbf{R}^2 = \bigcup_{\Lambda \in J} \Lambda, \tag{5.3.7}$$

where $\Lambda$ are compact connected sets, invariant under the $H_p$ flow. We assume (for simplicity) that $J$ has a natural structure of a graph whose edges correspond to families of regular leaves; Lagrangian tori (by the Arnold-Mineur-Liouville theorem [108]). The union of edges $J \setminus S$ possesses a natural real analytic structure.

Each torus $\Lambda \in J \setminus S$ carries real analytic coordinates $x_1, x_2$ identifying $\Lambda$ with $\mathbf{T}^2 = \mathbf{R}^2 / 2\pi \mathbf{Z}^2$, so that along $\Lambda$, we have

$$H_p = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2}, \tag{5.3.8}$$

where $a_1, a_2 \in \mathbf{R}$. The rotation number is defined as the ratio $\omega(\Lambda) = [a_1 : a_2] \in \mathbf{RP}^1$, and it depends analytically on $\Lambda \in J \setminus S$. We assume that $\omega(\Lambda)$ is not identically constant on any open edge.

We say that $\Lambda \in J \setminus S$ is respectively rational, irrational, diophantine if $a_1/a_2$ has the corresponding property. Diophantine means that there exist $\alpha > 0$, $d > 0$ such that

$$|(a_1, a_2) \cdot k| \ge \frac{\alpha}{|k|^{2+d}}, \ 0 \neq k \in \mathbf{Z}^2, \tag{5.3.9}$$

We introduce

$$\langle q \rangle_T = \frac{1}{T} \int_{-T/2}^{T/2} q \circ \exp(tH_p) dt, \ T > 0, \tag{5.3.10}$$

and consider the compact intervals $Q_\infty(\Lambda) \subset \mathbf{R}$, $\Lambda \in J$, defined by,

$$Q_\infty(\Lambda) = [\lim_{T\to\infty} \inf_\Lambda \langle q \rangle_T, \lim_{T\to\infty} \sup_\Lambda \langle q \rangle_T]. \tag{5.3.11}$$

A first localization of the spectrum $\sigma(P_\epsilon(x, hD_x; h))$ ([53]) is given by

$$\Im(\sigma(P_\epsilon) \cap \{z; |\Re z| \le \delta\}) \subset \epsilon[\inf \bigcup_{\Lambda \in J} Q_\infty(\Lambda) - o(1), \sup \bigcup_{\Lambda \in J} Q_\infty(\Lambda) + o(1)], \tag{5.3.12}$$

when $\delta, \epsilon, h \to 0$.

For each torus $\Lambda \in J \setminus S$, we let $\langle q\rangle(\Lambda)$ be the average of $q_{|\Lambda}$ with respect to the natural smooth measure on $\Lambda$, and assume that the analytic function $J \setminus S \ni \Lambda \mapsto \langle q\rangle(\Lambda)$ is not identically constant on any open edge.

By combining (5.3.8) with the Fourier series representation of $q$, we see that when $\Lambda$ is irrational then $Q_\infty(\Lambda) = \{\langle q\rangle(\Lambda)\}$, while in the rational case,

$$Q_\infty(\Lambda) \subset \langle q\rangle(\Lambda) + \mathcal{O}(\frac{1}{(|n|+|m|)^\infty})[-1,1], \tag{5.3.13}$$

when $\omega(\Lambda) = \frac{m}{n}$ and $m \in \mathbf{Z}$, $n \in \mathbf{N}$ are relatively prime.

Let $F_0 \in \cup_{\Lambda\in J} Q_\infty(\Lambda)$ and assume that there exists a Diophantine torus $\Lambda_d$ (or finitely many), such that

$$\langle q\rangle(\Lambda_d) = F_0, \quad d_\Lambda \langle q\rangle(\Lambda_d) \neq 0. \tag{5.3.14}$$

With M. Hitrik and S. Vũ Ngọc we obtained:

**Theorem 5.11.** *(*[53]*) Assume also that $F_0$ does not belong to $Q_\infty(\Lambda)$ for any other $\Lambda \in J$. Let $0 < \delta < K < \infty$. Then $\exists C > 0$ such that for $h > 0$ small enough, and $k^K \le \epsilon \le h^\delta$, the eigenvalues of $P_\epsilon$ in the rectangle $|\Re z| < h^\delta/C$, $|\Im z - \epsilon \Re F_0| < \epsilon h^\delta/C$ are given by*

$$P^{(\infty)}(h(k - \frac{k_0}{4}) - \frac{S}{2\pi}, \epsilon; h) + \mathcal{O}(h^\infty), \ k \in \mathbf{Z}^2,$$

*Here $P^{(\infty)}(\xi, \epsilon; h)$ is smooth, real-valued for $\epsilon = 0$ and when $h \to 0$ we have*

$$P^{(\infty)}(\xi, \epsilon; h) \sim \sum_{\ell=0}^{\infty} h^\ell p_\ell^{(\infty)}(\xi, \epsilon), \ p_0^{(\infty)} = p_\infty(\xi) + i\epsilon\langle q\rangle(\xi) + \mathcal{O}(\epsilon^2), \tag{5.3.15}$$

*corresponding to action angle coordinates.*

In [53] we also considered applications to small non-self-adjoint perturbations of the Laplacian on a surface of revolution. Thanks to (5.3.13) the total measure of the union of all $Q_\infty(\Lambda)$ over the rational tori is finite and sometimes small, and we could then show that there are plenty of values $F_0$, fulfilling the assumptions in the theorem.

With M. Hitrik we are currently studying the distribution of eigenvalues in sub-bands that are delimited by two different values "$F_0$" as in the theorem.

### 5.3.4. The case with rational tori

Let $F_0$ be as in (5.3.14) but now also allow for the possibility that there is a rational torus (or finitely many) $\Lambda_r$, such that

$$F_0 \in Q_\infty(\Lambda_r), \quad F_0 \neq \langle q\rangle(\Lambda_r), \tag{5.3.16}$$

$$d_\Lambda(\langle q\rangle)(\Lambda_r) \neq 0, \ d_\Lambda(\omega)(\Lambda_r) \neq 0. \tag{5.3.17}$$

Assume also that

$$F_0 \notin Q_\infty(\Lambda), \ \text{for all } \Lambda \in J \setminus \{\Lambda_d, \Lambda_r\}. \tag{5.3.18}$$

With M. Hitrik we showed the following result:

**Theorem 5.12.** *([52]) Let $\delta > 0$ be small and assume that $h \ll \epsilon \leq h^{\frac{2}{3}+\delta}$, or that the subprincipal symbol of $P$ vanishes and that $h^2 \ll \epsilon \leq h^{\frac{2}{3}+\delta}$ . Then the spectrum of $P_\epsilon$ in the rectangle*

$$[-\frac{\epsilon}{C}, \frac{\epsilon}{C}] + i\epsilon[F_0 - \frac{\epsilon^\delta}{C}, F_0 + \frac{\epsilon^\delta}{C}]$$

*is the union of two sets: $E_d \cup E_r$, where the elements of $E_d$ form a distorted lattice, given by the Bohr-Sommerfeld rule (5.3.15), with horizontal spacing $\asymp h$ and vertical spacing $\asymp \epsilon h$. The number of elements $\#(E_r)$ of $E_r$ is $\mathcal{O}(\epsilon^{3/2}/h^2)$.*

NB that $\#(E_d) \asymp \epsilon^{1+\delta}/h^2$.

This result can be applied to the damped wave equation on surfaces of revolution.

### 5.3.5. Outline of the proofs of Theorem 5.11 and 5.12

The principal symbol of $P_\epsilon$ is $p_\epsilon = p + i\epsilon q + \mathcal{O}(\epsilon^2)$. Put

$$\langle q \rangle_T = \frac{1}{T}\int_{-T/2}^{T/2} q \circ \exp(tH_p)dt.$$

As in Subsection 5.2 we will use an averaging of the imaginary part of the symbol. Let $J(t)$ be the piecewise affine function with support in $[-\frac{1}{2}, \frac{1}{2}]$, solving

$$J'(t) = \delta(t) - 1_{[-\frac{1}{2},\frac{1}{2}]}(t),$$

and introduce the weight

$$G_T(t) = \int J(-\frac{t}{T})q \circ \exp(tH_p)dt.$$

Then $H_pG_T = q - \langle q \rangle_T$, implying

$$p_\epsilon \circ \exp(i\epsilon H_{G_T}) = p + i\epsilon\langle q \rangle_T + \mathcal{O}_T(\epsilon^2). \tag{5.3.19}$$

The left hand side of (5.3.19) is the principal symbol of the isospectral operator $e^{-\frac{\epsilon}{h}G_T(x,hD_x)} \circ P_\epsilon \circ e^{\frac{\epsilon}{h}G_T(x,hD_x)}$ and under the assumptions of Theorem 5.11 resp. 5.12 its imaginary part will not take the value $i\epsilon F_0$ on $p^{-1}(0)$ away from $\Lambda_d$ resp. $\Lambda_d \cup \Lambda_r$. This means that we have localized the spectral problem to a neighborhood of $\Lambda_d$ resp. $\Lambda_d \cup \Lambda_r$.

Near $\Lambda_d$ we choose action-angle coordinates so that $\Lambda_d$ becomes the zero section in the cotangent space of the 2-torus, and

$$p_\epsilon(x, \xi) = p(\xi) + i\epsilon q(x, \xi) + \mathcal{O}(\epsilon^2). \tag{5.3.20}$$

We follow the quantized Birkhoff normal form procedure in the spirit of V.F. Lazutkin and Y. Colin de Verdière [65, 20]: solve first

$$H_pG = q(x, \xi) - \langle q(\cdot, \xi)\rangle, \tag{5.3.21}$$

where the bracket indicates that we take the average over the torus with respect to $x$. Composing with the corresponding complex canonical transformation, we get the new conjugated symbol

$$p(\xi) + i\epsilon\langle q(\cdot, \xi)\rangle + \mathcal{O}(\epsilon^2 + \xi^\infty).$$

Here the Diophantine condition is of course important.

Iterating the procedure we get for every $N$,

$$p_\epsilon \circ \exp(H_{G^{(N)}}) = \underbrace{p(\xi) + i\epsilon(\langle q\rangle(\xi) + \mathcal{O}(\epsilon,\xi))}_{\text{independent of } x} + \mathcal{O}((\xi,\epsilon)^{N+1})$$

This procedure can be continued on the operator level, and up to a small error we see that $P_\epsilon$ is microlocally equivalent to an operator $P_\epsilon(hD_\xi, \epsilon; h)$. At least formally, Theorem 5.11 then follows by considering Fourier series expansions, but in order to get a full proof we also have take into account that we have constructed complex canonical transformations that are quantized by Fourier integral operators with complex phase and study the action of these operators on suitable exponentially weighted spaces.

Near $\Lambda_r$ we can still use action-angle coordinates as in (5.3.20) but the homological equation (5.3.21) is no longer solvable. Instead, we use secular perturbation theory (cf the book [70]), which amounts to making a partial Birkhoff reduction.

After a linear change of $x$-variables, we may assume that $p(\xi) = \xi_2 + \mathcal{O}(\xi^2)$ and in order to fix the ideas $= \xi_2 + \xi_1^2$. Then we can make the averaging procedure only in the $x_2$-direction and reduce $p_\epsilon$ in (5.3.20) to

$$\widetilde{p}_\epsilon(x,\xi) = \underbrace{\xi_2 + \xi_1^2 + \mathcal{O}(\epsilon)}_{\substack{\text{independent of } x_2,\\ \approx \xi_2+\xi_1^2+i\epsilon\langle q\rangle_2(x_1,\xi)}} + \mathcal{O}((\epsilon,\xi)^\infty),$$

where $\langle q\rangle_2(x_1,\xi)$ denotes the average with respect to $x_2$.

Carrying out the reduction on the operator level, we obtain up to small errors an operator $\widetilde{P}_\epsilon(x_1, hD_{x_1}, hD_{x_2}; h)$ and after passing to Fourier series in $x_2$, a family of non-self-adjoint operators on $S^1_{x_1}$: $\widetilde{P}_\epsilon(x_1, hD_{x_1}, hk; h)$, $k \in \mathbf{Z}$.

The non-self-adjointness and the corresponding possible wild growth of the resolvent makes it hard to go all the way to study individual eigenvalues. However, it can be shown that in the region $|\xi_1| \gg \epsilon^{1/2}$ (inside the energy surface $p = 0$) we can go further and (as near $\Lambda_d$) get a sufficiently good elimination of the $x$-dependence. This leads to the conclusion that the contributions from a vicinity of $\Lambda_r$ to the spectrum of $P_\epsilon$ in the rectangle

$$|\Re z| \le \frac{\epsilon}{C},\ |\Im z - \epsilon F_0| \le \frac{\epsilon^{1+\delta}}{C},$$

come from a neighborhood of $\Lambda_r$ of phase space volume $\mathcal{O}(\epsilon^{3/2})$.

This explains heuristically why the rational torus will contribute with $\mathcal{O}(\epsilon^{3/2}/h^2)$ eigenvalues in the rectangle.

The actual proof is more complicated. We use a Grushin problem reduction in order to reduce the study near $\Lambda_r$ to that of a square matrix of size $\mathcal{O}(\epsilon^{3/2}/h^2)$. However, even if we avoid the eigenvalues of such a matrix, the inverse can only be bounded by

$$\exp \mathcal{O}(\epsilon^{3/2}/h^2). \tag{5.3.22}$$

What saves us is that away from $\Lambda_r \cup \Lambda_d$, we can conjugate the operator with exponential weights and show that the resolvent has an "off-diagonal decay" like $\exp(-1/(Ch))$. This implies that we can confine the growth in (5.3.22) to a small neighborhood of $\Lambda_r$, if

$$\frac{1}{Ch} \gg \frac{\epsilon^{\frac{3}{2}}}{h^2},$$

leading to the assumption $\epsilon \ll h^{2/3}$ in Theorem 5.12.

PART II

# Non-self-adjoint operators with random perturbations.

## 6. Zeros of holomorphic functions of exponential growth

We will need a result on the number of zeros in a domain of holomorphic functions $u(z) = u_h(z)$ that satisfy an exponential upper bound near the boundary of the domain as well as corresponding lower bounds at finitely many points, distributed along the boundary. Such a result (related to classical results for the zeros of entire functions, cf [68], Chapter III, Section 3, Theorem 3) was obtained by Hager [37, 38] under a rather strong regularity assumption on the exponent. With Hager [39] we obtained a more general result with a logarithmic loss however. Recently I revisited the proofs and was able to get a result that includes the earlier ones and allows to eliminate such losses, see [100].

Let $\Gamma \Subset \mathbf{C}$ be an open set and let $\gamma = \partial\Gamma$ be the boundary of $\Gamma$. Let $r : \gamma \to ]0, \infty[$ be a Lipschitz function of Lipschitz modulus $\leq 1/2$:

$$|r(x) - r(y)| \leq \frac{1}{2}|x - y|,\ x, y \in \gamma. \tag{6.0.1}$$

We further assume that $\gamma$ is Lipschitz in the following precise sense, where $r$ enters:

For every $x \in \gamma$ there exist new affine coordinates $\widetilde{y} = (\widetilde{y}_1, \widetilde{y}_2)$ of the form $\widetilde{y} = U(y - x)$, $y \in \mathbf{C} \simeq \mathbf{R}^2$ being the old coordinates, where $U = U_x$ is orthogonal, such that the intersection of $\Gamma$ and the rectangle $R_x := \{y \in \mathbf{C};\ |\widetilde{y}_1| < r(x),\ |\widetilde{y}_2| < C_0 r(x)\}$ takes the form

$$\{y \in R_x;\ \widetilde{y}_2 > f_x(\widetilde{y}_1),\ |\widetilde{y}_1| < r(x),\} \tag{6.0.2}$$

where $f_x(\widetilde{y}_1)$ is uniformly Lipschitz on $[-r(x), r(x)]$, and $C_0$ is a fixed constant, which is larger than the Lipschitz moduli of the functions $f_x$.

Notice that our assumption (6.0.2) remains valid if we decrease $r$. It will be convenient to extend the function to all of $\mathbf{C}$, by putting

$$r(x) = \inf_{y\in\gamma}(r(y) + \frac{1}{2}|x - y|). \tag{6.0.3}$$

The extended function is also Lipschitz with modulus $\leq \frac{1}{2}$:

$$|r(x) - r(y)| \leq \frac{1}{2}|x - y|,\ x, y \in \mathbf{C}.$$

Notice that

$$r(x) \geq \frac{1}{2}\mathrm{dist}\,(x, \gamma), \tag{6.0.4}$$

and that

$$|y - x| \leq r(x) \Rightarrow \frac{r(x)}{2} \leq r(y) \leq \frac{3r(x)}{2}. \tag{6.0.5}$$

**Theorem 6.1.** *Let $\Gamma \Subset \mathbf{C}$ be simply connnected, and have Lipschitz boundary $\gamma$ with an associated Lipschitz weight $r$ as in (6.0.1), (6.0.2), (6.0.3). Put $\widetilde{\gamma}_{\alpha r} = \cup_{x\in\gamma} D(x, \alpha r(x))$ for any constant $\alpha > 0$. Let $z_j^0 \in \gamma$, $j \in \mathbf{Z}/N\mathbf{Z}$ be distributed along the boundary in the positively oriented sense such that*

$$r(z_j^0)/4 \le |z_{j+1}^0 - z_j^0| \le r(z_j^0)/2.$$

*(Here "4" can be replaced by any fixed constant $> 2$.) Then if $C_1 > 0$ is large enough, depending only on the constant $C_0$ in (6.0.2) and if $C_1 \ge C_0$, $z_j \in D(z_j^0, r(z_j^0)/(2C_1))$, we have the following:*

*Let $\phi$ be a continuous subharmonic function on $\widetilde{\gamma}_{r/C_1}$ with a distribution extension to $\Gamma \cup \widetilde{\gamma}_{r/C_1}$ that will be denoted by the same symbol. Then there exists a constant $C_2 > 0$ such that if $u = u_h(z)$, $0 < h \le 1$, is a holomorphic function on $\Gamma \cup \widetilde{\gamma}_{r/C_1}$ satisfying*

$$h \ln |u| \le \phi(z) \text{ on } \widetilde{\gamma}_{r/C_1}, \tag{6.0.6}$$

$$h \ln |u(z_j)| \ge \phi(z_j) - \epsilon_j, \text{ for } j = 1, 2, ..., N, \tag{6.0.7}$$

*where $\epsilon_j \ge 0$, then the number of zeros of $u$ in $\Gamma$ satisfies*

$$\begin{aligned} &|\#(u^{-1}(0) \cap \Gamma) - \frac{1}{2\pi h}\mu(\Gamma)| \le \\ &\frac{C_2}{h}\left(\mu(\widetilde{\gamma}_{r/C_1}) + \sum_1^N \left(\epsilon_j + \int_{D(z_j, \frac{r(z_j)}{4C_1})} |\ln \frac{|w - z_j|}{r(z_j)}|\mu(dw)\right)\right). \end{aligned} \tag{6.0.8}$$

*Here $\mu := \Delta\phi \in \mathcal{D}'(\Gamma \cup \widetilde{\gamma}_{r/C_1})$ is a positive measure on $\widetilde{\gamma}_{r/C_1}$ so that $\mu(\Gamma)$ and $\mu(\widetilde{\gamma}_{r/C_1})$ are well-defined. Moreover, the constant $C_2$ only depends on $C_0$ in (6.0.2) and on $C_1$.*

We next discuss the elimination of the logarithmic integrals in (6.0.8).

Using (6.0.5), we get

$$\begin{aligned} &\int_{D(z_j^0, \frac{r(z_j^0)}{2C_1})} \int_{D(z, \frac{r(z)}{4C_1})} |\ln \frac{|w - z|}{r(z)}|\mu(dw) \frac{L(dz)}{L(D(z_j^0, \frac{r(z_j^0)}{2C_1}))} \le \\ &\int_{D(z_j^0, \frac{r(z_j^0)}{2C_1})} \int_{D(z_j^0, \frac{r(z_j^0)}{C_1})} |\ln \frac{|w - z|}{r(z)}|\mu(dw) \frac{L(dz)}{L(D(z_j^0, \frac{r(z_j^0)}{2C_1}))}, \end{aligned}$$

where $L$ denotes the Lebesgue measure. Here we use Fubini's theorem and the fact that

$$\int_{D(z_j^0, \frac{r(z_j^0)}{2C_1})} |\ln \frac{|z - w|}{r(z)}| L(dz) \le \mathcal{O}(1) L(D(z_j^0, \frac{r(z_j^0)}{2C_1}))$$

to conclude that the mean-value of

$$D(z_j^0, \frac{r(z_j^0)}{2C_1}) \ni z \mapsto \int_{D(z, \frac{r(z)}{4C_1})} |\ln \frac{|w - z|}{r(z)}|\mu(dw)$$

is $\mathcal{O}(1)\mu(D(z_j^0, \frac{r(z_j^0)}{C_1}))$. Thus we can find $\widetilde{z}_j \in D(z_j^0, \frac{r(z_j^0)}{2C_1})$ such that

$$\sum_{j=1}^N \int_{D(\widetilde{z}_j, \frac{r(\widetilde{z}_j)}{4C})} |\ln \frac{|w - \widetilde{z}_j|}{r(\widetilde{z}_j)}|\mu(dw) = \mathcal{O}(1)\mu(\widetilde{\gamma}_{r/C}).$$

This leads to the following variant of Theorem 6.1, where (6.0.8) is simplified but where the choice of $z_j$ is no more arbitrary.

**Theorem 6.2.** *Let $\Gamma$, $\gamma=\partial\Gamma$, $r$, $z_j^0$, $C_0$, $C_1$ be as in Theorem 6.1. Then $\exists\, \widetilde{z}_j \in D(z_j^0, \frac{r(z_j^0)}{2C_1})$ such that if $\phi$, $u$ are as in Theorem 6.1, satisfying (6.0.6), and*

$$h\ln|u(\widetilde{z}_j)| \geq \phi(\widetilde{z}_j)-\epsilon_j,\ \ j=1,2,...,N, \tag{6.0.9}$$

*instead of (6.0.7), then*

$$|\#(u^{-1}(0)\cap\Gamma)-\frac{1}{2\pi h}\mu(\Gamma)| \leq \frac{C_2}{h}(\mu(\widetilde{\gamma}_{r/C_1})+\sum\epsilon_j). \tag{6.0.10}$$

Of course, if we already know that

$$\int_{D(z_j,\frac{r(z_j)}{4C})} |\ln\frac{|w-z_j|}{r(z_j)}|\mu(dw) = \mathcal{O}(1)\mu(D(z_j,\frac{r(z_j)}{4C}), \tag{6.0.11}$$

then we can keep $\widetilde{z}_j=z_j$ in (6.0.8) and get (6.0.10). This is the case, if we assume that $\mu$ is equivalent to the Lebesgue measure in the following sense:

$$\frac{\mu(dw)}{\mu(D(z_j,\frac{r(z_j)}{2C_1}))} \asymp \frac{L(dw)}{L(D(z_j,\frac{r(z_j)}{2C_1}))} \text{ on } D(z_j,\frac{r(z_j)}{2C_1}),\ \text{uniformly for } j=1,2,...,N. \tag{6.0.12}$$

Then we get,

**Theorem 6.3.** *Make the assumptions of Theorem 6.1 as well as (6.0.11) or the stronger assumption (6.0.12). Then from (6.0.6), (6.0.7), we conclude (6.0.10).*

In particular, we recover the counting proposition of M. Hager [37, 38], where $\phi$ was independent of $h$ and of class $C^2$ in a fixed neighborhood of $\gamma$. Then $\mu \asymp L$ and if we choose $r \ll 1$ constant and assume (6.0.6), (6.0.7), we get from (6.0.9):

$$|\#(u^{-1}(0)\cap\Gamma)-\frac{1}{2\pi h}\mu(\Gamma)| \leq \frac{\widetilde{C}}{h}(r+\sum_1^N\epsilon_j). \tag{6.0.13}$$

Hager had $\epsilon_j=\epsilon$ independent of $j$, $r=\sqrt{\epsilon}$, $N\asymp\epsilon^{-1/2}$, so the remainder in (6.0.13) is $\mathcal{O}(\frac{\sqrt{\epsilon}}{h})$.

We next outline the proof of Theorem 6.1. Using a locally finite covering with discs $D(x,r(x))$ and a subordinated partition of unity, it is standard to find a smooth function $\widetilde{r}(x)$ satisfying

$$\frac{1}{C}r(x)\leq\widetilde{r}(x)\leq r(x),\ |\nabla\widetilde{r}(x)|\leq\frac{1}{2},\ \partial^\alpha\widetilde{r}(x)=\mathcal{O}(\widetilde{r}^{1-|\alpha|}). \tag{6.0.14}$$

From now on, we replace $r(x)$ by $\widetilde{r}(x)$ and the drop the tilde. The general estimates on $r$ remain valid and we have

$$r(x)\geq\frac{1}{C}\text{dist}\,(x,\gamma).$$

.

Consider the signed distance to $\gamma$:

$$g(x)=\begin{cases}\text{dist}\,(x,\gamma), & x\in\Gamma\\ -\text{dist}\,(x,\gamma), & x\in\mathbf{C}\setminus\Gamma\end{cases} \tag{6.0.15}$$

In the set $\cup_{x\in\gamma}R_x$, we consider the regularized function

$$g_\epsilon(x)=\int\frac{1}{(\epsilon r(x))^2}\chi(\frac{x-y}{\epsilon r(x)})g(y)L(dy), \tag{6.0.16}$$

where $0 \le \chi \in C_0^\infty(D(0,1))$, $\int \chi(x)L(dx) = 1$. Here $\epsilon > 0$ is small and we notice that $r(x) \asymp r(y)$, $g(y) = \mathcal{O}(r(y))$, when $\chi((x-y)/\epsilon r(x)) \neq 0$. It follows that $g_\epsilon(x) = \mathcal{O}(r(x))$ and more precisely, since $g$ is Lipschitz, that

$$g_\epsilon(x) - g(x) = \mathcal{O}(\epsilon r(x)). \tag{6.0.17}$$

Moreover one can show that if $(\nabla g)_\epsilon$ denotes the regularization of $\nabla g$, obtained as in (6.0.16), then

$$\nabla_x g_\epsilon(x) - (\nabla g)_\epsilon(x) = \mathcal{O}(1) \sup_{y \in D(x,\epsilon r(x))} \frac{|g(y)|}{r(x)}, \tag{6.0.18}$$

$$\partial^\alpha g_\epsilon(x) = \mathcal{O}_\alpha((\epsilon r(x))^{1-|\alpha|}),\ |\alpha| \ge 1. \tag{6.0.19}$$

Let $C > 0$ be large enough but independent of $\epsilon$. Put

$$\widetilde{\gamma}_{C\epsilon r} = \{x \in \cup_{y\in\gamma} R_y;\ |g_\epsilon(x)| < C\epsilon r(x)\}. \tag{6.0.20}$$

If $C > 0$ is sufficiently large, then in the coordinates associated to (6.0.2), $\widetilde{\gamma}_{C\epsilon r}$ takes the form

$$f_x^-(\widetilde{y}_1) < \widetilde{y}_2 < f_x^+(\widetilde{y}_1),\ |\widetilde{y}_1| < r(x), \tag{6.0.21}$$

where $f_x^\pm$ are smooth on $[-r(x), r(x)]$ and satisfy

$$\partial_{\widetilde{y}_1}^k f_x^\pm = \mathcal{O}_k((\epsilon r(x))^{1-k}),\ k \ge 1, \tag{6.0.22}$$

$$0 < f_x^+ - f_x,\ f_x - f_x^- \asymp \epsilon r(x). \tag{6.0.23}$$

Later, we will fix $\epsilon > 0$ small enough and write $\gamma_r = \widetilde{\gamma}_{C\epsilon r}$ and more generally, $\gamma_{\alpha r} = \widetilde{\gamma}_{C\alpha\epsilon r}$.

We shall next establish an exponentially weighted estimate for the Dirichlet Laplacian in $\gamma_r$ by adating the general approach of Agmon estimates to thin tubes (cf [42], [59]):

**Proposition 6.4.** *Let $C > 0$ be sufficiently large and $\epsilon > 0$ sufficiently small. Then if $\phi \in C^2(\overline{\gamma}_r)$ and*

$$|\phi_x'| \le \frac{1}{Cr}, \tag{6.0.24}$$

*we have*

$$\|e^\phi Du\| + \frac{1}{C}\|\frac{1}{r}e^\phi u\| \le C\|re^\phi \Delta u\|,\ u \in (H_0^1 \cap H^2)(\gamma_r), \tag{6.0.25}$$

*where we use the natural $L^2$ norms.*

**Outline of the Proof.** Let $\phi \in C^2(\overline{\gamma}_r; \mathbf{R})$ and put

$$-\Delta_\phi = e^\phi \circ (-\Delta) \circ e^{-\phi} = D_x^2 - (\phi_x')^2 + i(\phi_x' \circ D_x + D_x \circ \phi_x'),$$

where we make the usual observation that the last term is formally anti-self-adjoint. Then for every $u \in (H^2 \cap H_0^1)(\gamma_r)$:

$$(-\Delta_\phi u|u) = \|D_x u\|^2 - ((\phi_x')^2 u|u). \tag{6.0.26}$$

We need an apriori estimate for $D_x$. Let $v : \overline{\gamma}_r \to \mathbf{R}^n$ be sufficiently smooth. We sometimes consider $v$ as a vector field. Then for $u \in (H^2 \cap H_0^1)(\gamma_r)$:

$$(Du|vu) - (vu|Du) = i(\mathrm{div}\,(v)u|u).$$

Assume $\mathrm{div}\,(v) > 0$. Recall that if $v = \nabla w$, then $\mathrm{div}\,(v) = \Delta w$, so it suffices to take $w$ strictly subharmonic. Then

$$\int \mathrm{div}\,(v)|u|^2 dx \leq 2\|vu\|\|Du\| \leq \|Du\|^2 + \|vu\|^2,$$

which we write

$$\int (\mathrm{div}\,(v) - |v|^2)|u|^2 dx \leq \|Du\|^2.$$

Using this in (6.0.26), we get

$$\frac{1}{2}\|Du\|^2 + \int (\frac{1}{2}(\mathrm{div}\,(v) - |v|^2) - (\phi'_x)^2)|u|^2 dx \leq$$
$$\|\frac{1}{k}(-\Delta_\phi)u\|\|ku\| \leq \frac{1}{2}\|\frac{1}{k}(-\Delta_\phi)u\|^2 + \frac{1}{2}\|ku\|^2,$$

where $k$ is any positive continuous function on $\overline{\gamma}_r$. We write this as

$$\frac{1}{2}\|Du\|^2 + \int (\frac{1}{2}(\mathrm{div}\,(v) - |v|^2 - k^2) - (\phi'_x)^2) dx \leq \frac{1}{2}\|\frac{1}{k}(-\Delta_\phi)u\|^2. \tag{6.0.27}$$

The remaining work is then to see that we can choose $v$ so that

$$\mathrm{div}\,(v) \geq r^{-2},\ |v| \leq \mathcal{O}(r^{-1}). \tag{6.0.28}$$

and it turns out that this is possible with

$$v = \nabla(e^{\lambda g/r}), \tag{6.0.29}$$

where $\lambda > 0$ is sufficiently large and $g = g_\epsilon$. Then replace $v$ by a small multiple and finally choose $k$ to be a small multiple of $1/r$. □

If $\Omega \Subset \mathbf{C}$ has smooth boundary, let $G_\Omega$, $P_\Omega$ denote the Green and the Poisson kernels of $\Omega$, so that the Dirichlet problem,

$$\Delta u = v,\ u_{|\partial\Omega} = f, \quad u, v \in C^\infty(\overline{\Omega}),\ f \in C^\infty(\partial\Omega),$$

has the unique solution

$$u(x) = \int_\Omega G_\Omega(x,y)v(y)L(dy) + \int_{\partial\Omega} P_\Omega(x,y)f(y)|dy|.$$

Recall that $-G_\Omega \geq 0$, $P_\Omega \geq 0$. We have

$$-G_\Omega(x,y) \leq C - \frac{1}{2\pi}\ln|x-y|, \tag{6.0.30}$$

where $C > 0$ only depends on the diameter of $\Omega$.

We also have the scaling property:

$$G_\Omega(\frac{x}{t}, \frac{y}{t}) = G_{t\Omega}(x,y),\ x, y \in t\Omega, t > 0. \tag{6.0.31}$$

Moreover, $-G_\Omega$ is an increasing function of $\Omega$ in the natural sense. Using these facts with Proposition 6.4 one can show the following result:

**Proposition 6.5.** *For all $x, y \in \gamma_r$ (and $\epsilon > 0$ small enough), we have*

$$-G_{\gamma_r}(x,y) \leq C - \frac{1}{2\pi}\ln\frac{|x-y|}{r(y)},\ \textit{when } |x-y| \leq \frac{r(y)}{C}, \tag{6.0.32}$$

$$-G_{\gamma_r}(x,y) \leq C\exp(-\frac{1}{C}\int_{\pi_\gamma(y)}^{\pi_\gamma(x)} \frac{1}{r(t)}|dt|),\ \textit{when } |x-y| \geq \frac{r(y)}{C}, \tag{6.0.33}$$

*where it is understood that the integral is evaluated along $\gamma$ from $\pi_\gamma(y) \in \gamma$ to $\pi_\gamma(x) \in \gamma$, where $\pi_\gamma(y)$, $\pi_\gamma(x)$ denote points in $\gamma$ with $|x - \pi_\gamma(x)| = \mathrm{dist}\,(x,\gamma)$,*

$|y-\pi_\gamma(y)| = \mathrm{dist}\,(y,\gamma)$, *and we choose these two points (when they are not uniquely defined) and the intermediate segment in such a way that the integral is as small as possible.*

We will also need a lower bound on $G_{\gamma_r}$ on suitable subsets of $\gamma_r$. For $\epsilon > 0$ fixed and sufficiently small, we say that $M \Subset \gamma_r$ is an elementary piece of $\gamma_r$ if

- $M \subset \gamma_{(1-\frac{1}{Cr})}$,
- $\frac{1}{C} \le \frac{r(x)}{r(y)} \le C$, $x, y \in M$,
- $\exists y \in M$ such that $M = y + r(y)\widetilde{M}$, where $\widetilde{M}$ belongs to a bounded set of relatively compact subsets of $\mathbf{C}$ with smooth boundary.

In the following, it will be tacitly understood that we choose our elementary pieces with some uniform control ($C$ fixed and uniform control on the $\widetilde{M}$). Using Harnack's inequality one can show:

**Proposition 6.6.** *If $M$ is an elementary piece in $\gamma_r$, then*

$$-G_{\gamma_r}(x,y) \asymp 1 + |\ln \frac{|x-y|}{r(y)}|,\ \ x, y \in M. \tag{6.0.34}$$

Let $\phi$ be a continuous subharmonic function defined in some neighborhood of $\overline{\gamma_r}$. Let

$$\mu = \mu_\phi = \Delta\phi \tag{6.0.35}$$

be the corresponding locally finite positive measure.

Let $u$ be a holomorphic function defined in a neighborhood of $\Gamma \cup \overline{\gamma_r}$. We assume that

$$h \ln |u(z)| \le \phi(z),\ \ z \in \overline{\gamma_r}. \tag{6.0.36}$$

**Lemma 6.7.** *Let $z_0 \in M$, where $M$ is an elementary piece, such that*

$$h \ln |u(z_0)| \ge \phi(z_0) - \epsilon,\ \ 0 < \epsilon \ll 1. \tag{6.0.37}$$

*Then the number of zeros of $u$ in $M$ is*

$$\le \frac{C}{h}(\epsilon + \int_{\gamma_r} -G_{\gamma_r}(z_0,w)\mu(dw)). \tag{6.0.38}$$

*Proof.* Writing $\phi$ as a uniform limit of an increasing sequence of smooth functions, we may assume that $\phi \in C^\infty$. Let

$$n_u(dz) = \sum 2\pi\delta(z-z_j),$$

where $z_j$ are the zeros of $u$ counted with their multiplicity. We may assume that no $z_j$ are situated on $\partial\gamma_r$. Then, since $\Delta \ln |u| = n_u$,

$$\begin{aligned} h\ln|u(z)| &= \int_{\gamma_r} G_{\gamma_r}(z,w) h n_u(dw) + \int_{\partial\gamma_r} P_{\gamma_r}(z,w) h \ln |u(w)| |dw| \\ &\le \int_{\gamma_r} G_{\gamma_r}(z,w) h n_u(dw) + \int_{\partial\gamma_r} P_{\gamma_r}(z,w)\phi(w)|dw| \\ &= \int_{\gamma_r} G_{\gamma_r}(z,w) h n_u(dw) + \phi(z) - \int_{\gamma_r} G_{\gamma_r}(z,w)\mu(dw). \end{aligned} \tag{6.0.39}$$

Putting $z = z_0$ in (6.0.39) and using (6.0.37), we get

$$\int_{\gamma_r} -G_{\gamma_r}(z_0, w) h n_u(dw) \le \epsilon + \int_{\gamma_r} -G_{\gamma_r}(z_0, w)\mu(dw).$$

Now

$$-G_{\gamma_r}(z_0, w) \ge \frac{1}{C},\ w \in M,$$

and we get (6.0.38). □

Notice that this argument is basically the same as when using Jensen's formula to estimate the number of zeros of a holomorphic function in a disc.

Now we sharpen the assumption (6.0.37) and assume

$$h \ln |u(z_j)| \ge \phi(z_j) - \epsilon_j, \tag{6.0.40}$$

where $z_1, ..., z_N \in \gamma_{(1-\frac{1}{C_1})r}$ are points such that with the cyclic convention $N+1 = 1$:

$$|z_{j+1} - z_j| \asymp r(z_j),\ \frac{r(z_{j+1})}{r(z_j)} \asymp 1. \tag{6.0.41}$$

We also assume that $z_1, z_2, ..., z_N$ are arranged in such a way that

$$\mathbf{Z}/N\mathbf{Z} \mapsto \pi_\gamma(z_j) \text{ runs through the oriented boundary in the positive sense.} \tag{6.0.42}$$

Let $M_j \subset \gamma_r$ be elementary pieces such that

$$z_j \in M_j,\ \operatorname{dist}(z_j, M_k) \ge \frac{r(z_j)}{C} \text{ when } k \ne j,\ \gamma_{\widetilde{r}} \subset \cup_j M_j,\ \widetilde{r} = (1 - \frac{1}{C_1})r. \tag{6.0.43}$$

We will also assume for a while that $\phi$ is smooth.

According to Lemma 6.7, we have

$$\#(u^{-1}(0) \cap M_j) \le \frac{C_3}{h}(\epsilon_j + \int_{\gamma_r} -G_{\gamma_r}(z_j, w)\mu(dw)). \tag{6.0.44}$$

Consider the harmonic functions on $\gamma_{\widetilde{r}}$,

$$\Psi(z) = h(\ln |u(z)| + \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z, w) n_u(dw)), \tag{6.0.45}$$

$$\Phi(z) = \phi(z) + \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z, w)\mu(dw). \tag{6.0.46}$$

Then $\Phi(z) \ge \phi(z)$ with equality on $\partial\gamma_{\widetilde{r}}$. Similarly, $\Psi(z) \ge h \ln |u(z)|$ with equality on $\partial\gamma_{\widetilde{r}}$.

Consider the harmonic function

$$H(z) = \Phi(z) - \Psi(z),\ z \in \gamma_{\widetilde{r}}. \tag{6.0.47}$$

Then on $\partial\gamma_{\widetilde{r}}$, we have by (6.0.36) that

$$H(z) = \phi(z) - h \ln |u(z)| \ge 0,$$

so by the maximum principle,

$$H(z) \ge 0,\ \text{on } \gamma_{\widetilde{r}}. \tag{6.0.48}$$

By (6.0.40), we have

$$\begin{aligned} H(z_j) &= \Phi(z_j) - \Psi(z_j) \qquad (6.0.49)\\ &= \phi(z_j) - h\ln|u(z_j)| \\ &\quad + \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z_j,w)\mu(dw) - \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z_j,w)hn_u(dw) \\ &\leq \epsilon_j + \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z_j,w)\mu(dw). \end{aligned}$$

Harnack's inequality implies that

$$H(z) \leq \mathcal{O}(1)(\epsilon_j + \int -G_{\gamma_{\widetilde{r}}}(z_j,w)\mu(dw)) \text{ on } M_j \cap \gamma_{\widehat{r}},\ \widehat{r} = (1-\frac{1}{C_1})\widetilde{r}. \qquad (6.0.50)$$

Now assume that $u$ extends to a holomorphic function in a neighborhood of $\Gamma \cup \overline{\gamma_r}$. We then would like to evaluate the number of zeros of $u$ in $\Gamma$. Using (6.0.44), we first have

$$\#(u^{-1}(0)\cap\gamma_{\widehat{r}}) \leq \frac{C}{h}\sum_{j=1}^{N}\left(\epsilon_j + \int_{\gamma_r} -G_{\gamma_r}(z_j,w)\mu(dw)\right). \qquad (6.0.51)$$

Let $\chi \in C_0^\infty(\Gamma\cup\gamma_{\widehat{r}};[0,1])$ be equal to 1 on $\Gamma$. Of course $\chi$ will have to depend on $r$ but we may assume that for all $k \in \mathbf{N}$,

$$\nabla^k\chi = \mathcal{O}(r^{-k}). \qquad (6.0.52)$$

We are interested in

$$\int \chi(z)hn_u(dz) = \int_{\gamma_{\widetilde{r}}} h\ln|u(z)|\Delta\chi(z)L(dz). \qquad (6.0.53)$$

Here we have on $\gamma_{\widetilde{r}}$

$$\begin{aligned} h\ln|u(z)| &= \Psi(z) - \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z,w)hn_u(dw) \qquad (6.0.54)\\ &= \Phi(z) - H(z) - \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z,w)hn_u(dw) \\ &= \phi(z) + \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z,w)\mu(dw) - H(z) - \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z,w)hn_u(dw) \\ &= \phi(z) + R(z), \end{aligned}$$

where the last equality defines $R(z)$.

Inserting this in (6.0.53), we get

$$\int \chi(z)hn_u(dz) = \int \chi(z)\mu(dz) + \int R(z)\Delta\chi(z)L(dz). \qquad (6.0.55)$$

(Here we also used some extension of $\phi$ to $\Gamma$ with $\mu = \Delta\phi$.) The task is now to estimate $R(z)$ and the corresponding integral in (6.0.55). Put

$$\mu_j = \mu(M_j \cap \gamma_{\widetilde{r}}). \qquad (6.0.56)$$

Using the exponential decay property (6.0.33) (equally valid for $G_{\gamma_{\widetilde{r}}}$) we get for $z \in M_j \cap \gamma_{\widehat{r}}$, $\mathrm{dist}\,(z,\partial M_k) \geq r(z_j)/\mathcal{O}(1)$, $k \neq j$:

$$\int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z,w)\mu(dw) \leq \int_{M_j\cap\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z,w)\mu(dw) + \mathcal{O}(1)\sum_{k\neq j}\mu_k e^{-\frac{1}{C_0}|j-k|}, \qquad (6.0.57)$$

where $|j-k|$ denotes the natural distance from $j$ to $k$ in $\mathbf{Z}/N\mathbf{Z}$. Similarly from (6.0.50), we get

$$H(z) \le \mathcal{O}(1)(\epsilon_j + \int_{M_j\cap\widetilde{\gamma_r}} -G_{\widetilde{\gamma_r}}(z_j,w)\mu(dw) + \sum_{k\ne j} e^{-\frac{1}{C_0}|j-k|}\mu_k), \tag{6.0.58}$$

for $z\in M_j\cap\widetilde{\gamma_r}$.

This gives the following estimate on the contribution from the first two terms in $R(z)$ to the last integral in (6.0.55):

$$\begin{aligned}&\int_{\widetilde{\gamma_r}}\left(\int_{\widetilde{\gamma_r}} -G_{\widetilde{\gamma_r}}(z,w)\mu(dw) - H(z)\right)\Delta\chi(z)L(dz) \qquad (6.0.59)\\ &= \mathcal{O}(1)\sum_j(\epsilon_j + \int_{M_j\cap\widetilde{\gamma_r}} -G_{\widetilde{\gamma_r}}(z_j,w)\mu(dw)) + \sum_{k\ne j} e^{-\frac{1}{C_0}|j-k|}\mu_k)\\ &+\mathcal{O}(1)\sum_j\int_{M_j\cap\widetilde{\gamma_r}}\int_{M_j\cap\widetilde{\gamma_r}} -G_{\widetilde{\gamma_r}}(z,w)\mu(dw)|\Delta\chi(z)|L(dz).\end{aligned}$$

Here,

$$\int_{M_j\cap\widetilde{\gamma_r}} -G_{\widetilde{\gamma_r}}|\Delta\chi(z)|L(dz) = \mathcal{O}(1), \tag{6.0.60}$$

so (6.0.59) leads to

$$\begin{aligned}&\int_{\widetilde{\gamma_r}}\left(\int_{\widetilde{\gamma_r}} -G_{\widetilde{\gamma_r}}(z,w)\mu(dw) - H(z)\right)\Delta\chi(z)L(dz) \qquad (6.0.61)\\ &= \mathcal{O}(1)\left(\mu(\widetilde{\gamma_r}) + \sum_j\epsilon_j + \sum_j\int_{M_j\cap\widetilde{\gamma_r}} -G_{\widetilde{\gamma_r}}(z_j,w)\mu(dw)\right).\end{aligned}$$

The contribution from the last term in $R(z)$ (in (6.0.54)) to the last integral in (6.0.55) is

$$\int_{z\in\widetilde{\gamma_r}}\int_{w\in\widetilde{\gamma_r}} G_{\widetilde{\gamma_r}}(z,w)hn_u(dw)\Delta\chi(z)L(dz). \tag{6.0.62}$$

Here, by using an estimate similar to (6.0.57) with $\mu(dw)$ replaced by $L(dz)$ together with (6.0.60), we get

$$\int_{z\in\widetilde{\gamma_r}} G_{\widetilde{\gamma_r}}(z,w)(\Delta\chi)(z)L(dz) = \mathcal{O}(1),$$

so the expression (6.0.62) is by (6.0.51)

$$\begin{aligned}&\mathcal{O}(h)\#(u^{-1}(0)\cap\widetilde{\gamma_r}) \qquad (6.0.63)\\ =\ &\mathcal{O}(1)\sum_{j=1}^N(\epsilon_j + \int_{\gamma_r}(-G_{\gamma_r}(z_j,w))\mu(dw))\\ =\ &\mathcal{O}(1)(\mu(\gamma_r) + \sum_{j=1}^N(\epsilon_j + \int_{M_j} -G_{\gamma_r}(z_j,w)\mu(dw))).\end{aligned}$$

This is quite similar to (6.0.61). Using Proposition 6.5, we have

$$\int_{M_j\cap\widetilde{\gamma_r}} -G_{\widetilde{\gamma_r}}(z_j,w)\mu(dw) \le \mathcal{O}(1)(\int_{|w-z_j|\le\frac{r(z_j)}{C}} |\ln\frac{|z_j-w|}{r(z_j)}|\mu(dw) + \mu(M_j\cap\widetilde{\gamma_r})$$

and similarly for the last integral in (6.0.63) Using all this in (6.0.55), we get

$$\int \chi(z) h n_u(dz) = \int \chi(z)\mu(dz) \tag{6.0.64}$$
$$+\mathcal{O}(1)(\mu(\gamma_r) + \sum_j (\epsilon_j + \int_{|w-z_j|\le r(z_j)/C} |\ln(\frac{|z_j - w|}{r(z_j)})|\mu(dw)).$$

We replace the smoothness assumption on $\phi$ by the assumption that $\phi$ is continuous near $\Gamma$ and keep (6.0.40). Then by regularization, we still get (6.0.64).

Here, we observe that

$$|\#(u^{-1}(0)\cap\Gamma) - \frac{1}{2\pi h}\int \chi(z) h n_u(dz)| \le \#(u^{-1}(0)\cap\widetilde{\gamma_r}),$$

which can be estimated by means of (6.0.63), and combining this with (6.0.64), we get

$$|\#(u^{-1}(0)\cap\Gamma) - \frac{1}{2\pi h}\mu(\Gamma)| \le \tag{6.0.65}$$
$$\frac{\mathcal{O}(1)}{h}\left(\mu(\gamma_r) + \sum_j (\epsilon_j + \int_{|w-z_j|\le \frac{r(z_j)}{C}} |\ln \frac{|z_j - w|}{r(z_j)}|\mu(dw))\right).$$

## 7. The one-dimensional semi-classical case

In this section we consider a simple model operator in dimension 1 and show how random perturbations give rise to Weyl asymptotics in the interior of the range of $p$. We follow rather closely the work of Hager [38] with some inputs also from Bordeaux Montrieux [10] and Hager–Sj [39]. Some of the general ideas appear perhaps more clearly in this special situation.

Let $P = hD_x + g(x)$, $g \in C^\infty(S^1)$ with symbol $p(x,\xi) = \xi + g(x)$, and assume that $\Im g$ has precisely two critical points; a unique maximum and a unique minimum.

Let $\Omega \Subset \{z \in \mathbf{C};\ \min \Im g < \Im z < \max \Im g\}$ be open. Put

$$P_\delta = P_{\delta,\omega} = hD_x + g(x) + \delta Q_\omega, \tag{7.0.1}$$
$$Q_\omega u(x) = \sum_{|k|,|\ell|\le \frac{C_1}{h}} \alpha_{j,k}(\omega)(u|e^k)e^\ell(x),$$

where $C_1 > 0$ is sufficiently large, $e^k(x) = (2\pi)^{-1/2}e^{ikx}$, $k \in \mathbf{Z}$, and $\alpha_{j,k} \sim \mathcal{N}(0,1)$ are independent complex Gaussian random variables, centered with variance 1. $Q_\omega$ is compact, so $P_\delta$ has discrete spectrum. Let $\Gamma \Subset \Omega$ have smooth boundary.

**Theorem 7.1.** *Let $\kappa > 5/2$ and let $\epsilon_0 > 0$ be sufficiently small. Let $\delta = \delta(h)$ satisfy $e^{-\epsilon_0/h} \ll \delta \ll h^\kappa$ and put $\epsilon = \epsilon(h) = h\ln(1/\delta)$. Then for $h > 0$ small enough, we have with probability $\ge 1 - \mathcal{O}(\frac{\delta^2}{\sqrt{\epsilon}h^5})$ that the number of eigenvalues of $P_\delta$ in $\Gamma$ satisfies*

$$|\#(\sigma(P_\epsilon)\cap\Gamma) - \frac{1}{2\pi h}\mathrm{vol}(p^{-1}(\Gamma))| \le \mathrm{Const.}\,\frac{\sqrt{\epsilon}}{h}. \tag{7.0.2}$$

If instead, we let $\Gamma$ vary in a set of subsets that satisfy the assumptions uniformly, then with probability $\ge 1-\mathcal{O}(\frac{\delta^2}{\epsilon h^5})$ we have (7.0.2) uniformly for all $\Gamma$ in that subset. The remainder of the section is devoted to the (outline of) the proof of this result.

## 7.1. Preparations for the unperturbed operator

For $z \in \Omega$, let $x_+(z), x_-(z) \in S^1$ be the solutions of the equation $\Im g(x) = \Im z$, with $\pm \Im g'(x_\pm) < 0$, define $\xi_\pm(z)$ by $\xi_\pm + \Re g(x_\pm) = \Re z$. Then, with $\rho_\pm = (x_\pm, \xi_\pm)$, we have

$$p(\rho_\pm) = z, \quad \pm \frac{1}{i}\{p, \overline{p}\}(\rho_\pm) > 0.$$

We introduce quasimodes of the form

$$e_{\mathrm{wkb}}(x) = h^{-1/4} a(h) \chi(x - x_+(z)) e^{\frac{i}{h}\phi_+(x)},$$

where $a(h) \sim a_0 + h a_1 + ..,\ a_0 \neq 0$, $\phi_+(x) = \int_{x_+(z)}^x (z - g(y)) dy$, $\chi \in C_0^\infty(\mathrm{neigh}\,(0, \mathbf{R}))$ and $\chi(x) = 1$ in a neighborhood of 0. We can choose $a$ depending smoothly on $z$ such that all derivatives with respect to $z, \overline{z}$ are bounded when $h \to 0$ and $\|e_{\mathrm{wkb}}\| = 1$ where we take the $L^2$ norm over $]x_-(z), x_+(z) + 2\pi[$. We can assume that $e_{\mathrm{wkb}}$ is normalized in $L^2$ and

$$(P - z) e_{\mathrm{wkb}} = \mathcal{O}(e^{-\frac{1}{Ch}}).$$

Define $z$-dependent elliptic self-adjoint operators

$$Q = (P - z)^*(P - z),\ \widetilde{Q} = (P - z)(P - z)^* :\ L^2(S^1) \to L^2(S^1),$$

with domain $\mathcal{D}(Q)$, $\mathcal{D}(\widetilde{Q}) = H^2(S^1)$. They have discrete spectrum $\subset [0, +\infty[$. Using that $P - z$ is Fredholm of index zero, we see that $\dim \mathcal{N}(Q) = \dim \mathcal{N}(\widetilde{Q})$. If $\mu \neq 0$ is an eigenvalue of $Q$, with the corresponding eigenfunction $e \in C^\infty$, then $f := (P - z)e$ is an eigenfunction for $\widetilde{Q}$ with the same eigenvalue $\mu$. Pursuing this observation, we see that

$$\sigma(Q) = \sigma(\widetilde{Q}) = \{t_0^2, t_1^2, ...\},\ 0 \le t_j \nearrow +\infty.$$

**Proposition 7.2.** *There exists a constant $C > 0$ such that $t_0^2 = \mathcal{O}(e^{-1/(Ch)})$, $t_1^2 - t_0^2 \ge h/C$ for $h > 0$ small enough.*

*Proof.* We have $Q e_{\mathrm{wkb}} = r$, $\|r\| = \mathcal{O}(e^{-1/Ch})$ and since $Q$ is self adjoint we deduce that $t_0^2$ is exponentially small. If $e_0$ denotes the corresponding normalized eigenfunction, we see that $(P - z)e_0 =: v$ with $\|v\|$ exponentially small. Considering this ODE on $]x_-(z) - 2\pi, x_-(z)[$, we get

$$e_0(x) = C h^{-\frac{1}{4}} a(h) e^{\frac{i}{h}\phi_+(x)} + F v(x),$$

$$F v(x) = \frac{i}{h} \int_{x_+}^x e^{\frac{i}{h}(\phi_+(x) - \phi_+(y))} v(y) dy,$$

where $\phi_+(x) = \int_{x_+}^x (z - g(y)) dy$. We observe that $\Im(\phi_+(x) - \phi_+(y)) \ge 0$ on the domain of integration. With some more work Hager showed that $\|F\|_{\mathcal{L}(L^2, L^2)} = \mathcal{O}(h^{-1/2})$. Hence for our particular $v$, we see that $Fv$ is exponentially decaying in $L^2$. Recalling the form of $e_{\mathrm{wkb}}(x)$ we conclude that $\|e_0 - e_{\mathrm{wkb}}\|$ is exponentially small.

To show that $t_1^2 - t_0^2 \ge h/C$, it suffices to show that $(Qu|u) \ge \frac{h}{C}\|u\|^2$ when $u \perp e_0$ or in other words, that

$$\|u\| \le \sqrt{\frac{C}{h}} \|(P - z)u\|. \tag{7.1.1}$$

If $v := (P - z)u$, we again have

$$u = C h^{-\frac{1}{4}} a(h) e^{\frac{i}{h}\phi_+(x)} + F v$$

for some constant $C$ and the orthogonality requirement on $u$ implies that

$$0 = (1 + \mathcal{O}(h^\infty))C + (Fv|e_0),$$

where $(Fv|e_0) = \mathcal{O}(h^{-\frac{1}{2}})\|v\|$, so $C = \mathcal{O}(h^{-1/2})\|v\|$ and we get the desired estimate on $\|u\|$. ☐

## 7.2. Grushin (Shur, Feschbach, bifurcation) approach

Let $f_0$ be the normalized eigenfunction such that $\widetilde{Q} f_0 = t_0^2 f_0$. As observed prior to Proposition 7.2, we get

$$(P - z)e_0 = \alpha_0 f_0,\ (P - z)^* f_0 = \beta_0 e_0, \quad \alpha_0\beta_0 = t_0^2,$$

and combining this with $((P - z)e_0|f_0) = (e_0|(P - z)^* f_0)$, we see that $\alpha_0 = \overline{\beta}_0$. Define $R_+ : L^2(S^1) \to \mathbf{C}$, $R_- = \mathbf{C} \to L^2(S^1)$ by

$$R_+u = (u|e_0), \quad R_-u_- = u_- f_0.$$

Then

$$\mathcal{P}(z) := \begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} : H^1 \times \mathbf{C} \to L^2 \times \mathbf{C}$$

is bijective with the bounded inverse

$$\mathcal{E}(z) = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}.$$

Here $E = \mathcal{O}(h^{-1/2})$ in $L^2 \to L^2$ is basically the inverse of $P - z$ from $(f_0)^\perp$ to $(e_0)^\perp$, $E_-v = (v|f_0)$, $E_+v_+ = v_+e_0$, $E_{-+} = \mathcal{O}(e^{-1/(Ch)})$. It is a general feature of such auxiliary (Grushin) operators that

$$z \in \sigma(P) \Leftrightarrow E_{-+}(z) = 0.$$

## 7.3. d-bar equation for $E_{-+}$

**Proposition 7.3.** *We have*

$$\partial_{\overline{z}} E_{-+}(z) + f(z)E_{-+}(z) = 0, \tag{7.3.1}$$

*where*

$$f(z) = f_+(z) + f_-(z), \quad f_+(z) = (\partial_{\overline{z}} R_+)E_+,\ f_-(z) = E_-\partial_{\overline{z}} R_-. \tag{7.3.2}$$

*Thus,*

$$\partial_{\overline{z}}(e^{F(z)}E_{-+}(z)) = 0 \text{ if } \partial_{\overline{z}} F(z) = f(z). \tag{7.3.3}$$

*Moreover,*

$$\Re\Delta F(z) = \Re 4\partial_z f = \frac{2}{h}\Big(\frac{1}{\frac{1}{i}\{p,\overline{p}\}(\rho_+)} - \frac{1}{\frac{1}{i}\{p,\overline{p}\}(\rho_-)}\Big) + \mathcal{O}(1). \tag{7.3.4}$$

*Proof.* (7.3.1), (7.3.2) follow from the general formula for the differentiation of the inverse of an operator, here:

$$\partial_{\overline{z}}\mathcal{E} + \mathcal{E}(\partial_{\overline{z}}\mathcal{P})\mathcal{E} = 0.$$

Let $\Pi(z)$ be the spectral projection of $Q$: $L^2 \to \mathbf{C}e_0$. It is easy to see that the various $z$ and $\overline{z}$ derivatives of $e_{\rm wkb}$ and $\Pi(z)$ have at most temperate growth in $1/h$ and since $e_0$ is the normalization of $\Pi(z)e_{\rm wkb}$ we get the same fact for $e_0$ and hence for $e_0 - e_{\rm wkb}$.

This quantity is also exponentially small in $L^2$ and by elementary interpolation estimates for the successive derivatives in $z, \overline{z}$ we get the same conclusion for the higher $z$-gradients of $e_0 - e_{\mathrm{wkb}}$.
It follows that

$$f_+(z) = (e_0(z)|\partial_z e_0(z)) = (e_{\mathrm{wkb}}(z)|\partial_z e_{\mathrm{wkb}}(z)) + \mathcal{O}(e^{-\frac{1}{Ch}}),$$

and the various $z, \overline{z}$-derivatives of the remainder are also exponentially decaying.

Using that $e_{\mathrm{wkb}}$ behaves like a Gaussian, peaked at the point $x_+(\zeta)$, we can apply a variant of the method of stationary phase to get

$$(e_{\mathrm{wkb}}|\partial_z e_{\mathrm{wkb}}) = -\frac{i}{h}\overline{(\partial_z \phi_+)(x_+(z), z)} + \mathcal{O}(1), \tag{7.3.5}$$

where the remainder remains bounded after taking $z, \overline{z}$ derivatives.
Using that $\phi_+(x_+(z), z) = 0$, $(\phi_+)'_x(x_+(z), z) = \xi_+(z)$, we get after applying $\partial_z$ to the first of these relations, that

$$(\partial_z \phi_+)(x_+(z), z) = -\xi_+(z)\partial_z x_+(z).$$

On the other hand, if we apply $\partial_z$ and $\partial_{\overline{z}}$ to the equation, $p(x_+(z), \xi_+(z)) = z$ and use that $x_+$ and $\xi_+(z)$ are real valued we can show that

$$\partial_{\overline{z}} x_+ = \frac{p'_\xi}{\{p, \overline{p}\}}(\rho_+), \quad \partial_{\overline{z}} \xi_+ = \frac{-p'_x}{\{p, \overline{p}\}}(\rho_+).$$

Plugging this into (7.3.5), applying $\partial_z$ and taking real parts, we get the second (non-trivial) identity in (7.3.4) for the contribution from $f_+$. The one from $f_-$ can be treated similarly. ☐

Using the expressions for the $z$-derivatives of $x_+, \xi_+$ and the analogous ones for $x_-, \xi_-$, we have the following easy result relating (7.3.4) to the symplectic volume:

**Proposition 7.4.** *Writing $z = x + iy$, we have:*

$$d\xi_+(z) \wedge dx_+(z) = \frac{2}{\frac{1}{i}\{p, \overline{p}\}(\rho_+)} dy \wedge dx,$$

$$-d\xi_-(z) \wedge dx_-(z) = -\frac{2}{\frac{1}{i}\{p, \overline{p}\}(\rho_-)} dy \wedge dx,$$

*so by (7.3.4),*

$$\Re \Delta F(z) dy \wedge dx = \frac{1}{h}(d\xi_+ \wedge dx_+ - d\xi_- \wedge dx_-) + \mathcal{O}(1). \tag{7.3.6}$$

## 7.4. Adding the random perturbation

Let $X \sim \mathcal{N}_{\mathbf{C}}(0, \sigma^2)$ be a complex Gaussian random variable, meaning that $X$ has the probability distribution

$$X_*(P(d\omega)) = \frac{1}{\pi\sigma^2} e^{-\frac{|X|^2}{\sigma^2}} d(\Re X) d(\Im X). \tag{7.4.1}$$

Here $\sigma > 0$. For $t < 1/\sigma^2$, we have the expectation value

$$E(e^{t|X|^2}) = \frac{1}{1 - \sigma^2 t}. \tag{7.4.2}$$

Bordeaux Montrieux [10] observed that we have the following possibly classical result (improving a similar statement in [39]).

**Proposition 7.5.** *There exists $C_0 > 0$ such that the following holds: Let $X_j \sim \mathcal{N}_{\mathbf{C}}(0, \sigma_j^2)$, $1 \le j \le N < \infty$ be independent complex Gaussian random variables. Put $s_1 = \max \sigma_j^2$. Then for every $x > 0$, we have*

$$P(\sum_1^N |X_j|^2 \ge x) \le \exp(\frac{C_0}{2s_1} \sum_1^N \sigma_j^2 - \frac{x}{2s_1}).$$

*Proof.* For $t \le 1/(2s_1)$, we have

$$P(\sum |X_j|^2 \ge x) \le E(e^{t(\sum |X_j|^2 - x)}) = e^{-tx} \prod_1^N E(e^{t|X_j|^2})$$
$$= \exp(\sum_1^N \ln \frac{1}{1 - \sigma_j^2 t} - tx) \le \exp t(C_0 \sum \sigma_j^2 t - x).$$

It then suffices to take $t = (2s_1)^{-1}$. $\square$

Recall that

$$Q_\omega u(x) = \sum_{|k|,|j| \le C_1/h} \alpha_{j,k}(\omega)(u|e^k)e^j(x),\ e^k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}. \tag{7.4.3}$$

Since the Hilbert-Schmidt norm of $Q_\omega$ is given by $\|Q_\omega\|_{\mathrm{HS}}^2 = \sum |\alpha_{j,k}(\omega)|^2$, we get from the preceding proposition:

**Proposition 7.6.** *If $C > 0$ is large enough, then*

$$\|Q_\omega\|_{\mathrm{HS}} \le \frac{C}{h} \text{ with probability } \ge 1 - e^{-\frac{1}{Ch^2}}. \tag{7.4.4}$$

Now, we work under the assumption that $\|Q_\omega\|_{\mathrm{HS}} \le C/h$ and recall that $\|Q_\omega\| \le \|Q_\omega\|_{\mathrm{HS}}$. Assume that

$$\delta \ll h^{3/2}, \tag{7.4.5}$$

so that $\|\delta Q_\omega\| \ll h^{1/2}$. Then, by simple perturbation theory we see that

$$\mathcal{P}_\delta(z) = \begin{pmatrix} P_\delta - z & R_- \\ R_+ & 0 \end{pmatrix} : H^1 \times \mathbf{C} \to L^2 \times \mathbf{C}$$

is bijective with the bounded inverse

$$\mathcal{E}_\delta = \begin{pmatrix} E^\delta & E_+^\delta \\ E_-^\delta & E_{-+}^\delta \end{pmatrix}$$

$$\begin{aligned} E^\delta &= E + \mathcal{O}(\frac{\delta}{h}) = \mathcal{O}(h^{-1/2}) \text{ in } \mathcal{L}(L^2, L^2) \\ E_+^\delta &= E_+ + \mathcal{O}(\frac{\delta}{h^{3/2}}) = \mathcal{O}(1) \text{ in } \mathcal{L}(\mathbf{C}, L^2) \\ E_-^\delta &= E_- + \mathcal{O}(\frac{\delta}{h^{3/2}}) = \mathcal{O}(1) \text{ in } \mathcal{L}(L^2, \mathbf{C}) \\ E_{-+}^\delta &= E_{-+} - \delta E_- Q E_+ + \mathcal{O}(\frac{\delta^2}{h^{5/2}}). \end{aligned} \tag{7.4.6}$$

As before the eigenvalues of $P_\delta$ are the zeros of $E_{-+}^\delta$ and we have the d-bar equation

$$\partial_{\overline{z}} E_{-+}^\delta + f^\delta(z) E_{-+}^\delta = 0,$$

$$f^\delta(z) = \partial_{\overline{z}} R_+ E_+^\delta + E_-^\delta \partial_{\overline{z}} R_- = f(z) + \mathcal{O}(\frac{1}{h}\frac{\delta}{h^{3/2}}).$$

We can solve $\partial_{\overline{z}} F^\delta = f^\delta$ (making $e^{F^\delta} E_{-+}^\delta$ holomorphic) with

$$F^\delta = F + \mathcal{O}(\frac{\delta}{h^{5/2}}) = F + \mathcal{O}(\frac{\delta}{h^{3/2}})\frac{1}{h}. \tag{7.4.7}$$

**Proposition 7.7.** *Assume that* $0 < t \ll 1$, $\delta \ll h^{3/2}$,

$$\delta t \gg e^{-\frac{1}{C_0 h}}, \quad t \gg \frac{\delta}{h^{5/2}}, \tag{7.4.8}$$

*where* $C_0 \gg 1$ *is fixed. Then with probability* $\geq 1 - e^{-\frac{1}{Ch^2}}$, *we have*

$$|E_{-+}^\delta(z)| \leq e^{-\frac{1}{Ch}} + \frac{C\delta}{h}, \ \forall z \in \Omega. \tag{7.4.9}$$

*For every* $z \in \Omega$, *we have with probability* $\geq 1 - \mathcal{O}(t^2) - e^{-\frac{1}{Ch^2}}$, *that*

$$|E_{-+}^\delta(z)| \geq \frac{t\delta}{C}, \tag{7.4.10}$$

We only give the main idea of the proof which is to notice that $E_- Q_\omega E_+$ can be written as a sum of independent Gaussian random variables and is therefore itself a Gaussian random variable. Applying the standard formula for the variance of such a sum we get for the variance:

$$\sigma^2 = \sum_{|k|,|j| \leq \frac{C_1}{h}} |\widehat{e}_0(j)|^2 |\widehat{f}_0(k)|^2, \tag{7.4.11}$$

where $\widehat{e}_0(j)$, $\widehat{f}_0(j)$ are the Fourier coefficients of $e_0$, $f_0$. Now we can show that the Fourier coefficients are $\mathcal{O}((h/|j|)^N)$ for every $N \geq 0$, when $h|j|$ is sufficiently large, so if we take $C_1$ (in the definition of $Q_\omega$) large enough, we conclude that $\sigma^2 = 1 + \mathcal{O}(h^\infty)$.
The remainder of the proof then consists in showing that $|E_- Q_\omega E_+|$ is $\geq t$ with probability $\geq 1 - \mathcal{O}(t^2)$ and observing that when this happens, then the second term in the expression for $E_{-+}^\delta$ in (7.4.6) is dominant. □

**Proposition 7.8.** *Let* $\kappa > 5/2$ *and fix* $\epsilon_0 \in ]0,1[$ *sufficiently small. Let* $\delta = \delta(h)$ *satisfy* $e^{-\epsilon_0/h} \ll \delta \ll h^\kappa$, *and put* $\epsilon = \epsilon(h) = h \ln \frac{1}{\delta}$. *Then with probability* $\geq 1 - e^{-1/(Ch^2)}$ *we have* $|E_{-+}^\delta| \leq 1$ *for all* $z \in \Omega$.
*For any* $z \in \Omega$, *we have* $|E_{-+}^\delta| \geq e^{-C\epsilon/h}$ *with probability* $\geq 1 - \mathcal{O}(\delta^2/h^5)$.

This follows from Proposition 7.7 by choosing $t$ such that

$$\max(\frac{1}{\delta}e^{-\frac{1}{C_0 h}}, \frac{\delta}{h^{5/2}}, C\delta^{C-1}) \ll t \leq \mathcal{O}(\frac{\delta}{h^{5/2}}),$$

which is possible to do since

$$\frac{1}{\delta}e^{-\frac{1}{C_0 h}},\ C\delta^{C-1} \ll \frac{\delta}{h^{5/2}}.$$

Under the same assumptions, we also have

$$|F_\delta - F| \leq \mathcal{O}(\frac{\delta}{h^{3/2}})\frac{1}{h} \leq \mathcal{O}(\epsilon)\frac{1}{h}.$$

Thus for the holomorphic function $u(z) = e^{F_\delta(z)} E_{-+}^\delta(z)$ we have

- With probability $\geq 1 - e^{-1/(Ch^2)}$ we have $|u(z)| \leq \exp(\Re F(z) + C\epsilon/h)$ for all $z \in \Omega$.

- For every $z \in \Omega$, we have $|u(z)| \geq \exp(\Re F(z) - C\epsilon/h)$ with probability $\geq 1 - \mathcal{O}(\delta^2/h^5)$.

Theorem 7.1 on the Weyl asymptotics of small random perturbations of the operator $P = hD + g(x)$ is now a consequence of the following result of M. Hager, that we apply with $\phi = h\Re F$

**Proposition 7.9.** *Let $\Gamma \Subset \mathbf{C}$ have smooth boundary and let $\phi$ be a real valued $C^2$-function defined in a fixed neighborhood of $\overline{\Gamma}$. Let $z \mapsto u(z;h)$ be a family of holomorphic functions defined in a fixed neighborhood of $\overline{\Gamma}$, and let $0 < \epsilon = \epsilon(h) \ll 1$. Assume*

- $|u(z;h)| \leq \exp(\frac{1}{h}(\phi(z) + \epsilon))$ *for all $z$ in a fixed neighborhood of $\partial\Gamma$.*

- *There exist $z_1, ..., z_N$ depending on $h$, with $N = N(h) \asymp \epsilon^{-1/2}$ such that $\partial\Gamma \subset \cup_1^N D(z_k, \sqrt{\epsilon})$ such that $|u(z_k;h)| \geq \exp(\frac{1}{h}(\phi(z_k) - \epsilon))$, $1 \leq k \leq N(h)$.*

*Then, the number of zeros of $u$ in $\Gamma$ satisfies*

$$|\#(u^{-1}(0) \cap \Gamma) - \frac{1}{2\pi h} \int_\Gamma \Delta\phi(z) dxdy| \leq C\frac{\sqrt{\epsilon}}{h}.$$

This is essentially a special case of Theorem 6.3, but we outline the simple and direct proof of Hager in the next subsection.

## 7.5. Proof of Proposition 7.9, an outline

Define $\phi_j(z)$ by $i\phi_j(z) = \phi(z_j) + 2\partial_z\phi(z_j)(z - z_j)$. Then

$$\begin{aligned} \phi(z) &= \Re(i\phi_j(z)) + R_j(z),\ R_j(z) = \mathcal{O}((z - z_j)^2) \\ \phi_j'(z) &= \frac{2}{i}\partial_z\phi(z) + \mathcal{O}((z - z_j)). \end{aligned}$$

Consider the holomorphic function

$$v_j(z;h) = u(z;h)e^{-\frac{i}{h}\phi_j(z)}.$$

Then $|v_j(z;h)| \leq e^{\frac{1}{h}(\phi(z) - \Re i\phi_j(z))} = e^{\frac{1}{h}R_j} \leq e^{\frac{C\epsilon}{h}}$, when $z - z_j = \mathcal{O}(\sqrt{\epsilon})$, while

$$|v_j(z_j;h)| \geq e^{-\frac{C\epsilon}{h}}.$$

In a $\sqrt{\epsilon}$-neighborhood of $z_j$ we put $v = v_j$ and make the change of variables $w = (z - z_j)/\sqrt{\epsilon}$, $\widetilde{v}(w) = v(z)$, so that

$$|\widetilde{v}(w)| \leq e^{C\epsilon/h} \text{ on } D(0,2),\ |\widetilde{v}(0)| \geq e^{-C\epsilon/h}.$$

Using Jensen's formula we see that the number of zeros $w_1, ..., w_N$ of $\widetilde{v}$ in $D(0, 3/2)$ (repeated with their multiplicity) is $\mathcal{O}(\epsilon/h)$. Factorize:

$$\widetilde{v}(w) = e^{g(w)} \prod_1^N (w - w_k).$$

Using the maximum principle and a suitably chosen disc of radius between 4/3 and 3/2, and then also Harnack's inequality we can follow a standard procedure to show that

$$\Re g(w),\, g'(w) = \mathcal{O}(\epsilon/h) \text{ in } D(0,6/5).$$

Using finally that $\partial\Gamma$ is covered by the discs $D(z_j,\sqrt{\epsilon})$ and using the above representation of $u$ in each disc, we can show that the number of zeros of $u(\cdot;h)$ in $\Gamma$ is equal to

$$\begin{aligned}\Re\frac{1}{2\pi i}\int_{\partial\Gamma}\frac{u'(z)}{u(z)}dz &= \Re\frac{1}{2\pi h}\int_{\partial\Gamma}\frac{2}{i}\partial_z\phi(z)dz+\mathcal{O}(\frac{\sqrt{\epsilon}}{h})\\ &= \frac{1}{2\pi h}\int\Delta\phi(x)dxdy+\mathcal{O}(\frac{\sqrt{\epsilon}}{h}).\end{aligned} \qquad \square$$

# 8. The multi-dimensional semi-classical case

## 8.1. Introduction

In this section we consider general semi-classical operators with multiplicative random perturbations. We follow [97, 98] which make use of the work [39]. The use of Theorem 6.2 rather than the corresponding weaker result in [39] led us to improved remainder estimates in comparison with [98].

Let $X$ be a compact smooth manifold on which we choose a positive density of integration so that the scalar product on $L^2(X)$ is well-defined. On $X$ we consider an $h$-differential operator $P$ which in local coordinates takes the form,

$$P=\sum_{|\alpha|\le m}a_\alpha(x;h)(hD)^\alpha, \tag{8.1.1}$$

where we use standard multiindex notation and let $D=D_x=\frac{1}{i}\frac{\partial}{\partial x}$. We assume that the coefficients $a_\alpha$ are uniformly bounded in $C^\infty$ for $h\in]0,h_0]$, $0<h_0\ll 1$. (We will also discuss the case when we only have some Sobolev space control of $a_0(x)$.) Assume

$$\begin{aligned}&a_\alpha(x;h)=a_\alpha^0(x)+\mathcal{O}(h) \text{ in } C^\infty,\\ &a_\alpha(x;h)=a_\alpha(x) \text{ is independent of } h \text{ for } |\alpha|=m.\end{aligned} \tag{8.1.2}$$

Notice that this assumption is invariant under changes of local coordinates.

Also assume that $P$ is elliptic in the classical sense, uniformly with respect to $h$:

$$|p_m(x,\xi)|\ge\frac{1}{C}|\xi|^m, \tag{8.1.3}$$

for some positive constant $C$, where

$$p_m(x,\xi)=\sum_{|\alpha|=m}a_\alpha(x)\xi^\alpha \tag{8.1.4}$$

is invariantly defined as a function on $T^*X$. It follows that $p_m(T^*X)$ is a closed cone in $\mathbf{C}$ and we assume that

$$p_m(T^*X)\neq\mathbf{C}. \tag{8.1.5}$$

If $z_0 \in \mathbf{C} \setminus p_m(T^*X)$, we see that $\lambda z_0 \notin \Sigma(p)$ if $\lambda \geq 1$ is sufficiently large and fixed, where $\Sigma(p) := p(T^*X)$ and $p$ is the semiclassical principal symbol

$$p(x, \xi) = \sum_{|\alpha| \leq m} a^0_\alpha(x) \xi^\alpha. \tag{8.1.6}$$

Actually, (8.1.5) can be replaced by the weaker condition that $\Sigma(p) \neq \mathbf{C}$.

Standard elliptic theory and analytic Fredholm theory now show that if we consider $P$ as an unbounded operator: $L^2(X) \to L^2(X)$ with domain $\mathcal{D}(P) = H^m(X)$ (the Sobolev space of order $m$), then $P$ has purely discrete spectrum and each eigenvalue has finite algebraic multiplicity.

We will need the symmetry assumption

$$P^* = \Gamma P \Gamma, \tag{8.1.7}$$

where $P^*$ denotes the formal complex adjoint of $P$ in $L^2(X, dx)$, and $dx$ is the fixed smooth positive density of integration and $\Gamma$ is the antilinear operator of complex conjugation; $\Gamma u = \overline{u}$. Notice that this assumption implies that

$$p(x, -\xi) = p(x, \xi), \tag{8.1.8}$$

and conversely, if $p$ fulfills (8.1.8), then we get (8.1.7) if we replace $P$ by $\frac{1}{2}(P + \Gamma P^* \Gamma)$, which has the same semi-classical principal symbol $p$. Actually, (8.1.7) can be formulated more simply by saying that $P$ is symmetric for the bilinear form $\int_X u(x)v(x)dx$.

Let $V_z(t) := \mathrm{vol}\,(\{\rho \in T^*X;\ |p(\rho) - z|^2 \leq t\})$. For $\kappa \in ]0, 1]$, $z \in \mathbf{C}$, we consider the property that

$$V_z(t) = \mathcal{O}(t^\kappa),\ 0 \leq t \ll 1. \tag{8.1.9}$$

Since $r \mapsto p(x, r\xi)$ is a polynomial of degree $m$ in $r$ with non-vanishing leading coefficient, we see that (8.1.9) holds with $\kappa = 1/(2m)$.

The random potential will be of the form

$$q_\omega(x) = \sum_{0 < \mu_k \leq L} \alpha_k(\omega) \epsilon_k(x),\ |\alpha|_{\mathbf{C}^D} \leq R, \tag{8.1.10}$$

where $\epsilon_k$ is the orthonormal basis of eigenfunctions of $h^2 \widetilde{R}$, where $\widetilde{R}$ is an $h$-independent positive elliptic 2nd order operator on $X$ with smooth coefficients. Moreover, $h^2 \widetilde{R} \epsilon_k = \mu_k^2 \epsilon_k$, $\mu_k > 0$ and we may assume for simplicity that the $\mu_k$ form a (non-strictly) increasing sequence. We choose $L = L(h)$, $R = R(h)$ in the interval

$$h^{\frac{\kappa - 3n}{s - \frac{n}{2} - \epsilon}} \ll L \leq C h^{-M}, \qquad M \geq \frac{3n - \kappa}{s - \frac{n}{2} - \epsilon}, \tag{8.1.11}$$
$$\frac{1}{C} h^{-(\frac{n}{2} + \epsilon)M + \kappa - \frac{3n}{2}} \leq R \leq C h^{-\widetilde{M}}, \qquad \widetilde{M} \geq \frac{3n}{2} - \kappa + (\frac{n}{2} + \epsilon)M,$$

for some $\epsilon \in ]0, s - \frac{n}{2}[$, $s > \frac{n}{2}$, so by Weyl's law for the large eigenvalues of elliptic self-adjoint operators, the dimension $D$ is of the order of magnitude $(L/h)^n$. We introduce the small parameter $\delta = \tau_0 h^{N_1 + n}$, $0 < \tau_0 \leq \sqrt{h}$, where

$$N_1 := \widetilde{M} + sM + \frac{n}{2}. \tag{8.1.12}$$

The randomly perturbed operator is

$$P_\delta = P + \delta h^{N_1} q_\omega =: P + \delta Q_\omega. \tag{8.1.13}$$

The random variables $\alpha_j(\omega)$ will have a joint probability distribution

$$P(d\alpha) = C(h)e^{\Phi(\alpha;h)}L(d\alpha), \tag{8.1.14}$$

where for some $N_4 > 0$,

$$|\nabla_\alpha \Phi| = \mathcal{O}(h^{-N_4}), \tag{8.1.15}$$

$L(d\alpha)$ is the Lebesgue measure and we use the standard $\ell^2$ norm on $\mathbf{C}^D$. ($C(h)$ is the normalizing constant, assuring that the probability of $B_{\mathbf{C}^D}(0,R)$ is equal to 1.)

We also need the parameter

$$\epsilon_0(h) = (h^\kappa + h^n \ln \frac{1}{h})(\ln \frac{1}{\tau_0} + (\ln \frac{1}{h})^2) \tag{8.1.16}$$

and assume that $\tau_0 = \tau_0(h)$ is not too small, so that $\epsilon_0(h)$ is small. Let $\Omega \Subset \mathbf{C}$ be open, simply connected, not entirely contained in $\Sigma(p)$. The main result of this section is:

**Theorem 8.1.** *Under the assumptions above, let $\Gamma \Subset \Omega$ have smooth boundary, let $\kappa \in ]0,1]$ be the parameter in (8.1.10), (8.1.11), (8.1.16) and assume that (8.1.9) holds uniformly for $z$ in a neighborhood of $\partial\Gamma$. Then there exists a constant $C > 0$ such that for $C^{-1} \geq r > 0$, $\widetilde{\epsilon} \geq C\epsilon_0(h)$ we have with probability*

$$\geq 1 - \frac{C\epsilon_0(h)}{rh^{n+\max(n(M+1),N_4+\widetilde{M})}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} \tag{8.1.17}$$

*that:*

$$\begin{aligned} &|\#(\sigma(P_\delta)\cap\Gamma) - \frac{1}{(2\pi h)^n}\mathrm{vol}\,(p^{-1}(\Gamma))| \leq \\ &\frac{C}{h^n}\left(\frac{\widetilde{\epsilon}}{r} + \mathrm{vol}\,(p^{-1}(\partial\Gamma + D(0,r)))\right). \end{aligned} \tag{8.1.18}$$

*Here $\#(\sigma(P_\delta)\cap\Gamma)$ denotes the number of eigenvalues of $P_\delta$ in $\Gamma$, counted with their algebraic multiplicity.*

Actually, the theorem holds for the slightly more general operators, obtained by replacing $P$ by $P_0 = P + \delta_0(h^{\frac{n}{2}}q_1^0 + q_2^0)$, where $\|q_1^0\|_{H_h^s} \leq 1$, $\|q_2\|_{H^s} \leq 1$, $0 \leq \delta_0 \leq h$. Here, $H^s$ is the standard Sobolev space and $H_h^s$ is the same space with the natural semiclassical $h$-dependent norm. See Subsection 8.3. This allows us in principle to consider more general random perturbations and will be used in Section 9.

We also have a result valid simultaneously for a family $\mathcal{C}$ of domains $\Gamma \subset \Omega$ satisfying the assumptions of Theorem 8.1 uniformly in the natural sense: With a probability

$$\geq 1 - \frac{\mathcal{O}(1)\epsilon_0(h)}{r^2h^{n+\max(n(M+1),N_4+\widetilde{M})}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}}, \tag{8.1.19}$$

the estimate (8.1.18) holds simultaneously for all $\Gamma \in \mathcal{C}$.

**Remark 8.2.** If $\kappa > 1/2$, then $\mathrm{vol}\, p^{-1}(\partial\Gamma + D(0,r)) = \mathcal{O}(r^{2\kappa-1})$, where the exponent $2\kappa - 1$ is $> 0$. More generally, if

$$\mathrm{vol}\,(p^{-1}(\partial\Gamma + D(0,r))) = \mathcal{O}(r^\alpha),$$

for some $\alpha \in ]0,1]$, then we can choose $r = \widetilde{\epsilon}^{\frac{1}{\alpha+1}}$ and obtain that that the right hand side in (8.1.18) is $\mathcal{O}(1)h^{-n}\widetilde{\epsilon}^{\frac{\alpha}{\alpha+1}}$ showing that we have Weyl asymptotics. Notice here that if $z$ is not a critical value of $p$, in the sense that $d\Re p(\rho)$ and $\Im p(\rho)$ are independent whenever $p(\rho) = z$, then (8.1.9) holds with $\kappa = 1$.

In the proof we replace the zero counting proposition from [39] by the stronger Theorem 6.2 leading to an improved remainder estimate. It may be possible (though we have not yet checked the details) to replace the right hand side in (8.1.18) by

$$\frac{C}{h^n}\mathrm{vol}\,(p^{-1}(\partial\Gamma + D(0, h^{\frac{1}{2}-\epsilon}))),$$

for any fixed $\epsilon > 0$, and also to let $\Gamma$ be $h$-dependent of a suitable Lipschitz class as in section 6.

**Remark 8.3.** When $\widetilde{R}$ has real coefficients, we may assume that the eigenfunctions $\epsilon_j$ are real. Then (cf Remark 8.3 in [97]) we may restrict $\alpha$ in (8.1.10) to be in $\mathbf{R}^D$ so that $q_\omega$ is real, still with $|\alpha| \le R$, and change $C(h)$ in (8.1.14) so that $P$ becomes a probability measure on $B_{\mathbf{R}^D}(0, R)$. Then Theorem 8.1 remains valid.

**Remark 8.4.** The assumption (8.1.7) cannot be completely eliminated. Indeed, let $P = hD_x + g(x)$ on $\mathbf{T} = \mathbf{R}/(2\pi\mathbf{Z})$ where $g$ is smooth and complex valued. Then (cf Hager [37]) the spectrum of $P$ is contained in the line $\Im z = \int_0^{2\pi} \Im g(x)dx/(2\pi)$. This line will vary only very little under small multiplicative perturbations of $P$ so Theorem 8.1 cannot hold in this case. On the other hand, for other classes of perturbations, like the ones in Section 7 or in [39], the symmetry assumption can be dropped.

In the remainder of this section, we shall outline the proof of Theorem 8.1 following [98, 97].

## 8.2. Semiclassical Sobolev spaces and multiplication

We let $H_h^s(\mathbf{R}^n) \subset \mathcal{S}'(\mathbf{R}^n)$, $s \in \mathbf{R}$, denote the semiclassical Sobolev space of order $s$ equipped with the norm $\|\langle hD\rangle^s u\|$ where the norms are the ones in $L^2$, $\ell^2$ or the corresponding operator norms if nothing else is indicated. Here $\langle hD\rangle = (1 + (hD)^2)^{1/2}$. In [97] we recalled the following result:

**Proposition 8.5.** *Let $s > n/2$. Then there exists a constant $C = C(s)$ such that for all $u, v \in H_h^s(\mathbf{R}^n)$, we have $u \in L^\infty(\mathbf{R}^n)$, $uv \in H_h^s(\mathbf{R}^n)$ and*

$$\|u\|_{L^\infty} \le Ch^{-n/2}\|u\|_{H_h^s}, \tag{8.2.1}$$

$$\|uv\|_{H_h^s} \le Ch^{-n/2}\|u\|_{H_h^s}\|v\|_{H_h^s}. \tag{8.2.2}$$

We cover $X$ by finitely many coordinate neighborhoods $X_1, ..., X_p$ and for each $X_j$, we let $x_1, ..., x_n$ denote the corresponding local coordinates on $X_j$. Let $0 \le \chi_j \in C_0^\infty(X_j)$ have the property that $\sum_1^p \chi_j > 0$ on $X$. Define $H_h^s(X)$ to be the space of all $u \in \mathcal{D}'(X)$ such that

$$\|u\|_{H_h^s}^2 := \sum_1^p \|\chi_j \langle hD\rangle^s \chi_j u\|^2 < \infty. \tag{8.2.3}$$

It is standard to show that this definition does not depend on the choice of the coordinate neighborhoods or on $\chi_j$. With different choices of these quantities we get norms in (8.2.3) which are uniformly equivalent when $h \to 0$. In fact, this follows from the $h$-pseudodifferential calculus on manifolds with symbols in the Hörmander space $S^m_{1,0}$, that we quickly reviewed in the appendix in [97]. See also [98], Section 4. An equivalent definition of $H^s_h(X)$ is the following: Let

$$h^2\widetilde{R} = \sum (hD_{x_j})^* r_{j,k}(x) hD_{x_k} \tag{8.2.4}$$

be a non-negative elliptic operator with smooth coefficients on $X$, where the star indicates that we take the adjoint with respect to the fixed positive smooth density on $X$. Then $h^2\widetilde{R}$ is essentially self-adjoint with domain $H^2(X)$, so $(1+h^2\widetilde{R})^{s/2} : L^2 \to L^2$ is a closed densely defined operator for $s \in \mathbf{R}$, which is bounded precisely when $s \leq 0$. Standard methods allow to show that $(1+h^2\widetilde{R})^{s/2}$ is an $h$-pseudodifferential operator with symbol in $S^s_{1,0}$ and semiclassical principal symbol given by $(1+r(x,\xi))^{s/2}$, where $r(x,\xi) = \sum_{j,k} r_{j,k}(x)\xi_j\xi_k$ is the semiclassical principal symbol of $h^2\widetilde{R}$. See the appendix in [97]. The $h$-pseudodifferential calculus gives for every $s \in \mathbf{R}$:

**Proposition 8.6.** *$H^s_h(X)$ is the space of all $u \in \mathcal{D}'(X)$ such that $(1+h^2\widetilde{R})^{s/2}u \in L^2$ and the norm $\|u\|_{H^s_h}$ is equivalent to $\|(1+h^2\widetilde{R})^{s/2}u\|$, uniformly when $h \to 0$.*

**Remark 8.7.** From the first definition we see that Proposition 8.5 remains valid if we replace $\mathbf{R}^n$ by a compact $n$-dimensional manifold $X$.

Of course, $H^s_h(X)$ coincides with the standard Sobolev space $H^s(X)$ and the norms are equivalent for each fixed value of $h$, but not uniformly with respect to $h$. The following variant of Proposition 8.5 will be useful when studying the high energy limit in Section 9.

**Proposition 8.8.** *Let $s > n/2$. Then there exists a constant $C = C_s > 0$ such that*

$$\|uv\|_{H^s_h} \leq C\|u\|_{H^s}\|v\|_{H^s_h},\ \forall u \in H^s(\mathbf{R}^n),\ v \in H^s_h(\mathbf{R}^n). \tag{8.2.5}$$

*The result remains valid if we replace $\mathbf{R}^n$ by $X$.*

The proof is straight forward. We work in local coordinates and make a Fourier transform. Then we have to estimate convolutions in certain weighted $L^2$ spaces. See [98] for the details.

## 8.3. $H^s$-perturbations and eigenfunctions

Let $S^m(T^*X) = S^m_{1,0}(T^*X)$, $S^m(U \times \mathbf{R}^n) = S^m_{1,0}(U \times \mathbf{R}^n)$ denote the classical Hörmander symbol spaces, where $U \subset \mathbf{R}^n$ is open. The condition (8.1.5) implies that the closure of the image of $p$ is not equal to the whole complex plane and (as in [38, 39] we can find $\widetilde{p} \in S^m(T^*X)$ which is equal to $p$ outside any given fixed neighborhood of $p^{-1}(\overline{\Omega})$ such that $\widetilde{p} - z$ is non-vanishing, for any $z \in \overline{\Omega}$. Let $\widetilde{P} = P + \mathrm{Op}_h(\widetilde{p} - p)$, where $\mathrm{Op}_h(\widetilde{p} - p)$ denotes any reasonable quantization of $(\widetilde{p} - p)(x, h\xi)$. (See for instance the appendix in [97].) Then $\widetilde{P} - z : H^m_h(X) \to H^0_h(X)$ has a uniformly bounded inverse for $z \in \overline{\Omega}$ and $h > 0$ small enough. Now (see for instance [39, 97]) the eigenvalues of $P$ in $\Omega$, counted with their algebraic multiplicity, coincide with the zeros of the function $z \mapsto \det((\widetilde{P} - z)^{-1}(P - z)) = \det(1 - (\widetilde{P} - z)^{-1}(\widetilde{P} - P))$. Notice here that $(\widetilde{P} - z)^{-1}(\widetilde{P} - P)$ is of trace class so the determiant is well-defined ([33]).

Fix $s > n/2$ and consider the perturbed operator

$$P_\delta = P + \delta(h^{\frac{n}{2}}q_1 + q_2) = P + \delta(Q_1 + Q_2) = P + \delta Q, \tag{8.3.1}$$

where $q_j \in H^s(X)$,

$$\|q_1\|_{H_h^s} \le 1,\ \|q_2\|_{H^s} \le 1,\ 0 \le \delta \ll 1. \tag{8.3.2}$$

According to Propositions 8.5, 8.8, $Q = \mathcal{O}(1) : H_h^s \to H_h^s$ and hence by duality and interpolation,

$$Q = \mathcal{O}(1) : H_h^\sigma \to H_h^\sigma,\ -s \le \sigma \le s. \tag{8.3.3}$$

Again, the spectrum of $P_\delta$ in $\Omega$ is discrete and coincides with the set of zeros of

$$\det((\widetilde{P}_\delta - z)^{-1}(P_\delta - z)) = \det(1 - (\widetilde{P}_\delta - z)^{-1}(\widetilde{P} - P)), \tag{8.3.4}$$

where $\widetilde{P}_\delta := \widetilde{P} + \delta Q$. Here $(\widetilde{P} - z)^{-1} = \mathcal{O}(1) : H_h^\sigma \to H_h^\sigma$ for $\sigma$ in the same range and by an easy perturbation argument, we get the same conclusion for $(\widetilde{P}_\delta - z)^{-1}$.

Put

$$P_{\delta,z} := (\widetilde{P}_\delta - z)^{-1}(P_\delta - z) = 1 - (\widetilde{P}_\delta - z)^{-1}(\widetilde{P} - P) =: 1 - K_{\delta,z}, \tag{8.3.5}$$

$$S_{\delta,z} := P_{\delta,z}^* P_{\delta,z} = 1 - (K_{\delta,z} + K_{\delta,z}^* - K_{\delta,z}^* K_{\delta,z}) =: 1 - L_{\delta,z}. \tag{8.3.6}$$

Clearly,

$$K_{\delta,z}, L_{\delta,z} = \mathcal{O}(1) : H_h^{-s} \to H_h^s. \tag{8.3.7}$$

For $0 \le \alpha \le 1/2$, let $\pi_\alpha = 1_{[0,\alpha]}(S_{\delta,z})$. Then using some simple functional calculus we showed in [97], that

$$\pi_\alpha = \mathcal{O}(1) : H_h^{-s} \to H_h^s. \tag{8.3.8}$$

We have the corresponding result for $P_\delta - z$. Let

$$S_\delta = (P_\delta - z)^*(P_\delta - z) \tag{8.3.9}$$

be defined as the Friedrichs extension from $C^\infty(X)$ with quadratic form domain $H_h^m(X)$. For $0 \le \alpha \le \mathcal{O}(1)$, we now put $\pi_\alpha = 1_{[0,\alpha]}(S_\delta)$. Then as in [97], we see that this new spectral projection also fulfils (8.3.8), for $0 \le \alpha \ll 1$.

## 8.4. Some functional and pseudodifferential calculus

Let $P$ be of the form (8.1.1) and let $p$ in (8.1.6) be the corresponding semi-classical principal symbol. Assume classical ellipticity as in (8.1.3) and let $z \in \mathbf{C}$ be fixed throughout this subsection. Let

$$S = (P - z)^*(P - z), \tag{8.4.1}$$

viewed as the self-adjoint Friedrichs extension from $C^\infty$. Later on we will also consider a different choice of $S$, namely

$$S = P_z^* P_z,\ \text{where } P_z = (\widetilde{P} - z)^{-1}(P - z) \tag{8.4.2}$$

and $\widetilde{P}$ is defined prior to (8.3.1). The main goal is to make a trace class study of $\chi(\frac{1}{\alpha}S)$ when $0 < h \le \alpha \ll 1$, $\chi \in C_0^\infty(\mathbf{R})$. With the second choice of $S$, we shall also study $\ln\det(S + \alpha\chi(\frac{1}{\alpha}S))$, when $\chi \ge 0$, $\chi(0) > 0$. The main step will be to get enough information about the resolvent $(w - \frac{1}{\alpha}S)^{-1}$ for $w = \mathcal{O}(1)$, $\Im w \neq 0$ and then apply the Cauchy-Riemann-Green-Stokes formula

$$\chi(\frac{1}{\alpha}S) = -\frac{1}{\pi}\int \frac{\partial\widetilde{\chi}(w)}{\partial\overline{w}}(w - \frac{1}{\alpha}S)^{-1}L(dw), \tag{8.4.3}$$

where $\widetilde{\chi} \in C_0^\infty(\mathbf{C})$ is an almost holomorphic extension of $\chi$, so that

$$\frac{\partial\widetilde{\chi}}{\partial\overline{w}} = \mathcal{O}(|\Im w|^\infty). \tag{8.4.4}$$

Thanks to (8.4.4) we can work in symbol classes with some temparate but otherwise unspecified growth in $1/|\Im w|$.

Let
$$s = |p - z|^2 \tag{8.4.5}$$
be the semiclassical principal symbol of $S$ in (8.4.1). A basic weight function in our calculus will be
$$\Lambda := \left(\frac{\alpha + s}{1 + s}\right)^{\frac{1}{2}}, \tag{8.4.6}$$
satisfying $\sqrt{\alpha} \le \Lambda \le 1$.

As a preparation and motivation for the calculus, we first consider symbol properties of $1 + \frac{s}{\alpha}$ and its powers.

**Proposition 8.9.** *For every choice of local coordinates $x$ on $X$, let $(x, \xi)$ denote the corresponding canonical coordinates on $T^*X$. Then for all $\ell \in \mathbf{R}$, $\widetilde{\alpha}, \beta \in \mathbf{N}^n$, we have uniformly in $\xi$ and locally uniformly in $x$:*
$$\partial_x^{\widetilde{\alpha}} \partial_\xi^\beta (1 + \frac{s}{\alpha})^\ell = \mathcal{O}(1)(1 + \frac{s}{\alpha})^\ell \Lambda^{-|\widetilde{\alpha}| - |\beta|} \langle \xi \rangle^{-|\beta|}. \tag{8.4.7}$$

The proof ([98]) is straight forward and the same can be said about the proof of

**Proposition 8.10.** *([98]) Let $w$ vary in some bounded subset of $\mathbf{C}$. For all $\ell \in \mathbf{R}$, $\widetilde{\alpha}, \beta \in \mathbf{N}^n$, there exists $J \in \mathbf{N}$, such that*
$$\partial_x^{\widetilde{\alpha}} \partial_\xi^\beta (w - \frac{s}{\alpha})^\ell = \mathcal{O}(1)(1 + \frac{s}{\alpha})^\ell \Lambda^{-|\widetilde{\alpha}| - |\beta|} \langle \xi \rangle^{-|\beta|} |\Im w|^{-J}, \tag{8.4.8}$$
*uniformly in $\xi$ and locally uniformly in $x$.*

We now define our new symbol spaces.

**Definition 8.11.** *Let $\widetilde{m}(x, \xi)$ be a weight function of the form $\widetilde{m}(x, \xi) = \langle \xi \rangle^k \Lambda^\ell$. We say that the family $a = a_w \in C^\infty(T^*X)$, $w \in D(0, C)$, belongs to $S_\Lambda(\widetilde{m})$ if for all $\widetilde{\alpha}, \beta \in \mathbf{N}^n$ there exists $J \in \mathbf{N}$ such that*
$$\partial_x^{\widetilde{\alpha}} \partial_\xi^\beta a = \mathcal{O}(1) \widetilde{m}(x, \xi) \Lambda^{-|\widetilde{\alpha}| - |\beta|} \langle \xi \rangle^{-|\beta|} |\Im w|^{-J}. \tag{8.4.9}$$

Here, as in Proposition 8.10, it is understood that the estimate is expressed in canonical coordinates and is locally uniform in $x$ and uniform in $\xi$. Notice that the set of estimates (8.4.9) is invariant under changes of local coordinates in $X$.

Let $U \subset X$ be a coordinate neighborhood that we shall view as a subset of $\mathbf{R}^n$ in the natural way. Let $a \in S_\Lambda(T^*U, \widetilde{m})$ be a symbol as in Definition 8.11 so that (8.4.9) holds uniformly in $\xi$ and locally uniformly in $x$. For fixed values of $\alpha$, $w$ the symbol $a$ belongs to $S^k_{1,0}(T^*U)$, so the classical $h$-quantization
$$Au = \mathrm{Op}_h(a)u(x) = \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}(x-y)\cdot\eta} a(x, \eta; h) u(y) dy d\eta \tag{8.4.10}$$
is a well-defined operator $C_0^\infty(U) \to C^\infty(U)$, $\mathcal{E}'(U) \to \mathcal{D}'(U)$. In order to develop our rudimentary calculus on $X$ we need a pseudolocal property for the distribution kernel $K_A(x, y)$, whose proof is also routine (see [98]).

**Proposition 8.12.** *For all $\widetilde{\alpha}, \beta \in \mathbf{N}^n$, $N \in \mathbf{N}$, there exists $M \in \mathbf{N}$ such that*
$$\partial_x^{\widetilde{\alpha}} \partial_y^\beta K_A(x, y) = \mathcal{O}(h^N |\Im w|^{-M}), \tag{8.4.11}$$
*locally uniformly on $U \times U \setminus \mathrm{diag}(U \times U)$.*

This means that if $\phi, \psi \in C_0^\infty(U)$ have disjoint supports, then for every $N \in \mathbf{N}$, there exists $M \in \mathbf{N}$ such that $\phi A \psi : H^{-N}(\mathbf{R}^n) \to H^N(\mathbf{R}^n)$ with norm $\mathcal{O}(h^N |\Im w|^{-M})$, and this leads to a simple way of introducing pseudodifferential operators on $X$: Let $U_1, ..., U_s$ be coordinate neighborhoods that cover $X$. Let $\chi_j \in C_0^\infty(U_j)$ form a partition of unity and let $\widetilde{\chi}_j \in C_0^\infty(U_j)$ satisfy $\chi_j \prec \widetilde{\chi}_j$ in the sense that $\widetilde{\chi}_j$ is equal to 1 near supp $(\chi_j)$. Let $a = (a_1, ..., a_s)$, where $a_j \in S_\Lambda(\widetilde{m})$. Then we quantize $a$ by the formula:

$$A = \sum_1^s \widetilde{\chi}_j \circ \mathrm{Op}_h(a_j) \circ \chi_j. \tag{8.4.12}$$

This is not an invariant quantization procedure but it will suffice for our purposes.

Using integration by parts and stationary phase we can study the composition to the left with non-exotic pseudodifferential operators and we obtain the following result for a coordinate neighborhood:

**Proposition 8.13.** *([98]). Let $A = \mathrm{Op}_h(a)$, $a \in S_{1,0}(m_1)$, $B = \mathrm{Op}_h(b)$, $b \in S_\Lambda(m_2)$ and assume that $b$ has uniformly compact support in $x$. Then $A \circ B = \mathrm{Op}_h(c)$, where $c$ belongs to $S_\Lambda(m_1 m_2)$ and has the asymptotic expansion*

$$c \sim \sum \frac{h^{|\beta|}}{\beta!} \partial_\xi^\beta a(x,\xi) D_x^\beta b(x,\xi),$$

*in the sense that for every $N \in \mathbf{N}$,*

$$c = \sum_{|\beta|<N} \frac{h^{|\beta|}}{\beta!} \partial_\xi^\beta a(x,\xi) D_x^\beta b(x,\xi) + r_N(x,\xi;h),$$

*where $r_N \in S_\Lambda(\frac{m_1 m_2}{(\Lambda\langle\xi\rangle)^N} h^N)$.*

We have a parametrix construction for $w - \frac{1}{\alpha}S$, still with $S$ as in (8.4.1). Let us first work in a coordinate neighborhood $U$, viewed as an open set in $\mathbf{R}^n$. Then for every $N \in \mathbf{N}$ we can construct a symbol

$$E_N \equiv \frac{1}{w - \frac{s}{\alpha}} \mod S_\Lambda\left(\frac{\alpha}{\Lambda^2\langle\xi\rangle^{2m}} \frac{h}{\Lambda^2\langle\xi\rangle}\right), \tag{8.4.13}$$

such that on the symbol level

$$(w - \frac{1}{\alpha}S)\#E_N = 1 + r_N,\ \ r_N \in S_\Lambda\left(\left(\frac{h}{\Lambda^2\langle\xi\rangle}\right)^{N+1}\right), \tag{8.4.14}$$

$$E_N \text{ is a holomorphic function of } w, \text{ for } |\xi| \geq C, \tag{8.4.15}$$

where $C$ is independent of $N$.

Now we return to the manifold situation and denote by $E_N^{(j)}, r_N^{(j)}$ the corresponding symbols on $T^*U_j$, constructed above. Denote the operators by the same symbols, and put on the operator level:

$$E_N = \sum_{j=1}^s \widetilde{\chi}_j E_N^{(j)} \chi_j, \tag{8.4.16}$$

with $\chi_j$, $\widetilde{\chi_j}$ as in (8.4.12). Then

$$\begin{aligned}(w-\frac{1}{\alpha}S)E_{N-1} &= 1-\sum_{j=1}^{s}\frac{1}{\alpha}[S,\widetilde{\chi}_j]E_{N-1}^{(j)}\chi_j+\sum_{j=1}^{s}\widetilde{\chi}_j r_N^{(j)}\chi_j \qquad (8.4.17)\\ &=: 1+R_N^{(1)}+R_N^{(2)}\\ &=: 1+R_N.\end{aligned}$$

Proposition 8.12 implies that for every $\widetilde{N}$, there exists an $\widetilde{M}$ such that the trace class norm of $R_N^{(1)}$ satisfies

$$\|R_N^{(1)}\|_{\rm tr}\le \mathcal{O}(h^{\widetilde{N}}|\Im w|^{-\widetilde{M}}). \qquad (8.4.18)$$

As for the trace class norm of $R_N^{(2)}$, we can combine standard facts about such norms for pseudodifferential operators and scaling to get

$$\|R_N\|_{\rm tr}\le Ch^{-n}|\Im w|^{-M(N)}\iint\left(\frac{h}{\Lambda^2\langle\xi\rangle}\right)^N dxd\xi. \qquad (8.4.19)$$

The contribution to this expression from the region where $\Lambda\ge 1/C$ is $\mathcal{O}(h^{N-n})|\Im w|^{-M(N)}$.

The volume growth assumption (8.1.9), that we now assume for our fixed $z$, says that

$$V(t):=\mathrm{vol}\,(\{\rho\in T^*X;\ s\le t\})=\mathcal{O}(t^\kappa),\ 0\le t\ll 1, \qquad (8.4.20)$$

for $0<\kappa\le 1$. Using this and (8.4.19) one can show that

$$\|R_N\|_{\rm tr}\le \mathcal{O}(1)h^{-n}\alpha^\kappa\left(\frac{h}{\alpha}\right)^N|\Im w|^{-M(N)}. \qquad (8.4.21)$$

From (8.4.17), we get

$$(w-\frac{1}{\alpha}S)^{-1}=E_{N-1}-(w-\frac{1}{\alpha}S)^{-1}R_N.$$

Write

$$E_{N-1}=\frac{1}{w-\frac{s}{\alpha}}+F_{N-1},\quad F_{N-1}\in S_\Lambda(\frac{\alpha h}{\Lambda^4\langle\xi\rangle^{2m+1}}).$$

More precisely we do this for each $E_{N-1}^{(j)}$ in (8.4.16). Then quantize and plug this into (8.4.3):

$$\begin{aligned}\chi(\frac{1}{\alpha}S) &= -\frac{1}{\pi}\int\frac{\partial\widetilde{\chi}}{\partial\overline{w}}\mathrm{Op}_h(\frac{1}{w-\frac{s}{\alpha}})L(dw)-\frac{1}{\pi}\int\frac{\partial\widetilde{\chi}}{\partial\overline{w}}F_{N-1}L(dw) \qquad (8.4.22)\\ &\quad -\frac{1}{\pi}\int\frac{\partial\widetilde{\chi}}{\partial\overline{w}}(w-\frac{1}{\alpha}S)^{-1}R_N L(dw)=:\mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

Here by definition,

$$\mathrm{Op}_h\left(\frac{1}{w-\frac{s}{\alpha}}\right)=\sum_{j=1}^{s}\widetilde{\chi}_j\mathrm{Op}_h\left(\frac{1}{w-\frac{s}{\alpha}}\right)\chi_j$$

with the coordinate dependent quantization appearing to the right.

After some further estimates we get

$$\mathrm{tr}\,(\mathrm{I})=\frac{1}{(2\pi h)^n}\iint\chi(\frac{s(x,\xi)}{\alpha})dxd\xi. \qquad (8.4.23)$$

As at the last estimate in the proof of Proposition 4.4 in [39] we see that this quantity is $\mathcal{O}(\alpha^\kappa h^{-n})$ and more generally,

$$\|\mathrm{I}\|_{\mathrm{tr}} = \mathcal{O}(\alpha^\kappa h^{-n}).$$

For II, one can show, using the fact that the symbol is holomorphic in $w$ for large $\xi$, that

$$\|\mathrm{II}\|_{\mathrm{tr}} = \mathcal{O}(1)\frac{\alpha^\kappa}{h^n}\frac{h}{\alpha}.$$

It is also clear that

$$\|\mathrm{III}\|_{tr} = \mathcal{O}(1)\frac{\alpha^\kappa}{h^n}\left(\frac{h}{\alpha}\right)^N.$$

Summing up our estimates, we get the following result:

**Proposition 8.14.** *Let $\chi \in C_0^\infty(\mathbf{R})$. For $0 < h \leq \alpha < 1$, we have*

$$\|\chi(\frac{1}{\alpha}S)\|_{\mathrm{tr}} = \mathcal{O}(1)\frac{\alpha^\kappa}{h^n}, \tag{8.4.24}$$

$$\mathrm{tr}\,\chi(\frac{1}{\alpha}S) = \frac{1}{(2\pi h)^n}\iint \chi(\frac{s(x,\xi)}{\alpha})dxd\xi + \mathcal{O}(\frac{\alpha^\kappa}{h^n}\frac{h}{\alpha}). \tag{8.4.25}$$

**Remark 8.15.** *Using simple $h$-pseudodifferential calculus (for instance as in the appendix of* [97], *we see that if we redefine $S$ as in (8.4.2), then in each local coordinate chart, $S = \mathrm{Op}_h(S)$, where $S \equiv s \bmod S_{1,0}(h\langle\xi\rangle^{-1})$ and $s$ is now redefined as*

$$s(x,\xi) = \left(\frac{|p(x,\xi)-z|}{|\widetilde{p}(x,\xi)-z|}\right)^2. \tag{8.4.26}$$

*The discussion goes through without any changes (now with $m = 0$) and we still have Proposition 8.14 with the new choice of $S$, $s$.*

*In both cases it follows from (8.4.24) that the number $N(\alpha)$ of eigenvalues of $S$ in the interval $[0,\alpha]$ satisfies*

$$\mathcal{O}(\alpha^\kappa/h^n). \tag{8.4.27}$$

In the remainder of this subsection, we choose $S$, $s$ as in (8.4.2), (8.4.26). In this case we notice that $S$ is a trace class perturbation of the identity, whose symbol is $1 + \mathcal{O}(h^\infty/\langle\xi\rangle^\infty)$ and similarly for all its derivatives, in a region $|\xi| \geq \mathrm{Const.}$

Let $0 \leq \chi \in C_0^\infty([0,\infty[)$ with $\chi(0) > 0$ and let $\alpha_0 > 0$ be small and fixed. Using standard pseudodifferential calculus in the spirit of [77], we get

$$\ln\det(S + \alpha_0\chi(\frac{1}{\alpha_0}S)) = \frac{1}{(2\pi h)^n}(\iint \ln(s + \alpha_0\chi(\frac{1}{\alpha_0}s))dxd\xi + \mathcal{O}(h)). \tag{8.4.28}$$

Extend $\chi$ to be an element of $C_0^\infty(\mathbf{R};\mathbf{C})$ in such a way that $t + \chi(t) \neq 0$ for all $t \in \mathbf{R}$. As in [39], we use that

$$\frac{d}{dt}\ln(E + t\chi(\frac{E}{t})) = \frac{1}{t}\psi(\frac{E}{t}), \tag{8.4.29}$$

where

$$\psi(E) = \frac{\chi(E) - E\chi'(E)}{E + \chi(E)}, \tag{8.4.30}$$

so that $\psi \in C_0^\infty(\mathbf{R})$. By standard functional calculus for self-adjoint operators, we have

$$\frac{d}{dt}\ln\det(S+t\chi(\frac{S}{t})) = \mathrm{tr}\,\frac{1}{t}\psi(\frac{S}{t}). \tag{8.4.31}$$

Using (8.4.25), we then get for $t \geq \alpha \geq h > 0$:

$$\frac{d}{dt}\ln\det(S+t\chi(\frac{1}{t}S)) = \frac{1}{(2\pi h)^n}(\iint \frac{1}{t}\psi(\frac{s}{t})dxd\xi + \mathcal{O}(ht^{\kappa-2})).$$

Integrating this from $t=\alpha_0$ to $t=\alpha$ and using (8.4.28), (8.4.29), leads to

**Proposition 8.16.** *If* $0 \leq \chi \in C_0^\infty([0,\infty[)$, $\chi(0) > 0$, *we have uniformly for* $0 < h \leq \alpha \ll 1$

$$\ln\det(S+\alpha\chi(\frac{1}{\alpha}S)) = \frac{1}{(2\pi h)^n}(\iint \ln s(x,\xi)dxd\xi + \mathcal{O}(\alpha^\kappa \ln\alpha)). \tag{8.4.32}$$

*Here the remainder term can be replaced by* $\mathcal{O}(\alpha^\kappa)$ *when* $\kappa < 1$ *and by* $\mathcal{O}(\alpha + h\ln\alpha)$ *when* $\kappa = 1$.

Notice that (8.4.32) implies the upper bound,

$$\ln\det P_z^* P_z \leq \frac{1}{(2\pi h)^n}(\iint \ln(s)dxd\xi + \mathcal{O}(\alpha^\kappa \ln\frac{1}{\alpha})). \tag{8.4.33}$$

We next consider $P_{\delta,z} = (\widetilde{P}_\delta - z)^{-1}(P_\delta - z) = 1 - K_{\delta,z}$ with $P_\delta = P + \delta Q$, $\widetilde{P}_\delta = \widetilde{P} + \delta Q$ as in Subsection 8.3 and under the assumptions (8.3.4), (8.3.6). Put

$$S_{\delta,z} = P_{\delta,z}^* P_{\delta,z} = 1 - K_{\delta,z} - K_{\delta,z}^* + K_{\delta,z}^* K_{\delta,z},$$

where $K_{\delta,z}$ is given by (8.3.6), so that

$$\|K_{\delta,z}\| \leq \mathcal{O}(1),\ \|K_{\delta,z}\|_{\mathrm{tr}} \leq \|(\widetilde{P}_\delta - z)^{-1}\| \|\widetilde{P} - P\|_{\mathrm{tr}} \leq \mathcal{O}(h^{-n}).$$

Here $\|\cdot\|_{\mathrm{tr}}$ denotes the trace class norm, and we refer for instance to [27] for the standard estimate on the trace class norm of an $h$-pseudodifferential operator with compactly supported symbol, that we used for the last estimate.

Write $\dot{K}_{\delta,z} = \frac{\partial}{\partial\delta}K_{\delta,z}$. Then

$$\dot{K}_{\delta,z} = -(z - \widetilde{P}_\delta)^{-1}Q(z - \widetilde{P}_\delta)^{-1}(\widetilde{P} - P),$$

so

$$\|\dot{K}_{\delta,z}\| \leq \mathcal{O}(\|Q\|), \quad \|\dot{K}_{\delta,z}\|_{\mathrm{tr}} \leq \mathcal{O}(\|Q\| h^{-n}).$$

It follows that

$$\|\dot{S}_{\delta,z}\| \leq \mathcal{O}(\|Q\|), \quad \|\dot{S}_{\delta,z}\|_{\mathrm{tr}} \leq \mathcal{O}(\|Q\| h^{-n}).$$

Let $N = N(\alpha,\delta)$ denote the number of singular values of $P_{\delta,z}$ in the interval $[0,\sqrt{\alpha}[$ for $h \ll \alpha \ll 1$. Assume

$$\delta \leq \mathcal{O}(h). \tag{8.4.34}$$

Then $\|S_{\delta,z} - S_{0,z}\| \leq \mathcal{O}(h)$ and from (8.4.27) we get

$$N(\alpha,\delta) = \mathcal{O}(\alpha^\kappa h^{-n}). \tag{8.4.35}$$

Let $1_\alpha(t) = \max(\alpha, t)$. For $0 < \epsilon \ll 1$, let $C^\infty(\overline{\mathbf{R}}_+) \ni 1_{\alpha,\epsilon} \geq 1_\alpha$ be equal to $t$ outside a small neighborhood of $t=0$ and converge to $1_\alpha$ uniformly when $\epsilon \to 0$. For any fixed $\epsilon > 0$, we put $f(t) = 1_{\alpha,\epsilon}(t)$ for $t \geq 0$ and extend $f$ to $\mathbf{R}$ in such a way that $f(t) = t + g(t)$, $g \in C_0^\infty(\mathbf{R})$. Let $\widetilde{f}(t) = t + \widetilde{g}(t)$ be an almost holomorphic extension of $f$ with $\widetilde{g} \in C_0^\infty(\mathbf{C})$. Then we have:

$$f(S_{\delta,z}) = S_\delta - \frac{1}{\pi}\int (w - S_{\delta,z})^{-1}\overline{\partial}\widetilde{g}(w)L(dw).$$

Differentiating with respect to $\delta$ one can show the identity

$$\frac{\partial}{\partial\delta}\ln\det f(S_{\delta,z}) = \mathrm{tr}\,(f(S_{\delta,z})^{-1}f'(S_{\delta,z})\dot{S}_{\delta,z}).$$

Now we can choose $f = 1_{\alpha,\epsilon}$ such that $|f'(t)| \le 1$ for $t \ge 0$. Then we get the estimate

$$\begin{aligned}\frac{\partial}{\partial\delta}\ln\det(1_{\alpha,\epsilon}(S_{\delta,z})) &= \mathrm{tr}\,(1_{\alpha,\epsilon}(S_{\delta,z})^{-1}1'_{\alpha,\epsilon}(S_{\delta,z})\dot{S}_{\delta,z}) \\ &= \mathcal{O}(\frac{\|\dot{S}_{\delta,z}\|_{\mathrm{tr}}}{\alpha}) \\ &= \mathcal{O}(1)\frac{\|Q\|}{\alpha h^n}.\end{aligned}$$

Since $\ln\det 1_\alpha(S_{\delta,z}) = \lim_{\epsilon\to 0}\ln\det 1_{\alpha,\epsilon}(S_{\delta,z})$, we can integrate the above estimate, pass to the limit and obtain

$$\ln\det 1_\alpha(S_{\delta,z}) = \ln\det 1_\alpha(S_{0,z}) + \mathcal{O}(\frac{\delta\|Q\|}{\alpha h^n}).$$

With some more work, we also get

$$\ln\det 1_\alpha(S_{\delta,z}) = \frac{1}{(2\pi h)^n}(\iint \ln s(x,\xi)dxd\xi + \mathcal{O}(\alpha^\kappa\ln\alpha) + \mathcal{O}(\frac{\delta\|Q\|}{\alpha})). \qquad (8.4.36)$$

## 8.5. Grushin problems

Let $P : \mathcal{H} \to \mathcal{H}$ be a bounded operator, where $\mathcal{H}$ is a complex separable Hilbert space. Following the standard definitions (see [33]) we define the singular values of $P$ to be the decreasing sequence $s_1(P) \ge s_2(P) \ge \ldots$ of eigenvalues of the self-adjoint operator $(P^*P)^{1/2}$ as long as these eigenvalues lie above the supremum of the essential spectrum. If there are only finitely many such eigenvalues, $s_1(P), \ldots, s_k(P)$ then we define $s_{k+1}(P) = s_{k+2}(P) = \ldots$ to be the supremum of the essential spectrum of $(P^*P)^{1/2}$. When $\dim\mathcal{H} = M < \infty$ our sequence is finite (by definition); $s_1 \ge s_2 \ge \ldots \ge s_M$, otherwise it is infinite. Using that if $P^*Pu = s_j^2u$, then $PP^*(Pu) = s_j^2Pu$ and similarly with $P$ and $P^*$ permuted, we see that $s_j(P^*) = s_j(P)$. Strictly speaking, $P^*P : \mathcal{N}(P)^\perp \to \mathcal{N}(P)^\perp$ and $PP^* : \mathcal{N}(P^*)^\perp \to \mathcal{N}(P^*)^\perp$ are unitarily equivalent via the map $P(P^*P)^{-1/2} : \mathcal{N}(P)^\perp \to \mathcal{N}(P^*)^\perp$ and its inverse $P^*(PP^*)^{-1/2} : \mathcal{N}(P^*)^\perp \to \mathcal{N}(P)^\perp$. (To check this, notice that the relation $P(P^*P) = (PP^*)P$ on $\mathcal{N}(P)^\perp$ implies $P(P^*P)^\alpha = (PP^*)^\alpha P$ on the same space for every $\alpha \in \mathbf{R}$.)

In the case when $P$ is a Fredholm operator of index 0, it will be convenient to introduce the increasing sequence $0 \le t_1(P) \le t_2(P) \le \ldots$ consisting first of all eigenvalues of $(P^*P)^{1/2}$ below the infimum of the essential spectrum and then, if there are only finitely many such eigenvalues, we repeat indefinitely that infimum. (The length of the resulting sequence is the dimension of $\mathcal{H}$.) When $\dim\mathcal{H} = M <$

$\infty$, we have $t_j(P) = s_{M+1-j}(P)$. Again, we have $t_j(P^*) = t_j(P)$ (as reviewed in [39]). Moreover, in the case when $P$ has a bounded inverse, we see that

$$s_j(P^{-1}) = \frac{1}{t_j(P)}. \tag{8.5.1}$$

Let $P$ be a Fredholm operator of index 0. Let $1 \le N < \infty$ and let $R_+ : \mathcal{H} \to \mathbf{C}^N$, $R_- : \mathbf{C}^N \to \mathcal{H}$ be bounded operators. Assume that

$$\mathcal{P} = \begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H} \times \mathbf{C}^N \to \mathcal{H} \times \mathbf{C}^N \tag{8.5.2}$$

is bijective with a bounded inverse

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} \tag{8.5.3}$$

Recall (for instance from [103]) that $P$ has a bounded inverse precisely when $E_{-+}$ has, and when this happens we have the relations,

$$P^{-1} = E - E_+ E_{-+}^{-1} E_-, \quad E_{-+}^{-1} = -R_+ P^{-1} R_-. \tag{8.5.4}$$

Recall ([33]) that if $A, B$ are bounded operators, then we have the general estimates of Ky Fan,

$$s_{n+k-1}(A+B) \le s_n(A) + s_k(B), \tag{8.5.5}$$

$$s_{n+k-1}(AB) \le s_n(A) s_k(B), \tag{8.5.6}$$

in particular for $k = 1$, we get

$$s_n(AB) \le \|A\| s_n(B),\ s_n(AB) \le s_n(A)\|B\|,\ s_n(A+B) \le s_n(A) + \|B\|.$$

Applying this to the second part of (8.5.4), we get

$$s_k(E_{-+}^{-1}) \le \|R_-\| \|R_+\| s_k(P^{-1}),\ 1 \le k \le N$$

implying

$$t_k(P) \le \|R_-\| \|R_+\| t_k(E_{-+}),\ 1 \le k \le N. \tag{8.5.7}$$

By a perturbation argument, we see that this holds also in the case when $P$, $E_{-+}$ are non-invertible.

Similarly from the first part of (8.5.4), we get

$$s_k(P^{-1}) \le \|E\| + \|E_+\| \|E_-\| s_k(E_{-+}^{-1}),$$

leading to

$$t_k(P) \ge \frac{t_k(E_{-+})}{\|E\| t_k(E_{-+}) + \|E_+\| \|E_-\|}. \tag{8.5.8}$$

Again this can be extended to the non-necessarily invertible case by means of small perturbations.

Next, we recall from [39] a natural construction of an associated Grushin problem to a given operator. Let $P_0 : \mathcal{H} \to \mathcal{H}$ be a Fredholm operator of index 0 as above. Assume that the first $N$ singular values $t_1(P_0) \le t_2(P_0) \le ... \le t_N(P_0)$ correspond to discrete eigenvalues of $P_0^* P_0$ and assume that $t_{N+1}(P_0)$ is strictly positive. In the following we sometimes write $t_j$ instead of $t_j(P_0)$ for short.

Recall that $t_j^2$ are the first eigenvalues both for $P_0^* P_0$ and $P_0 P_0^*$. Let $e_1, ..., e_N$ and $f_1, ..., f_N$ be corresponding orthonormal systems of eigenvectors of $P_0^* P_0$ and $P_0 P_0^*$ respectively. They can be chosen so that

$$P_0 e_j = t_j f_j,\ P_0^* f_j = t_j e_j. \tag{8.5.9}$$

Define $R_+ : L^2 \to \mathbf{C}^N$ and $R_- : \mathbf{C}^N \to L^2$ by

$$R_+u(j) = (u|e_j),\ R_-u_- = \sum_1^N u_-(j)f_j. \tag{8.5.10}$$

As in [39], the Grushin problem

$$\begin{cases} P_0u + R_-u_- = v, \\ R_+u = v_+, \end{cases} \tag{8.5.11}$$

has a unique solution $(u, u_-) \in L^2 \times \mathbf{C}^N$ for every $(v, v_+) \in L^2 \times \mathbf{C}^N$, given by

$$\begin{cases} u = E^0v + E^0_+v_+, \\ u_- = E^0_-v + E^0_{-+}v_+, \end{cases} \tag{8.5.12}$$

where

$$\begin{aligned} E^0_+v_+ &= \sum_1^N v_+(j)e_j, \quad E^0_-v(j) = (v|f_j), \\ E^0_{-+} &= -\mathrm{diag}\,(t_j), \qquad \|E^0\| \le \tfrac{1}{t_{N+1}}. \end{aligned} \tag{8.5.13}$$

$E^0$ can be viewed as the inverse of $P_0$ as an operator from the orthogonal space $(e_1, e_2, ..., e_N)^\perp$ to $(f_1, f_2, ..., f_N)^\perp$.

We notice that in this case, the norms of $R_+$ and $R_-$ are equal to 1, so (8.5.7) tells us that $t_k(P_0) \le t_k(E^0_{-+})$ for $1 \le k \le N$, but of course the expression for $E^0_{-+}$ in (8.5.13) implies equality.

Let $Q \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ and put $P_\delta = P_0 - \delta Q$ (where we sometimes put a minus sign in front of the perturbation for notational convenience). We are particularly interested in the case when $Q = Q_\omega u = q_\omega u$ is the operator of multiplication with a random function $q_\omega$. Here $\delta > 0$ is a small parameter. Choose $R_\pm$ as in (8.5.10). Then if $\delta < t_{N+1}$ and $\|Q\| \le 1$, the perturbed Grushin problem

$$\begin{cases} P_\delta u + R_-u_- = v, \\ R_+u = v_+, \end{cases} \tag{8.5.14}$$

is well posed and has the solution

$$\begin{cases} u = E^\delta v + E^\delta_+v_+, \\ u_- = E^\delta_- + E^\delta_{-+}v_+, \end{cases} \tag{8.5.15}$$

where

$$\mathcal{E}^\delta = \begin{pmatrix} E^\delta & E^\delta_+ \\ E^\delta_- & E^\delta_{-+} \end{pmatrix} \tag{8.5.16}$$

is obtained from $\mathcal{E}^0$ by

$$\mathcal{E}^\delta = \mathcal{E}^0 \left(1 - \delta \begin{pmatrix} QE^0 & QE^0_+ \\ 0 & 0 \end{pmatrix}\right)^{-1}. \tag{8.5.17}$$

Using the Neumann series, we get

$$E^\delta_{-+} = E^0_{-+} + \delta E^0_- Q E^0_+ + \delta^2 E^0_- Q E^0 Q E^0_+ + \delta^3 E^0_- Q (E^0 Q)^2 E^0_+ + ... \tag{8.5.18}$$

We also get

$$E^\delta = E^0 + \sum_1^\infty \delta^k E^0 (QE^0)^k \tag{8.5.19}$$

$$E_+^\delta = E_+^0 + \sum_1^\infty \delta^k (E^0 Q)^k E_+^0 \tag{8.5.20}$$

$$E_-^\delta = E_-^0 + \sum_1^\infty \delta^k E_-^0 (Q E^0)^k. \tag{8.5.21}$$

The leading perturbation in $E_{-+}^\delta$ is $\delta M$, where $M = E_-^0 Q E_+^0 : \mathbf{C}^N \to \mathbf{C}^N$ has the matrix

$$M(\omega)_{j,k} = (Qe_k|f_j), \tag{8.5.22}$$

which in the multiplicative case reduces to

$$M(\omega)_{j,k} = \int q(x) e_k(x) \overline{f_j(x)} dx. \tag{8.5.23}$$

Put $\tau_0 = t_{N+1}(P_0)$ and recall the assumption

$$\|Q\| \le 1. \tag{8.5.24}$$

Then, if $\delta \le \tau_0/2$, the new Grushin problem is well posed with an inverse $\mathcal{E}^\delta$ given in (8.5.16)–(8.5.21). We get

$$\|E^\delta\| \le \frac{1}{1-\frac{\delta}{\tau_0}} \|E^0\| \le \frac{2}{\tau_0}, \quad \|E_\pm^\delta\| \le \frac{1}{1-\frac{\delta}{\tau_0}} \le 2, \tag{8.5.25}$$

$$\|E_{-+}^\delta - (E_{-+}^0 + \delta E_-^0 Q E_+^0)\| \le \frac{\delta^2}{\tau_0} \frac{1}{1-\frac{\delta}{\tau_0}} \le 2\frac{\delta^2}{\tau_0}. \tag{8.5.26}$$

Using this in (8.5.7), (8.5.8) together with the fact that $t_k(E_{-+}^\delta) \le 2\tau_0$, we get

$$\frac{t_k(E_{-+}^\delta)}{8} \le t_k(P_\delta) \le t_k(E_{-+}^\delta). \tag{8.5.27}$$

**Remark 8.17.** under suitable assumptions, the preceding discussion can be extended to the case of unbounded operators. This turns out to be the case for our elliptic operator $P_\delta$.

We next collect some facts from [39]. The first result follows from Section 2 in that paper.

**Proposition 8.18.** *Let $P : \mathcal{H} \to \mathcal{H}$ be bounded and assume that $P - 1$ is of trace class, so that $P$ is Fredholm of index 0. Let $R_+, R_-, \mathcal{P}, \mathcal{E} = \mathcal{P}^{-1}$ be as in (8.5.2), (8.5.3). Then $\mathcal{P}$ is also a trace class perturbation of the identity operator and*

$$\det P = \det \mathcal{P} \det E_{-+}. \tag{8.5.28}$$

Now consider the operator $P_z = P_{0,z}$ in (8.3.5) for $z \in \Omega$, and keep the assumption (8.4.20).

Define

$$\mathcal{P}_\delta = \begin{pmatrix} P_{\delta,z} & R_{-,\delta} \\ R_{+,\delta} & 0 \end{pmatrix}$$

as in (8.5.9)–(8.5.11), so that $\mathcal{P} = \mathcal{P}_0$. As in (5.10) in [39] we have

$$|\det \mathcal{P}_\delta|^2 = \alpha^{-N} \det 1_\alpha(S_{\delta,z}), \quad 2 \ln |\det \mathcal{P}_\delta| = \ln \det 1_\alpha(S_{\delta,z}) + N \ln \frac{1}{\alpha}, \tag{8.5.29}$$

where $1_\alpha(t) = \max(\alpha, t)$, $t \geq 0$, which with (8.4.36) and the bound $N = \mathcal{O}(\alpha^\kappa h^{-n})$ gives

$$\ln|\det \mathcal{P}_\delta| = \frac{1}{(2\pi h)^n}(\iint \ln|p_z| dx d\xi + \mathcal{O}(\alpha^\kappa \ln\frac{1}{\alpha} + \frac{\delta}{\alpha}\|Q\|)). \tag{8.5.30}$$

## 8.6. Singular values and determinants for certain matrices associated to $\delta$ potentials

**Lemma 8.19.** *([97, 98]) Let $e_1, ..., e_N \in C^0(X)$ and put*

$$\overrightarrow{e}(x) = \begin{pmatrix} e_1(x) \\ e_2(x) \\ .. \\ e_N(x) \end{pmatrix}, \ x \in X.$$

*Let $L \subset \mathbf{C}^N$ be a linear subspace of dimension $M-1$, for some $1 \leq M \leq N$. Then there exists $x \in X$ such that*

$$\operatorname{dist}(\overrightarrow{e}(x), L)^2 \geq \frac{1}{\operatorname{vol}(X)} \operatorname{tr}((1-\pi_L)\mathcal{E}_X), \tag{8.6.1}$$

*where $\mathcal{E}_X = ((e_j|e_k)_{L^2(X)})_{1\leq j,k\leq N}$ and $\pi_L$ is the orthogonal projection from $\mathbf{C}^N$ onto $L$.*

*Proof.* Let $\nu_1, ..., \nu_N$ be an orthonormal basis in $\mathbf{C}^N$ such that $L$ is spanned by $\nu_1, ..., \nu_{M-1}$ (and equal to 0 when $M = 1$). Then by direct calculations,

$$\int_X \operatorname{dist}(\overrightarrow{e}(x), L)^2 dx = \sum_{\ell=M}^{N} (\mathcal{E}_X \nu_\ell|\nu_\ell) = \operatorname{tr}((1-\pi_L)\mathcal{E}_X).$$

It then suffices to estimate the integral from above by

$$\operatorname{vol}(X) \sup_{x\in X} \operatorname{dist}(\overrightarrow{e}(x), L)^2,$$

and we can find an $x \in X$ satisfying (8.6.1). □

If we make the assumption that

$$e_1, ..., e_N \text{ is an orthonormal family in } L^2(X), \tag{8.6.2}$$

then $\mathcal{E}_X = 1$ and (8.6.1) simplifies to

$$\max_{x\in X} \operatorname{dist}(\overrightarrow{e}(x), L)^2 \geq \frac{N-M+1}{\operatorname{vol}(X)}. \tag{8.6.3}$$

In the general case, let $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq ... \leq \varepsilon_N$ denote the eigenvalues of $\mathcal{E}_X$. Then, using the mini-max principle, one can show that

$$\inf_{\dim L = M-1} \operatorname{tr}((1-\pi_L)\mathcal{E}_X) = \varepsilon_1 + \varepsilon_2 + ... + \varepsilon_{N-M+1} =: E_M. \tag{8.6.4}$$

Now, we can use the lemma to choose successively $a_1, ..., a_N \in X$ such that

$$\begin{aligned} \|\overrightarrow{e}(a_1)\|^2 &\geq \frac{E_1}{\mathrm{vol}\,(X)}, \\ \mathrm{dist}\,(\overrightarrow{e}(a_2), \mathbf{C}\overrightarrow{e}(a_1))^2 &\geq \frac{E_2}{\mathrm{vol}\,(X)}, \\ &\dots \\ \mathrm{dist}\,(\overrightarrow{e}(a_M), \mathbf{C}\overrightarrow{e}(a_1) \oplus ... \oplus \mathbf{C}\overrightarrow{e}(a_{M-1}))^2 &\geq \frac{E_M}{\mathrm{vol}\,(X)}, \\ &\dots \end{aligned}$$

Let $\nu_1, \nu_2, ..., \nu_N$ be the Gram-Schmidt orthonormalization of the basis $\overrightarrow{e}(a_1), \overrightarrow{e}(a_2), ..., \overrightarrow{e}(a_N)$, so that

$$\overrightarrow{e}(a_M) \equiv c_M \nu_M \mathrm{mod}\,(\nu_1, ..., \nu_{M-1}), \text{ where } |c_M| \geq \left(\frac{E_M}{\mathrm{vol}\,(X)}\right)^{\frac{1}{2}}. \tag{8.6.5}$$

Consider the $N \times N$ matrix $E = (\overrightarrow{e}(a_1)\ \overrightarrow{e}(a_2) \dots \overrightarrow{e}(a_N))$ where $\overrightarrow{e}(a_j)$ are viewed as columns. Expressing these vectors in the basis $\nu_1, ..., \nu_N$ will not change the absolute value of the determinant and $E$ now becomes an upper triangular matrix with diagonal entries $c_1, ..., c_N$. Hence

$$|\det E| = |c_1 \cdot ... \cdot c_N|, \tag{8.6.6}$$

and (8.6.5) implies that

$$|\det E| \geq \frac{(E_1 E_2 ... E_N)^{1/2}}{(\mathrm{vol}\,(X))^{N/2}}. \tag{8.6.7}$$

Let $f_1, f_2, ..., f_N$ be a second family of continuous functions on $X$. Define $M = \mathbf{C}^N \to \mathbf{C}^N$ by

$$Mu = \sum_1^N (u|\overrightarrow{f}(a_\nu))\overrightarrow{e}(a_\nu), \ u \in \mathbf{C}^N. \tag{8.6.8}$$

Then

$$M = E \circ F^*, \tag{8.6.9}$$

where

$$F = (\overrightarrow{f}(a_1) ... \overrightarrow{f}(a_N)). \tag{8.6.10}$$

Now, we assume

$$f_j = \overline{e}_j, \ \forall j. \tag{8.6.11}$$

Then $F^* = {}^tE$, so

$$M = E \circ {}^tE. \tag{8.6.12}$$

We get from (8.6.7), (8.6.12), that

$$|\det M| \geq \frac{E_1 E_2 ... E_N}{\mathrm{vol}\,(X)^N}. \tag{8.6.13}$$

Under the assumption (8.6.2), this simplifies to

$$|\det M| \geq \frac{N!}{\mathrm{vol}\,(X)^N}. \tag{8.6.14}$$

Using that

$$|\det M| = \prod_1^N s_j \leq s_1^{k-1} s_k^{N)k+1} \leq s_1^N, \text{ where } s_j = s_j(M), \tag{8.6.15}$$

we get

**Proposition 8.20.** *Under the above assumptions,*

$$s_1 \geq \frac{(E_1 ... E_N)^{\frac{1}{N}}}{\mathrm{vol}\,(X)}, \tag{8.6.16}$$

$$s_k \geq s_1 \left( \prod_1^N \left( \frac{E_j}{s_1 \mathrm{vol}\,(X)} \right) \right)^{\frac{1}{N-k+1}}. \tag{8.6.17}$$

## 8.7. Singular values of matrices associated to suitable admissible potentials

We let $P, \widetilde{P}, p, \widetilde{p}$ be as in the introduction to this section. We also choose $\epsilon_k$, $\mu_k$, $D = D(h)$, $L = L(h)$ as in and around (8.1.10), (8.1.11).

**Definition 8.21.** *An admissible potential is a potential of the form*

$$q(x) = \sum_{0<\mu_k \leq L} \alpha_k \epsilon_k(x),\ \ \alpha \in \mathbf{C}^D. \tag{8.7.1}$$

We shall approximate $\delta$-potentials in $H_h^{-s}$ with admissible ones and then apply the results of the preceding subsection. As in the introduction we let $s > n/2$, $0 < \epsilon < s - n/2$.

**Proposition 8.22.** *Let $a \in X$. Then $\exists \alpha \in \mathbf{C}^D$, $r \in H_h^{-s}$ such that*

$$\delta_a(x) = \sum_{\mu_k \leq L} \alpha_k \epsilon_k + r(x), \tag{8.7.2}$$

*where*

$$\|r\|_{H_h^{-s}} \leq C_{s,\epsilon} L^{-(s-\frac{n}{2}-\epsilon)} h^{-\frac{n}{2}}, \tag{8.7.3}$$

$$(\sum |\alpha_k|^2)^{\frac{1}{2}} \leq \langle L \rangle^{\frac{n}{2}+\epsilon} (\sum_{\mu_k \leq L} \langle \mu_k \rangle^{-2(\frac{n}{2}+\epsilon)} |\alpha_k|^2)^{\frac{1}{2}} \leq C L^{\frac{n}{2}+\epsilon} h^{-\frac{n}{2}}. \tag{8.7.4}$$

The proof uses that $\delta_a \in H^{-(\frac{n}{2}+\epsilon)}$ with norm $\mathcal{O}(h^{-n/2})$ and the fact that for any $\widetilde{s} \in \mathbf{R}$,

$$\| \sum_1^\infty \alpha_k \epsilon_k \|^2_{H_h^{\widetilde{s}}} \asymp \sum \langle \mu_k \rangle^{2\widetilde{s}} |\alpha_k|^2.$$

It then suffices to truncate the expansion of $\delta_a$ in the basis $\epsilon_1, \epsilon_2, ....$

Let $P_\delta$ be as in (8.3.3) and assume (8.3.4), (8.3.6). Let $\mathcal{R}(\pi_\alpha) = \mathbf{C}e_1 \oplus ... \oplus \mathbf{C}e_N$ be as in one of the two cases of subsection 8.3. By the mini-max principle and standard spectral asymptotics (see [27]), we know that $N = \mathcal{O}(h^{-n})$ and if we want to use the assumption (8.1.9) we even have $N = \mathcal{O}((\max(\alpha, h))^\kappa h^{-n})$ by(8.4.35). For the moment we shall only use that $N$ is bounded by a negative power of $h$. Let $a = (a_1, ..., a_N) \in X^N$ and put

$$q_a(x) = \sum_1^N \delta(x - a_j), \tag{8.7.5}$$

$$M_{q_a;j,k} = \int q_a(x) e_k(x) e_j(x) dx,\ \ 1 \leq j, k \leq N. \tag{8.7.6}$$

Now (8.3.8) implies that $\| \sum \lambda_k e_k \|_{H_h^s} \le \mathcal{O}(\|\lambda\|_{\ell^2})$ so (8.2.2) and the fact that

$$\|q_a\|_{H_h^{-s}} = \mathcal{O}(1) N h^{-n/2},$$

imply that for all $\lambda, \mu \in \mathbf{C}^n$,

$$\begin{aligned} \langle M_{q_a}\lambda, \mu\rangle &= \int q_a(x)(\sum \lambda_k e_k)(\sum \mu_j e_j) dx \\ &= \mathcal{O}(1) N h^{-n} \|\lambda\| \|\mu\| \end{aligned}$$

and hence

$$s_1(M_{q_a}) = \|M_{q_a}\|_{\mathcal{L}(\mathbf{C}^N, \mathbf{C}^N)} = \mathcal{O}(1) N h^{-n}. \tag{8.7.7}$$

We now choose $a$ so that (8.6.16), (8.6.17) hold.

The $e_j$ form and orthonormal system, so $\mathcal{E}_X = 1$ and

$$E_j = N - j + 1. \tag{8.7.8}$$

Then, choosing the $a_j$ as in Subsection 8.6, (8.6.16) gives the lower bound

$$s_1 \ge \frac{(N!)^{\frac{1}{N}}}{\mathrm{vol}\,(X)} = (1 + \mathcal{O}(\frac{\ln N}{N})) \frac{N}{e\,\mathrm{vol}\,(X)}, \tag{8.7.9}$$

where the last identity follows from Stirling's formula.

Rewriting (8.6.17) as

$$s_k \ge s_1^{-\frac{k-1}{N-k+1}} \left( \prod_1^N \frac{E_j}{\mathrm{vol}\,(X)} \right)^{\frac{1}{N-k+1}},$$

and using (8.7.7), we get

$$s_k \ge \frac{1}{C^{\frac{k-1}{N-k+1}} (\mathrm{vol}\,(X))^{\frac{N}{N-k+1}}} \left( \frac{h^n}{N} \right)^{\frac{k-1}{N-k+1}} (N!)^{\frac{1}{N-k+1}}. \tag{8.7.10}$$

Summing up, we get

**Proposition 8.23.** *We can find $a_1, ..., a_N \in X$ such that if $q_a = \sum_1^N \delta(x - a_j)$ and $M_{q_a;j,k} = \int q_a(x) e_k(x) e_j(x) dx$, then the singular values $s_1 \ge s_2 \ge ... \ge s_N$ of $M_{q_a}$, satisfy (8.7.7), (8.7.9) and (8.7.10).*

We next approximate $q_a$ with an admissible potential by applying Proposition 8.22 to each $\delta$-function in $q_a$:

$$q_a = q + r,\ q = \sum_{\mu_k \le L} \alpha_k \epsilon_k, \tag{8.7.11}$$

where

$$\|q\|_{H_h^{-s}} \le C h^{-\frac{n}{2}} N, \tag{8.7.12}$$

$$\|r\|_{H_h^{-s}} \le C_\epsilon L^{-(s - \frac{n}{2} - \epsilon)} h^{-\frac{n}{2}} N, \tag{8.7.13}$$

$$(\sum |\alpha_k|^2)^{\frac{1}{2}} \le C L^{\frac{n}{2} + \epsilon} h^{-\frac{n}{2}} N. \tag{8.7.14}$$

Below, we shall have $N = \mathcal{O}(h^{\kappa - n})$ so if we choose $L$ as in (8.1.11), we get

$$|\alpha|_{\mathbf{C}^D} \le C h^{-(\frac{n}{2} + \epsilon) M + \kappa - \frac{3n}{2}}$$

and $q$ satisfies (8.1.10), (8.1.11). We get

$$\|M_r\| \le C_\epsilon L^{-(s - \frac{n}{2} - \epsilon)} h^{-n} N. \tag{8.7.15}$$

For the admissible potential $q$ in (8.7.11), we thus obtain from (8.7.10), (8.7.15):

$$s_k(M_q) \geq \frac{1}{C^{\frac{k-1}{N-k+1}}(\mathrm{vol}\,(X))^{\frac{N}{N-k+1}}}\left(\frac{h^n}{N}\right)^{\frac{k-1}{N-k+1}}(N!)^{\frac{1}{N-k+1}} - C_\epsilon L^{-(s-\frac{n}{2}-\epsilon)}h^{-n}N. \tag{8.7.16}$$

Similarly, from (8.7.7), (8.7.15) we get for $L \geq 1$:

$$\|M_q\| \leq CNh^{-n}. \tag{8.7.17}$$

Using Proposition 8.6, we get for every $\epsilon > 0$,

$$\|q\|_{H_h^s} \leq \mathcal{O}(1)NL^{s+\frac{n}{2}+\epsilon}h^{-\frac{n}{2}},\ \forall \epsilon > 0. \tag{8.7.18}$$

Summing up, we have obtained

**Proposition 8.24.** *Fix $s > n/2$ and $P_\delta$ as in (8.3.3), (8.3.4), (8.3.6) and let $\pi_\alpha$, $e_1, ..., e_N$ be as in one of the two cases in Subsection 8.3. Choose the $h$-dependent parameter $L$ with $1 \ll L \leq \mathcal{O}(h^{-N_0})$ for some fixed $N_0 > 0$. Then we can find an admissible potential $q$ as in (8.7.11) (different from the one in (8.3.3), (8.3.4)) such that the matrix $M_q$, defined by*

$$M_{q;j,k} = \int qe_ke_jdx,$$

*satisfies (8.7.16), (8.7.17). Moreover the $H_h^s$-norm of $q$ satisfies (8.7.18).*

Notice also that if we choose $\widetilde{R}$ with real coefficients, then we can choose $q$ real-valued.

## 8.8. Lower bounds on the small singular values for suitable perturbations

In this subsection, we fix a $z \in \Omega$. We will use Proposition 8.24 iteratively to construct a special admissible perturbation $P_\delta$ for which we have nice lower bounds on the small singular values of $P_\delta - z$, that will lead to similar bounds for the ones of $P_{\delta,z}$ and to a lower bound on $|\det P_{\delta,z}|$.

We will need the symmetry assumption (8.1.7):

$$P^* = \Gamma \circ P \circ \Gamma, \tag{8.8.1}$$

This property remains unchanged if we add a multiplication operator to $P$.

As in the introduction, we let

$$V_z(t) = \mathrm{vol}\,(\{\rho \in T^*X;\ |p(\rho) - z|^2 \leq t\}), \tag{8.8.2}$$

and assume (for our fixed value of $z$) that (8.1.9) holds:

$$V_z(t) = \mathcal{O}(t^\kappa),\ 0 \leq t \ll 1, \tag{8.8.3}$$

for some $\kappa \in ]0, 1]$. Proposition 8.14 gives:

**Proposition 8.25.** *Assume (8.8.3) and recall Remark 8.17. For $0 < h \ll \alpha \ll 1$, the number $N(\alpha)$ of eigenvalues of $(P - z)^*(P - z)$ in $[0, \alpha]$ satisfies*

$$N(\alpha) = \mathcal{O}(\alpha^\kappa h^{-n}). \tag{8.8.4}$$

Let $\epsilon > 0$, $s > \frac{n}{2} + \epsilon$ be fixed as in the introduction and consider

$$P_0 = P + \delta_0(h^{\frac{n}{2}} q_1^0 + q_2^0), \text{ with } 0 \le \delta_0 \ll h, \ \|q_1^0\|_{H_h^s}, \|q_2^0\|_{H^s} \le 1. \qquad (8.8.5)$$

From the mini-max principle, we see that Proposition 8.25 still applies after replacing $P$ by $P_0$.

Choose $\tau_0 \in ]0, (Ch)^{\frac{1}{2}}]$ and let $N = \mathcal{O}(h^{\kappa-n})$ be the number of singular values of $P_0 - z$; $0 \le t_1(P_0 - z) \le \ldots \le t_N(P_0 - z) < \tau_0$ in the interval $[0, \tau_0[$. As in the introduction we put

$$N_1 = \widetilde{M} + sM + \frac{n}{2}, \qquad (8.8.6)$$

where $M, \widetilde{M}$ are the parameters in (8.1.11). Fix $\theta \in ]0, \frac{1}{4}[$ and recall that $N$ is determined by the property $t_N(P_0 - z) < \tau_0 \le t_{N+1}(P_0 - z)$. Fix $\epsilon_0 > 0$.

**Proposition 8.26.** *a) If $q$ is an admissible potential as in (8.1.10), (8.1.11), we have*

$$\|q\|_\infty \le Ch^{-n/2}\|q\|_{H_h^s} \le \widetilde{C}h^{-N_1}. \qquad (8.8.7)$$

*b) If $N$ is sufficiently large, there exists such an admissible potential $q$, such that if*

$$P_\delta = P_0 + \frac{\delta h^{N_1}}{\widetilde{C}} q =: P_0 + \delta Q, \ \delta = \frac{\tau_0}{C} h^{N_1+n}$$

*(so that $\|Q\| \le 1$) then*

$$t_\nu(P_\delta - z) \ge t_\nu(P_0 - z) - \frac{\tau_0 h^{N_1+n}}{C} \ge (1 - \frac{h^{N_1+n}}{C}) t_\nu(P_0 - z), \ \nu > N, \qquad (8.8.8)$$

$$t_\nu(P_\delta - z) \ge \tau_0 h^{N_2}, \ [N - \theta N] + 1 \le \nu \le N. \qquad (8.8.9)$$

*Here, we put*

$$N_2 = 2(N_1 + n) + \epsilon_0, \qquad (8.8.10)$$

*and we let $[a] = \max(\mathbf{Z} \cap ] - \infty, a])$ denote the integer part of the real number $a$. When $N = \mathcal{O}(1)$, we have the same result provided that we replace (8.8.9) by*

$$t_N(P_\delta) \ge \tau_0 h^{N_2}. \qquad (8.8.11)$$

*Proof.* The part a) follows from Subsection 8.2, the definition of admissible potentials in the introduction and from the definition of $N_1$ in (8.8.6). (See also (8.7.18).) We shall therefore concentrate on the proof of b).

Let $e_1, \ldots, e_N$ be an orthonormal family of eigenfunctions corresponding to $t_\nu(P_0 - z)$, so that

$$(P_0 - z)^*(P_0 - z)e_j = (t_j(P_0 - z))^2 e_j. \qquad (8.8.12)$$

Using (8.1.6) $\Leftrightarrow$ (8.8.1), we see that a corresponding family of eigenfunctions of $(P - z)(P - z)^*$ is given by

$$\widetilde{f}_j = \Gamma e_j. \qquad (8.8.13)$$

$\widetilde{f}_1, \ldots, \widetilde{f}_N$ and $f_1, \ldots, f_N$ are orthonormal families that span the same space $F_N$. Let $E_N$ be the span of $e_1, \ldots, e_N$. We then know that

$$(P_0 - z) : E_N \to F_N \text{ and } (P_0 - z)^* : F_N \to E_N \qquad (8.8.14)$$

have the same singular values $0 \le t_1 \le t_2 \le \ldots \le t_N$.

Define $R_+ : L^2 \to \mathbf{C}^N$, $R_- : \mathbf{C}^N \to L^2$ by

$$R_+ u(j) = (u|e_j), \quad R_- u_- = \sum_1^N u_-(j)\widetilde{f}_j. \qquad (8.8.15)$$

Then

$$\mathcal{P} = \begin{pmatrix} P_0 - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{D}(P_0) \times \mathbf{C}^N \to L^2 \times \mathbf{C}^N \tag{8.8.16}$$

has a bounded inverse

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}.$$

The singular values of $E_{-+}$ are given by $t_j(E_{-+}) = t_j(P_0 - z)$, $1 \le j \le N$, or equivalently by $s_j(E_{-+}) = t_{N+1-j}(P_0 - z)$, for $1 \le j \le N$.

We will apply Subsection 8.5, and recall that $N$ is assumed to be sufficiently large and that $\theta$ has been fixed in $]0, 1/4[$. (The case of bounded $N$ will be treated later.) Let $N_2$ be given in (8.8.10). Since $z$ is fixed it will also be notationally convenient to assume that $z = 0$.

*Case 1.* $s_j(E_{-+}) \ge \tau_0 h^{N_2}$, for $1 \le j \le N - [(1-\theta)N]$. Then we get the desired conclusion with $q = 0$, $P_\delta = P_0$.

*Case 2.*

$$s_j(E_{-+}) < \tau_0 h^{N_2} \text{ for some } j \text{ such that } 1 \le j \le N - [(1-\theta)N]. \tag{8.8.17}$$

Recall that for the special admissible potential $q$ in (8.7.11), we have (8.7.16). For $k \le N/2$, we have $N - k + 1 > N/2$, so

$$\frac{k-1}{N-k+1} \le 1,$$

and (8.7.16) gives

$$s_k(M_q) \ge \frac{h^n}{CN}(N!)^{\frac{1}{N}} - C_\epsilon L^{-(s-\frac{n}{2}-\epsilon)} \frac{N}{h^n}.$$

By Stirling's formula, we have $(N!)^{\frac{1}{N}} \ge N/\mathrm{Const}$, so for $1 \le k \le N/2$, we obtain with a new constant $C > 0$:

$$s_k(M_q) \ge \frac{h^n}{C} - C_\epsilon L^{-(s-\frac{n}{2}-\epsilon)} \frac{N}{h^n}.$$

Here, we recall from Proposition 8.25 (which also applies to $P_0$) that $N = \mathcal{O}(h^{\kappa-n})$ and choose $L$ so that

$$L^{-(s-\frac{n}{2}-\epsilon)} h^{\kappa-2n} \ll h^n,$$

i.e. so that (in agreement with (8.1.11))

$$L \gg h^{\frac{\kappa-3n}{s-\frac{n}{2}-\epsilon}}. \tag{8.8.18}$$

We then get

$$s_k(M_q) \ge \frac{h^n}{C},\ 1 \le k \le \frac{N}{2}, \tag{8.8.19}$$

for a new constant $C > 0$.

From (8.7.17) and the fact that $N = \mathcal{O}(h^{\kappa-n})$ we get

$$s_1(M_q) \le \|M_q\| \le CNh^{-n} \le \widetilde{C} h^{\kappa-2n}. \tag{8.8.20}$$

In addition to the lower bound (8.8.18) we assume as in (8.1.11) (in all cases) that

$$L \le Ch^{-M},\ \text{for some } M \ge \frac{3n-\kappa}{s-\frac{n}{2}-\epsilon}. \tag{8.8.21}$$

As we saw after (8.7.14), $q$ is indeed an admissible potential as in (8.1.10), (8.1.11), so that by (8.8.7)

$$\|q\|_\infty \leq Ch^{-\frac{n}{2}}\|q\|_{H_h^s} \leq \widetilde{C}h^{-N_1}. \tag{8.8.22}$$

Put

$$P_\delta = P_0 + \frac{\delta h^{N_1}}{\widetilde{C}}q = P_0 + \delta Q,\ Q = \frac{h^{N_1}}{\widetilde{C}}q,\ \|Q\| \leq 1. \tag{8.8.23}$$

Then, if $\delta \leq \tau_0/2$, we can replace $P_0$ by $P_\delta$ in (8.8.16) and we still have a well-posed problem as in Subsection 8.5 with $Q_\omega = Q$ as above. Here $E_-^0 Q E_+^0 = h^{N_1}M_q/\widetilde{C}$ so according to (8.8.19), we have with a new constant $C$

$$s_k(\delta E_-^0 Q E_+^0) \geq \frac{\delta h^{N_1+n}}{C},\ 1 \leq k \leq \frac{N}{2}. \tag{8.8.24}$$

Playing with the general estimate (8.5.5), we get

$$s_\nu(A+B) \geq s_{\nu+k-1}(A) - s_k(B)$$

and for a sum of three operators

$$s_\nu(A+B+C) \geq s_{\nu+k+\ell-2}(A) - s_k(B) - s_\ell(C).$$

We apply this to $E_{-+}^\delta$ in (8.5.26) and get

$$s_\nu(E_{-+}^\delta) \geq s_{\nu+k-1}(\delta E_-^0 Q E_+^0) - s_k(E_{-+}^0) - 2\frac{\delta^2}{\tau_0}. \tag{8.8.25}$$

Here we use (8.8.17) with $j = k = N - [(1-\theta)N]$ as well as (8.8.24), to get for $\nu \leq N - [(1-\theta)N]$

$$s_\nu(E_{-+}^\delta) \geq \frac{\delta h^{N_1+n}}{C} - \tau_0 h^{N_2} - 2\frac{\delta^2}{\tau_0}. \tag{8.8.26}$$

Recall that $\theta < \frac{1}{4}$.

Choose

$$\delta = \frac{1}{C}\tau_0 h^{N_1+n}, \tag{8.8.27}$$

where (the new constant) $C > 0$ is sufficiently large.

Then, with a new constant $C > 0$, we get (for $h > 0$ small enough)

$$s_\nu(E_{-+}^\delta) \geq \frac{\delta}{C}h^{N_1+n},\ 1 \leq \nu \leq N - [(1-\theta)N], \tag{8.8.28}$$

implying

$$s_\nu(E_{-+}^\delta) \geq 8\tau_0 h^{N_2},\ 1 \leq \nu \leq N - [(1-\theta)N]. \tag{8.8.29}$$

For the corresponding operator $P_\delta$, we have for $\nu > N$:

$$t_\nu(P_\delta) \geq t_\nu(P_0) - \delta = t_\nu(P_0) - \frac{\tau_0 h^{N_1+n}}{C}.$$

Since $t_\nu(P) \geq \tau_0$ in this case, we get (8.8.8).

From (8.8.29) and (8.5.27), we get (8.8.9).

When $N = \mathcal{O}(1)$, we still get (8.8.26) with $\nu = 1$ and this leads to (8.8.11).

□

The construction can now be iterated. Assume that $N \gg 1$ and replace $(P_0, N, \tau_0)$ by $(P_\delta, [(1-\theta)N], \tau_0 h^{N_2}) =: (P^{(1)}, N^{(1)}, \tau_0^{(1)})$ and keep on, using the same values for the exponents $N_1, N_2$. Then we get a sequence $(P^{(k)}, N^{(k)}, \tau_0^{(k)})$, $k = 0, 1, ..., k(N)$, where the last value $k(N)$ is determined by the fact that $N^{(k(N))}$ is of the order of magnitude of a large constant. Moreover,

$$t_\nu(P^{(k)}) \geq \tau_0^{(k)},\ N^{(k)} < \nu \leq N^{(k-1)}, \tag{8.8.30}$$

$$t_\nu(P^{(k+1)}) \geq t_\nu(P^{(k)}) - \frac{\tau_0^{(k)} h^{N_1+\nu}}{C},\ \nu > N^{(k)}, \tag{8.8.31}$$

$$\tau_0^{(k+1)} = \tau_0^{(k)} h^{N_2}, \tag{8.8.32}$$

$$N^{(k+1)} = [(1-\theta)N^{(k)}], \tag{8.8.33}$$

$$P^{(0)} = P,\ N^{(0)} = N,\ \tau_0^{(0)} = \tau_0.$$

Here,

$$P^{(k+1)} = P^{(k)} + \delta^{(k+1)} Q^{(k+1)} = P^{(k)} + \frac{\delta^{(k+1)} h^{N_1}}{\widetilde{C}} q^{(k+1)},$$
$$\|Q^{(k+1)}\| \leq 1,\ \delta^{(k+1)} = \frac{1}{C} \tau_0^{(k)} h^{N_1+n}.$$

Notice that $N^{(k)}$ decays exponentially fast with $k$:

$$N^{(k)} \leq (1-\theta)^k N, \tag{8.8.34}$$

so we get the condition on $k$ that $(1-\theta)^k N \geq C \gg 1$ which gives,

$$k \leq \frac{\ln \frac{N}{C}}{\ln \frac{1}{1-\theta}}. \tag{8.8.35}$$

We also have

$$\tau_0^{(k)} = \tau_0 \left(h^{N_2}\right)^k. \tag{8.8.36}$$

For $\nu > N$, we iterate (8.8.31), to get

$$\begin{aligned} t_\nu(P^{(k)}) &\geq t_\nu(P) - \tau_0 \frac{h^{N_1+n}}{C} \left(1 + h^{N_2} + h^{2N_2} + ...\right) \\ &\geq t_\nu(P) - \tau_0 \mathcal{O}(\frac{h^{N_1+n}}{C}). \end{aligned} \tag{8.8.37}$$

For $1 \ll \nu \leq N$, let $\ell = \ell(N)$ be the unique value for which $N^{(\ell)} < \nu \leq N^{(\ell-1)}$, so that

$$t_\nu(P^{(\ell)}) \geq \tau_0^{(\ell)}, \tag{8.8.38}$$

by (8.8.30). If $k > \ell$, we get

$$t_\nu(P^{(k)}) \geq t_\nu(P^{(\ell)}) - \tau_0^{(\ell)} \mathcal{O}(\frac{h^{N_1+n}}{C}). \tag{8.8.39}$$

The iteration above works until we reach a value $k = k_0 = \mathcal{O}(\frac{\ln \frac{N}{C}}{\ln \frac{1}{1-\theta}})$ for which $N^{(k_0)} = \mathcal{O}(1)$. After that, we continue the iteration further by decreasing $N^{(k)}$ by one unit at each step.

Summing up the discussion so far, we have obtained

**Proposition 8.27.** *Let $(P, z)$ satisfy the assumptions in the beginning of this subsection and choose $P_0$ as in (8.8.5). Let $s > \frac{n}{2}$, $0 < \epsilon < s - \frac{n}{2}$, $M \geq \frac{3n-\kappa}{s-\frac{n}{2}-\epsilon}$, $N_1 = \widetilde{M} + sM + \frac{n}{2}$, $N_2 = 2(N_1 + n) + \epsilon_0$, where $\epsilon_0 > 0$. Let $L$ be an $h$-dependent parameter satisfying*

$$h^{\frac{\kappa-3n}{s-\frac{n}{2}-\epsilon}} \ll L \leq Ch^{-M}. \tag{8.8.40}$$

*Let $0 < \tau_0 \leq \sqrt{h}$ and let $N^{(0)} = \mathcal{O}(h^{\kappa-n})$ be the number of singular values of $P_0 - z$ in $[0, \tau_0[$. Let $0 < \theta < \frac{1}{4}$ and let $N(\theta) \gg 1$ be sufficiently large. Define $N^{(k)}$, $1 \leq k \leq k_1$ iteratively in the following way. As long as $N^{(k)} \geq N(\theta)$, we put $N^{(k+1)} = [(1-\theta)N^{(k)}]$. Let $k_0 \geq 0$ be the last $k$ value we get in this way. For $k > k_0$ put $N^{(k+1)} = N^{(k)} - 1$, until we reach the value $k_1$ for which $N^{(k_1)} = 1$.*

*Put $\tau_0^{(k)} = \tau_0 h^{kN_2}$, $1 \leq k \leq k_1 + 1$. Then there exists an admissible potential $q = q_h(x)$ as in (8.1.10), (8.1.11), satisfying (8.7.14), (8.7.18), so that,*

$$\|q\|_{H_h^s} \leq \mathcal{O}(1)h^{-N_1+\frac{n}{2}}, \ \|q\|_{L^\infty} \leq \mathcal{O}(1)h^{-N_1},$$

*such that if $P_\delta = P_0 + \frac{1}{C}\tau_0 h^{2N_1+n} q = P_0 + \delta Q$, $\delta = \frac{1}{C}h^{N_1+n}\tau_0$, $Q = h^{N_1}q$, we have the following estimates on the singular values of $P_\delta - z$:*

- *If $\nu > N^{(0)}$, we have $t_\nu(P_\delta - z) \geq (1 - \frac{h^{N_1+n}}{C})t_\nu(P_0 - z)$.*
- *If $N^{(k)} < \nu \leq N^{(k-1)}$, $1 \leq k \leq k_1$, then $t_\nu(P_\delta - z) \geq (1 - \mathcal{O}(h^{N_1+n}))\tau_0^{(k)}$.*
- *Finally, for $\nu = N^{(k_1)} = 1$, we have $t_1(P_\delta - z) \geq (1 - \mathcal{O}(h^{N_1+n}))\tau_0^{(k_1+1)}$.*

Now it is possible to pass from the Grushin problem for $P_\delta - z$ to a suitable one for $P_{\delta,z}$ and follow up with estimates on the singular values (cf (8.5.8)) and obtain:

**Proposition 8.28.** *Proposition 8.27 remains valid if we replace $P_\delta - z$ there with $P_{\delta,z}$.*

Taking a suitable Grushin problem for $P_{\delta,z}$ and using Proposition 8.18 we can show when $\tau_0 = \sqrt{h}$:

**Proposition 8.29.** *For the special admissible perturbation $P_\delta$ in the propositions 8.27, 8.28, we have*

$$\begin{aligned} &\ln|\det P_{\delta,z}| \geq \\ &\frac{1}{(2\pi h)^n}\left(\iint \ln|p_z| dx d\xi - \mathcal{O}\left(h^{N_1+n-\frac{1}{2}} + (h^\kappa + h^n \ln\frac{1}{h})(\ln\frac{1}{\tau_0} + (\ln\frac{1}{h})^2)\right)\right). \end{aligned} \tag{8.8.41}$$

We also have the upper bound

$$|\det E_{-+}| \leq \|E_{-+}\|^{N^{(0)}} \leq \exp(CN^{(0)}),$$

which leads to

$$\ln|\det P_{\delta,z}| \leq \frac{1}{(2\pi h)^n}\left(\iint \ln|p_z| dx d\xi + \mathcal{O}\left(h^{N_1+n-\frac{1}{2}} + h^\kappa \ln\frac{1}{h}\right)\right). \tag{8.8.42}$$

Notice that this bound is more general, it only depends on the fact that the perturbation of $P$ is of the form $\delta Q$ with $\delta = \tau_0 h^{N_1+n}/C$ and with $\|Q\| = \mathcal{O}(1)$.

When $\tau_0 \leq \sqrt{h}$ we keep the same Grushin problem as before and notice that the singular values of $E_{-+}$ that are $\leq \tau_0$, obey the estimates in Proposition 8.27. Their contribution to $\ln|\det E_{-+}|$ can still be estimated from below. The contribution from the singular values of $E_{-+}$ that are $> \tau_0$ can be estimated from below by $-\mathcal{O}(h^{\kappa-n}\ln(1/\tau_0))$ and this leads to the conclusion that *Proposition 8.29 remains valid when* $0 < \tau_0 \leq \sqrt{h}$. *The same holds for the upper bound (8.8.42).*

## 8.9. Estimating the probability that $\det E_{-+}^{\delta}$ is small

In this subsection we keep the assumptions on $(P, z)$ of the beginning of Subsection 8.8 and choose $P_0$ as in (8.8.5). We consider general $P_\delta$ of the form

$$P_\delta = P_0 + \delta Q,\ \delta Q = \delta h^{N_1} q(x),\ \delta = \frac{1}{C} h^{N_1+n}\tau_0, \tag{8.9.1}$$

where $q$ is an admissible potential as in (8.1.10), (8.1.11). Notice that $D := \#\{k;\ \mu_k \leq L\}$ satisfies:

$$D \leq \mathcal{O}(L^n h^{-n}) \leq \mathcal{O}(h^{-N_3}),\ N_3 := n(M+1). \tag{8.9.2}$$

With $R$ as in (8.1.10), we allow $\alpha$ to vary in the ball

$$|\alpha|_{\mathbf{C}^D} \leq 2R = \mathcal{O}(h^{-\widetilde{M}}). \tag{8.9.3}$$

(Our probability measure will be supported in $B_{\mathbf{C}^D}(0, R)$ but we will need to work in a larger ball.)

We consider the holomorphic function

$$F(\alpha) = (\det P_{\delta,z}) \exp\Big(-\frac{1}{(2\pi h)^n} \iint \ln|p_z| dx d\xi\Big). \tag{8.9.4}$$

Then by (8.8.42), we have

$$\ln|F(\alpha)| \leq \epsilon_0(h) h^{-n},\ |\alpha| < 2R, \tag{8.9.5}$$

and for one particular value $\alpha = \alpha^0$ with $|\alpha^0| \leq \frac{1}{2}R$, corresponding to the special potential in Proposition 8.27:

$$\ln|F(\alpha^0)| \geq -\epsilon_0(h) h^{-n}, \tag{8.9.6}$$

where $\epsilon_0(h)$ is given in (8.1.16).

Let $\alpha^1 \in \mathbf{C}^D$ with $|\alpha^1| = R$ and consider the holomorphic function of one complex variable

$$f(w) = F(\alpha^0 + w\alpha^1). \tag{8.9.7}$$

We will mainly consider this function for $w$ in the disc determined by the condition $|\alpha^0 + w\alpha^1| < R$:

$$D_{\alpha^0,\alpha^1}: \left| w + \left(\frac{\alpha^0}{R} | \frac{\alpha^1}{R}\right)\right|^2 < 1 - \left|\frac{\alpha^0}{R}\right|^2 + \left|\left(\frac{\alpha^0}{R} | \frac{\alpha^1}{R}\right)\right|^2 =: r_0^2, \tag{8.9.8}$$

whose radius is between $\frac{\sqrt{3}}{2}$ and 1.

From (8.9.5), (8.9.6) we get

$$\ln|f(0)| \geq -\epsilon_0(h) h^{-n},\ \ln|f(w)| \leq \epsilon_0(h) h^{-n}. \tag{8.9.9}$$

By (8.9.5), we may assume that the last estimate holds in a larger disc, say $D(-(\frac{\alpha^0}{R}|\frac{\alpha^1}{R}), 2r_0)$. Let $w_1, ..., w_M$ be the zeros of $f$ in $D(-(\frac{\alpha^0}{R}|\frac{\alpha^1}{R}), 3r_0/2)$. Then it is standard to get the factorization

$$f(w) = e^{g(w)} \prod_1^M (w - w_j),\ w \in D\Big(-\big(\frac{\alpha^0}{R}|\frac{\alpha^1}{R}\big), 4r_0/3\Big), \tag{8.9.10}$$

together with the bounds

$$|\Re g(w)| \le \mathcal{O}(\epsilon_0(h)h^{-n}),\ M = \mathcal{O}(\epsilon_0(h)h^{-n}). \tag{8.9.11}$$

See for instance Section 5 in [99] where further references are also given.

For $0 < \epsilon \ll 1$, put

$$\Omega(\epsilon) = \{r \in [0, r_0[;\ \exists w \in D_{\alpha^0,\alpha^1} \text{ such that } |w| = r \text{ and } |f(w)| < \epsilon\}. \tag{8.9.12}$$

If $r \in \Omega(\epsilon)$ and $w$ is a corresponding point in $D_{\alpha^0,\alpha^1}$, we have with $r_j = |w_j|$,

$$\prod_1^M |r - r_j| \le \prod_1^M |w - w_j| \le \epsilon \exp(\mathcal{O}(\epsilon_0(h)h^{-n})). \tag{8.9.13}$$

Then at least one of the factors $|r - r_j|$ is bounded by $(\epsilon e^{\mathcal{O}(\epsilon_0(h)h^{-n})})^{1/M}$. In particular, the Lebesgue measure $\lambda(\Omega(\epsilon))$ of $\Omega(\epsilon)$ is bounded by $2M(\epsilon e^{\mathcal{O}(\epsilon_0(h)h^{-n})})^{1/M}$. Noticing that the last bound increases with $M$ when the last member of (8.9.13) is $\le 1$, we get

**Proposition 8.30.** *Let $\alpha^1 \in \mathbf{C}^D$ with $|\alpha^1| = R$ and assume that $\epsilon > 0$ is small enough so that the last member of (8.9.13) is $\le 1$. Then*

$$\lambda(\{r \in [0, r_0];\ |\alpha^0 + r\alpha^1| < R,\ |F(\alpha^0 + r\alpha^1)| < \epsilon\}) \le \frac{\epsilon_0(h)}{h^n}\exp(\mathcal{O}(1) + \frac{h^n}{\mathcal{O}(1)\epsilon_0(h)}\ln \epsilon). \tag{8.9.14}$$

*Here and in the following, the symbol $\mathcal{O}(1)$ in a denominator indicates a bounded positive quantity.*

Typically, we can choose $\epsilon = \exp -\frac{\epsilon_0(h)}{h^{n+\alpha}}$ for some small $\alpha > 0$ and then the upper bound in (8.9.14) becomes

$$\frac{\epsilon_0(h)}{h^n}\exp(\mathcal{O}(1) - \frac{1}{\mathcal{O}(1)h^\alpha}).$$

Now we equip $B_{\mathbf{C}^D}(0, R)$ with a probability measure of the form

$$P(d\alpha) = C(h)e^{\Phi(\alpha)}L(d\alpha), \tag{8.9.15}$$

where $L(d\alpha)$ is the Lebesgue measure, $\Phi$ is a $C^1$ function which depends on $h$ and satisfies

$$|\nabla\Phi| = \mathcal{O}(h^{-N_4}), \tag{8.9.16}$$

and $C(h)$ is the appropriate normalization constant.

Writing $\alpha = \alpha^0 + Rr\alpha^1$, $0 \le r < r_0(\alpha^1)$, $\alpha^1 \in S^{2D-1}$, $\frac{\sqrt{3}}{2} \le r_0 \le 1$, we get

$$P(d\alpha) = \widetilde{C}(h)e^{\phi(r)}r^{2D-1}drS(d\alpha^1), \tag{8.9.17}$$

where $\phi(r) = \phi_{\alpha^0,\alpha^1}(r) = \Phi(\alpha^0 + rR\alpha^1)$ so that $\phi'(r) = \mathcal{O}(h^{-N_5})$, $N_5 = N_4 + \widetilde{M}$. Here $S(d\alpha^1)$ denotes the Lebesgue measure on $S^{2D-1}$.

For a fixed $\alpha^1$, we consider the normalized measure

$$\mu(dr) = \widehat{C}(h)e^{\phi(r)}r^{2D-1}dr \tag{8.9.18}$$

on $[0, r_0(\alpha^1)]$ and we want to show an estimate similar to (8.9.14) for $\mu$ instead of $\lambda$. Write $e^{\phi(r)}r^{2D-1} = \exp(\phi(r) + (2D-1)\ln r)$ and consider the derivative of the exponent,

$$\phi'(r) + \frac{2D-1}{r}.$$

This derivative is $\geq 0$ for $r \leq \frac{h^{[N_5-N_3]_+}}{C} =: 2\widetilde{r}_0$, where we may assume that $2\widetilde{r}_0 \leq r_0$. Introduce the measure $\widetilde{\mu} \geq \mu$ by

$$\widetilde{\mu}(dr) = \widehat{C}(h)e^{\phi(r_{\max})}r_{\max}^{2D-1}dr,\ r_{\max} := \max(r, \widetilde{r}_0). \tag{8.9.19}$$

Since $\widetilde{\mu}([0, \widetilde{r}_0]) \leq \mu([\widetilde{r}_0, 2\widetilde{r}_0])$, we get

$$\widetilde{\mu}([0, r(\alpha^1)]) \leq \mathcal{O}(1). \tag{8.9.20}$$

We can write

$$\widetilde{\mu}(dr) = \widehat{C}(h)e^{\psi(r)}dr, \tag{8.9.21}$$

where

$$\psi'(r) = \mathcal{O}(1)(h^{-N_5} + h^{-N_3+[N_5-N_3]_+}) = \mathcal{O}(h^{-N_6}), \quad N_6 = \max(N_3, N_5). \tag{8.9.22}$$

Cf (8.9.2).

We now decompose $[0, r_0(\alpha^1)]$ into $\asymp h^{-N_6}$ intervals of length $\asymp h^{N_6}$. If $I$ is such an interval, we see that

$$\frac{\lambda(dr)}{C\lambda(I)} \leq \frac{\widetilde{\mu}(dr)}{\widetilde{\mu}(I)} \leq C\frac{\lambda(dr)}{\lambda(I)} \text{ on } I. \tag{8.9.23}$$

From (8.9.14), (8.9.23) we get when the right hand side of (8.9.13) is $\leq 1$,

$$\begin{aligned}\widetilde{\mu}(\{r \in I;\ |F(\alpha^0 + rR\alpha^1)| < \epsilon\})/\widetilde{\mu}(I) &\leq \frac{\mathcal{O}(1)}{\lambda(I)}\frac{\epsilon_0(h)}{h^n}\exp(\frac{h^n}{\mathcal{O}(1)\epsilon_0(h)}\ln\epsilon) \\ &= \mathcal{O}(1)h^{-N_6}\frac{\epsilon_0(h)}{h^n}\exp(\frac{h^n}{\mathcal{O}(1)\epsilon_0(h)}\ln\epsilon).\end{aligned}$$

Multiplying with $\widetilde{\mu}(I)$ and summing the estimates over $I$ we get

$$\widetilde{\mu}(\{r \in [0, r(\alpha^1)];\ |F(\alpha^0 + rR\alpha^1)| < \epsilon\}) \leq \mathcal{O}(1)h^{-N_6}\frac{\epsilon_0(h)}{h^n}\exp(\frac{h^n}{\mathcal{O}(1)\epsilon_0(h)}\ln\epsilon). \tag{8.9.24}$$

Since $\mu \leq \widetilde{\mu}$, we get the same estimate with $\widetilde{\mu}$ replaced by $\mu$. Then from (8.9.17) we get

**Proposition 8.31.** *Let $\epsilon > 0$ be small enough for the right hand side of (8.9.13) to be $\leq 1$. Then*

$$P(|F(\alpha)| < \epsilon) \leq \mathcal{O}(1)h^{-N_6}\frac{\epsilon_0(h)}{h^n}\exp(\frac{h^n}{\mathcal{O}(1)\epsilon_0(h)}\ln\epsilon). \tag{8.9.25}$$

**Remark 8.32.** In the case when $\widetilde{R}$ has real coefficients, we may assume that the eigenfunctions $\epsilon_j$ are real, and from the observation after Proposition 8.24 we see that we can choose $\alpha_0$ above to be real. The discussion above can then be restricted to the case of real $\alpha^1$ and hence to real $\alpha$. We can then introduce the probability measure $P$ as in (8.9.15) on the real ball $B_{\mathbf{R}^D}(0, R)$. The subsequent discussion goes through without any changes, and we still have the conclusion of Proposition 8.31.

## 8.10. End of the proof of the main result

We now work under the assumptions of Theorem 8.1. For $z$ in a fixed neighborhood of $\Gamma$, we rephrase (8.8.42) as

$$|\det P_{\delta,z}| \le \exp \frac{1}{h^n}(\frac{1}{(2\pi)^n}\iint \ln |p_z| dx d\xi + \epsilon_0(h)), \tag{8.10.1}$$

where $\epsilon_0(h)$ is given in (8.1.16). Moreover, Proposition 8.31 shows that with probability

$$\ge 1 - \mathcal{O}(1) h^{-N_6-n}\epsilon_0(h) e^{-\frac{h^n}{\mathcal{O}(1)\epsilon_0(h)}\ln\frac{1}{\epsilon}}, \tag{8.10.2}$$

we have

$$|\det P_{\delta,z}| \ge \epsilon \exp(\frac{1}{h^n}(\frac{1}{(2\pi)^n})\iint \ln |p_z| dx d\xi), \tag{8.10.3}$$

provided that $\epsilon > 0$ is small enough so that

$$\text{The right hand side of (8.9.13) is } \le 1, \forall \alpha^1 \in S^{2D-1}. \tag{8.10.4}$$

Write $\epsilon = e^{-\widetilde{\epsilon}/h^n}$, $\widetilde{\epsilon} = h^n \ln \frac{1}{\epsilon}$. Then (8.10.4) holds if

$$\widetilde{\epsilon} \ge C\epsilon_0(h), \tag{8.10.5}$$

for some large constant $C$. (8.10.2), (8.10.3) can be rephrased by saying that with probability

$$\ge 1 - \mathcal{O}(1) h^{-N_6-n}\epsilon_0(h) e^{-\frac{1}{C}\frac{\widetilde{\epsilon}}{\epsilon_0(h)}}, \tag{8.10.6}$$

we have

$$|\det P_{\delta,z}| \ge \exp \frac{1}{h^n}(\frac{1}{(2\pi)^n}\iint \ln |p_z| dx d\xi - \widetilde{\epsilon}). \tag{8.10.7}$$

This is of interest for $\widetilde{\epsilon}$ in the range

$$\epsilon_0(h) \ll \widetilde{\epsilon} \ll 1. \tag{8.10.8}$$

Now, let $\Gamma \Subset \Omega$ be connected with smooth boundary. Recall that $0 < \kappa \le 1$ and that

$$\text{(8.1.9) holds uniformly for all } z \text{ in some neighborhood of } \partial\Gamma. \tag{8.10.9}$$

Then the function

$$\phi(z) = \frac{1}{(2\pi)^n}\iint \ln |p_z| dx d\xi \tag{8.10.10}$$

is continuous and subharmonic in a neighborhood of $\partial\Gamma$. We shall apply Theorem 6.2, with $0 < r \ll 1$ constant, to the holomorphic function

$$u(z) = \det P_{\delta,z}.$$

Then, according to (8.10.6), (8.10.7) we know that with probability

$$\ge 1 - \frac{\mathcal{O}(1)\epsilon_0(h)}{r h^{N_6+n}} e^{-\frac{\widetilde{\epsilon}}{\mathcal{O}(1)\epsilon_0(h)}} \tag{8.10.11}$$

we have

$$h^n \ln |u(\widetilde{z}_j)| \ge \phi(\widetilde{z}_j) - \widetilde{\epsilon},\ j = 1, ..., N, \quad N \asymp \frac{1}{r}. \tag{8.10.12}$$

In a full neighborhood of $\partial\Gamma$ we also have

$$h^n \ln |u(z)| \le \phi(z) + C\widetilde{\epsilon}. \tag{8.10.13}$$

We conclude from Theorem 6.2 that with probability bounded from below as in (8.10.11) we have for every $\widehat{M} > 0$:

$$|\#(u^{-1}(0) \cap \Gamma) - \frac{1}{h^n 2\pi} \int_\Gamma \Delta\phi L(dz)| \le \tag{8.10.14}$$
$$\frac{\mathcal{O}(1)}{h^n} \left( \frac{\widetilde{\epsilon}}{r} + \mu(\partial\Gamma + D(0,r)) \right),$$

where $\mu$ denotes the measure $\Delta\phi L(dz)$.

According to Section 10 in [39], the measure $\frac{1}{2\pi}\Delta\phi L(dz)$ is the push forward under $p$ of $(2\pi)^{-n}$ times the symplectic volume element, and we can replace $\frac{1}{2\pi}\Delta\phi L(dz)$ by this push forward in (8.10.14). Moreover $u^{-1}(0)$ is the set of eigenvalues of $P_\delta$ so we can rephrase (8.10.14) as

$$|\#(\sigma(P_\delta) \cap \Gamma) - \frac{1}{(2\pi h)^n} \mathrm{vol}\,(p^{-1}(\Gamma))| \le \tag{8.10.15}$$
$$\frac{\mathcal{O}(1)}{h^n} \left( \frac{\widetilde{\epsilon}}{r} + \mathrm{vol}\,(p^{-1}(\partial\Gamma + D(0,r))) \right).$$

This concludes the proof of Theorem 8.1, with $P$ replaced by the slightly more general operator $P_0$.

# 9. Almost sure Weyl asymptotics of large eigenvalues

## 9.1. Introduction

W. Bordeaux Montrieux [10] has studied elliptic systems of differential operators on $S^1$ with random perturbations of the coefficients, and under some additional assumptions, he showed that the large eigenvalues obey the Weyl law *almost surely.* His analysis was based on a reduction to the semi-classical case (using essentially the Borel-Cantelli lemma), where he could use and extend the methods of Hager [38].

The purpose of this section is to describe the work of Bordeaux Montrieux and the author [11] on the almost sure Weyl asymptotics of the large eigenvalues of elliptic operators on compact manifolds. For simplicity, we treat only the scalar case and the random perturbation is a potential.

Let $X$ be a smooth compact manifold of dimension $n$. Let $P^0$ be an elliptic differential operator on $X$ of order $m \ge 2$ with smooth coefficients and with principal symbol $p(x,\xi)$. In local coordinates we get, using standard multi-index notation,

$$P^0 = \sum_{|\alpha| \le m} a^0_\alpha(x) D^\alpha, \quad p(x,\xi) = \sum_{|\alpha|=m} a^0_\alpha(x)\xi^\alpha. \tag{9.1.1}$$

Recall that the ellipticity of $P^0$ means that $p(x,\xi) \ne 0$ for $\xi \ne 0$. We assume that

$$p(T^*X) \ne \mathbf{C}. \tag{9.1.2}$$

Fix a strictly positive smooth density of integration $dx$ on $X$, so that the $L^2$ norm $\|\cdot\|$ and inner product $(\cdot|\cdot)$ are unambiguously defined. Let $\Gamma : L^2(X) \to L^2(X)$ be the antilinear operator of complex conjugation, given by $\Gamma u = \overline{u}$. We need the symmetry assumption

$$P^* = \Gamma P \Gamma, \ \text{(or equivalently, } P^{\mathrm{t}} = P) \tag{9.1.3}$$

where $P^*$ is the formal complex adjoint of $P$. As in [98] we observe that the property (9.1.3) implies that

$$p(x,-\xi) = p(x,\xi), \tag{9.1.4}$$

and conversely, if (9.1.4) holds, then the operator $\frac{1}{2}(P+\Gamma P\Gamma)$ has the same principal symbol $p$ and satisfies (9.1.3).

Let $\widetilde{R}$ be an elliptic differential operator on $X$ with smooth coefficients, which is self-adjoint and strictly positive. Let $\epsilon_0, \epsilon_1, \ldots$ be an orthonormal basis of eigenfunctions of $\widetilde{R}$ so that

$$\widetilde{R}\epsilon_j = (\mu_j^0)^2 \epsilon_j, \quad 0 < \mu_0^0 < \mu_1^0 \le \mu_2^0 \le \ldots \tag{9.1.5}$$

Our randomly perturbed operator is

$$P_\omega^0 = P + q_\omega^0(x), \tag{9.1.6}$$

where $\omega$ is the random parameter and

$$q_\omega^0(x) = \sum_0^\infty \alpha_j^0(\omega)\epsilon_j. \tag{9.1.7}$$

Here we assume that $\alpha_j^0(\omega)$ are independent complex Gaussian random variables of variance $\sigma_j^2$ and mean value 0:

$$\alpha_j^0 \sim \mathcal{N}(0, \sigma_j^2), \tag{9.1.8}$$

where

$$(\mu_j^0)^{-\rho} e^{-(\mu_j^0)^{\frac{\beta}{M+1}}} \lesssim \sigma_j \lesssim (\mu_j^0)^{-\rho}, \tag{9.1.9}$$

$$M = \frac{3n - \frac{1}{2}}{s - \frac{n}{2} - \epsilon},\ 0 \le \beta < \frac{1}{2},\ \rho > n, \tag{9.1.10}$$

where $s$, $\rho$, $\epsilon$ are fixed constants such that

$$\frac{n}{2} < s < \rho - \frac{n}{2},\ 0 < \epsilon < s - \frac{n}{2}.$$

Let $H^s(X)$ be the standard Sobolev space of order $s$. As will follow from considerations below, we have $q_\omega^0 \in H^s(X)$ almost surely since $s < \rho - \frac{n}{2}$. Hence $q_\omega^0 \in L^\infty$ almost surely, implying that $P_\omega^0$ has purely discrete spectrum.

Consider the function $F(\omega) = \arg p(\omega)$ on $S^*X$. For given $\theta_0 \in S^1 \simeq \mathbf{R}/(2\pi\mathbf{Z})$, $N_0 \in \dot{\mathbf{N}} := \mathbf{N} \setminus \{0\}$, we introduce the property:

$$P(\theta_0, N_0): \quad \sum_1^{N_0} |\nabla^k F(\omega)| \neq 0 \text{ on } \{\omega \in S^*X;\ F(\omega) = \theta_0\}. \tag{9.1.11}$$

Notice that if $P(\theta_0, N_0)$ holds, then $P(\theta, N_0)$ holds for all $\theta$ in some neighborhood of $\theta_0$.

We can now state our main result.

**Theorem 9.1.** *([11]) Assume that $m \ge 2$. Let $0 \le \theta_1 \le \theta_2 \le 2\pi$ and assume that $P(\theta_1, N_0)$ and $P(\theta_2, N_0)$ hold for some $N_0 \in \dot{\mathbf{N}}$. Let $g \in C^\infty([\theta_1,\theta_2];]0,\infty[)$ and put*

$$\Gamma^g_{\theta_1,\theta_2;0,\lambda} = \{re^{i\theta}; \theta_1 \le \theta \le \theta_2,\ 0 \le r \le \lambda g(\theta)\}.$$

*Then for every* $\delta \in ]0, \frac{1}{2} - \beta[$ *there exists* $C > 0$ *such that almost surely:* $\exists C(\omega) < \infty$ *such that for all* $\lambda \in [1, \infty[$*:*

$$|\#(\sigma(P_\omega^0) \cap \Gamma^g_{\theta_1,\theta_2;0,\lambda}) - \frac{1}{(2\pi)^n} \mathrm{vol}\, p^{-1}(\Gamma^g_{\theta_1,\theta_2;0,\lambda})| \le C(\omega) + C\lambda^{\frac{n}{m} - \frac{1}{m}(\frac{1}{2} - \beta - \delta)\frac{1}{N_0+1}}. \quad (9.1.12)$$

*Here* $\sigma(P_\omega^0)$ *denotes the spectrum and* $\#(A)$ *denotes the number of elements in the set* $A$*. In (9.1.12) the eigenvalues are counted with their algebraic multiplicity.*

The proof actually allows to have almost surely a simultaneous conclusion for a whole family of $\theta_1, \theta_2, g$:

**Theorem 9.2.** *Assume that* $m \ge 2$*. Let* $\Theta$ *be a compact subset of* $[0, 2\pi]$*. Let* $N_0 \in \mathbf{N}$ *and assume that* $P(\theta, N_0)$ *holds uniformly for* $\theta \in \Theta$*. Let* $\mathcal{G}$ *be a subset of* $\{(g, \theta_1, \theta_2);\ \theta_j \in \Theta, \theta_1 \le \theta_2,\ g \in C^\infty([\theta_1, \theta_2]; ]0, \infty[)\}$ *with the property that* $g$ *and* $1/g$ *are uniformly bounded in* $C^\infty([\theta_1, \theta_2]; ]0, \infty[)$ *when* $(g, \theta_1, \theta_2)$ *varies in* $\mathcal{G}$*. Then for every* $\delta \in ]0, \frac{1}{2} - \beta[$ *there exists* $C > 0$ *such that almost surely:* $\exists C(\omega) < \infty$ *such that for all* $\lambda \in [1, \infty[$ *and all* $(g, \theta_1, \theta_2) \in \mathcal{G}$*, we have the estimate (9.1.12).*

The condition (9.1.9) allows us to choose $\sigma_j$ decaying faster than any negative power of $\mu_j^0$. Then from the discussion below, it will follow that $q_\omega(x)$ is almost surely a smooth function. A rough and somewhat intuitive interpretation of Theorem 9.2 is then that for almost every elliptic operator of order $\ge 2$ with smooth coefficients on a compact manifold which satisfies the conditions (9.1.2), (9.1.3), the large eigenvalues distribute according to Weyl's law in sectors with limiting directions that satisfy a weak non-degeneracy condition.

## 9.2. Volume considerations

In the next subsection we shall perform a reduction to a semi-classical situation and work with $h^m P_0$ which has the semi-classical principal symbol $p$ in (9.1.1). Again,

$$V_z(t) = \mathrm{vol}\,\{\rho \in T^*X;\ |p(\rho) - z|^2 \le t\},\ t \ge 0. \quad (9.2.1)$$

**Proposition 9.3.** *For any compact set* $K \subset \dot{\mathbf{C}} = \mathbf{C} \setminus \{0\}$*, we have*

$$V_z(t) = \mathcal{O}(t^\kappa),\ \textit{uniformly for } z \in K,\ 0 \le t \ll 1, \quad (9.2.2)$$

*with* $\kappa = 1/2$*.*

This follows from:

**Proposition 9.4.** *Let* $\gamma$ *be the curve* $\{re^{i\theta} \in \mathbf{C};\ r = g(\theta),\ \theta \in S^1\}$*, where* $0 < g \in C^1(S^1)$*. Then*

$$\mathrm{vol}\,(p^{-1}(\gamma + D(0, t))) = \mathcal{O}(t),\ t \to 0.$$

This follows from the fact that the radial derivative of $p$ is $\neq 0$.

Using (9.1.11), we can prove:

**Proposition 9.5.** *Let* $\theta_0 \in S^1$*,* $N_0 \in \dot{\mathbf{N}}$ *and assume that* $P(\theta_0, N_0)$ *holds. Then if* $0 < r_1 < r_2$ *and* $\gamma$ *is the radial segment* $[r_1, r_2]e^{i\theta_0}$*, we have*

$$\mathrm{vol}\,(p^{-1}(\gamma + D(0, t))) = \mathcal{O}(t^{1/N_0}),\ t \to 0.$$

Now, let $0 \leq \theta_1 < \theta_2 \leq 2\pi$, $g \in C^\infty([\theta_1, \theta_2]; ]0, \infty[)$ and put

$$\Gamma^g_{\theta_1,\theta_2;r_1,r_2} = \{re^{i\theta};\ \theta_1 \leq \theta \leq \theta_2,\ r_1 g(\theta) \leq r \leq r_2 g(\theta)\}, \tag{9.2.3}$$

for $0 \leq r_1 \leq r_2 < \infty$. If $0 < r_1 < r_2 < +\infty$ and $P(\theta_j, N_0)$ hold for $j = 1, 2$, then the last two propositions imply that

$$\operatorname{vol} p^{-1}(\partial\Gamma^g_{\theta_1,\theta_2;r_1,r_2} + D(0,t)) = \mathcal{O}(t^{1/N_0}),\ t \to 0. \tag{9.2.4}$$

## 9.3. Semiclassical reduction

We are interested in the distribution of large eigenvalues $\zeta$ of $P^0_\omega$, so we make a standard reduction to a semi-classical problem by letting $0 < h \ll 1$ satisfy

$$\zeta = \frac{z}{h^m},\ |z| \asymp 1,\ h \asymp |\zeta|^{-1/m}, \tag{9.3.1}$$

and write

$$h^m(P^0_\omega - \zeta) = h^m P^0_\omega - z =: P + h^m q^0_\omega - z, \tag{9.3.2}$$

where

$$P = h^m P^0 = \sum_{|\alpha| \leq m} a_\alpha(x; h)(hD)^\alpha. \tag{9.3.3}$$

Here

$$\begin{aligned} a_\alpha(x; h) &= \mathcal{O}(h^{m-|\alpha|}) \text{ in } C^\infty, \\ a_\alpha(x; h) &= a^0_\alpha(x) \text{ when } |\alpha| = m. \end{aligned} \tag{9.3.4}$$

So $P$ is a standard semi-classical differential operator with semi-classical principal symbol $p(x, \xi)$.

Our strategy will be to decompose the random perturbation

$$h^m q^0_\omega = \delta Q_\omega + k_\omega(x),$$

where the two terms are independent, and with probability very close to 1, $\delta Q_\omega$ will be a semi-classical random perturbation as in Section 8 while

$$\|k_\omega\|_{H^s} \leq h, \tag{9.3.5}$$

and

$$s \in ]\frac{n}{2}, \rho - \frac{n}{2}[ \tag{9.3.6}$$

is fixed. Then $h^m P^0_\omega$ will be viewed as a random perturbation of $h^m P^0 + k_\omega$. In order to achieve this without extra assumptions on the order $m$, we will also have to represent some of our eigenvalues $\alpha^0_j(\omega)$ as sums of two independent Gaussian random variables.

We start by examining when

$$\|h^m q^0_\omega\|_{H^s} \leq h. \tag{9.3.7}$$

**Proposition 9.6.** *There is a constant $C > 0$ such that (9.3.7) holds with probability*

$$\geq 1 - \exp(C - \frac{1}{2Ch^{2(m-1)}}).$$

Here is a brief outline of the proof. We have

$$h^m q_\omega^0 = \sum_0^\infty \alpha_j(\omega)\epsilon_j, \quad \alpha_j = h^m\alpha_j^0 \sim \mathcal{N}(0,(h^m\sigma_j)^2), \tag{9.3.8}$$

and the $\alpha_j$ are independent. Now, by the functional characterization of $H^s$ in Subsection 8.2, we get

$$\|h^m q_\omega^0\|_{H^s}^2 \asymp \sum_0^\infty |(\mu_j^0)^s\alpha_j(\omega)|^2, \tag{9.3.9}$$

where $(\mu_j^0)^s\alpha_j \sim \mathcal{N}(0,(\widetilde{\sigma}_j)^2)$ are independent random variables and $\widetilde{\sigma}_j = (\mu_j^0)^s h^m\sigma_j$.

Combining this with Proposition 7.5 and standard Weyl asymptotics for $\widetilde{R}$ leads to the result.

Write

$$q_\omega^0 = q_\omega^1 + q_\omega^2, \tag{9.3.10}$$

$$q_\omega^1 = \sum_{0<h\mu_j^0\le L}\alpha_j^0(\omega)\epsilon_j,\ q_\omega^2 = \sum_{h\mu_j^0>L}\alpha_j^0(\omega)\epsilon_j. \tag{9.3.11}$$

From Proposition 9.6 and its proof, we know that

$$\|h^m q_\omega^2\|_{H^s} \le h \text{ with probability } \ge 1-\exp(C_0 - \frac{1}{2Ch^{2(m-1)}}). \tag{9.3.12}$$

We write

$$P + h^m q_\omega^0 = (P + h^m q_\omega^2) + h^m q_\omega^1,$$

Theorem 8.1 can be applied with $P$ replaced by the perturbation $P+h^m q_\omega^2$, provided that we have $\|h^m q_\omega^2\|_{H^s}\le h$.

The next question is then wether $h^m q_\omega^1$ can be written as $\tau_0 h^{2N_1+n}q_\omega$ where $q_\omega = \sum_{0<h\mu_j^0\le L}\alpha_j\epsilon_j$ and $|\alpha|_{\mathbf{C}^D}\le R$ with probability close to 1. This turns out to be impossible without extra assumptions.

In order to avoid such an extra assumption, we shall now represent $\alpha_j^0$ for $h\mu_j^0\le L$ as the sum of two independent Gaussian random variables. Let $j_0 = j_0(h)$ be the largest $j$ for which $h\mu_j^0\le L$. Put

$$\sigma' = \frac{1}{C}h^K e^{-Ch^{-\beta}}, \text{ where } K\ge\rho(M+1),\ C\gg 1 \tag{9.3.13}$$

so that $\sigma'\le\frac{1}{2}\sigma_j$ for $1\le j\le j_0(h)$. The factor $h^K$ is needed only when $\beta=0$.

For $j\le j_0$, we may assume that $\alpha_j^0(\omega)=\alpha_j'(\omega)+\alpha_j''(\omega)$, where $\alpha_j'\sim\mathcal{N}(0,(\sigma')^2)$, $\alpha_j''\sim\mathcal{N}(0,(\sigma_j'')^2)$ are independent random variables and

$$\sigma_j^2 = (\sigma')^2 + (\sigma_j'')^2,$$

so that

$$\sigma_j'' = \sqrt{\sigma_j^2-(\sigma')^2}\asymp\sigma_j.$$

Put $q_\omega^1 = q_\omega' + q_\omega''$, where

$$q_\omega' = \sum_{h\mu_j^0\le L}\alpha_j'(\omega)\epsilon_j,\ q_\omega'' = \sum_{h\mu_j^0\le L}\alpha_j''(\omega)\epsilon_j.$$

Now (cf (9.3.10)) we write

$$P + h^m q_\omega^0 = (P + h^m(q_\omega''+q_\omega^2)) + h^m q_\omega'.$$

Theorem 8.1 is valid for random perturbations of

$$P_0 := P + h^m(q_\omega''+q_\omega^2),$$

provided that $\|h^m(q''_\omega + q^2_\omega)\|_{H^s} \leq h$, which again holds with a probability as in (9.3.12). The new random perturbation is now $h^m q'_\omega$ which we write as $\tau_0 h^{2N_1+n}\widetilde{q}_\omega$, where $\widetilde{q}_\omega$ takes the form

$$\widetilde{q}_\omega(x) = \sum_{0<h\mu_j^0\leq L} \beta_j(\omega)\epsilon_j, \tag{9.3.14}$$

with new independent random variables

$$\beta_j = \frac{1}{\tau_0}h^{m-2N_1-n}\alpha'_j(\omega) \sim \mathcal{N}(0, (\frac{1}{\tau_0}h^{m-2N_1-n}\sigma'(h))^2). \tag{9.3.15}$$

Now, by Proposition 7.5,

$$\mathbf{P}(|\beta|^2_{\mathbf{C}^D} > R^2) \leq \exp(\mathcal{O}(1)D\frac{h^{m-2N_1-n}\sigma'(h)}{\tau_0} - \frac{R^2\tau_0^2}{\mathcal{O}(1)(h^{m-2N_1-n}\sigma'(h))^2}).$$

Here by Weyl's law for the distribution of eigenvalues of elliptic self-adjoint differential operators, we have $D \asymp (L/h)^n$. Moreover, $L, R$ behave like certain powers of $h$.

- In the case when $\beta = 0$, we choose $\tau_0 = h^{1/2}$. Then for any $a > 0$ we get
$$\mathbf{P}(|\beta|_{\mathbf{C}^D} > R) \leq C\exp(-\frac{1}{Ch^a})$$
for any given fixed $a$, provided we choose $K$ large enough in (9.3.13).
- In the case $\beta > 0$ we get the same conclusion with $\tau_0 = h^{-K}\sigma'$ if $K$ is large enough.

In both cases, we see that the independent random variables $\beta_j$ in (9.3.14), (9.3.15) have a joint probability density $C(h)e^{\Phi(\alpha;h)}L(d\alpha)$, satisfying (8.1.15) for some $N_4$ depending on $K$.

With $\kappa = 1/2$, we put

$$\epsilon_0(h) = h^\kappa((\ln\frac{1}{h})^2 + \ln\frac{1}{\tau_0}),$$

where $\tau_0$ is chosen as above. Notice that $\epsilon_0(h)$ is of the order of magnitude $h^{\kappa-\beta}$ up to a power of $\ln\frac{1}{h}$. Then Theorem 8.1 gives:

**Proposition 9.7.** *There exists a constant $N_4 > 0$ depending on $\rho, n, m$ such that the following holds: Let $\Gamma \Subset \dot{\mathbf{C}}$ have piecewise smooth boundary. Then $\exists C > 0$ such that for $0 < r \leq 1/C$, $\widetilde{\epsilon} \geq C\epsilon_0(h)$, we have with probability*

$$\geq 1 - \frac{C\epsilon_0(h)}{rh^{n+\max(n(M+1),N_4+\widetilde{M})}}e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} - Ce^{-\frac{1}{Ch}}, \tag{9.3.16}$$

*that*

$$|\#(h^m P^0_\omega) \cap \Gamma) - \frac{1}{(2\pi h)^n}\mathrm{vol}\,(p^{-1}(\Gamma))| \leq \frac{C}{h^n}(\frac{\widetilde{\epsilon}}{r} + \mathrm{vol}\,(p^{-1}(\partial\Gamma + D(0,r)))). \tag{9.3.17}$$

As already noted, this gives Weyl asymptotics provided that

$$\mathrm{vol}\, p^{-1}(\partial\Gamma + D(0,r)) = \mathcal{O}(r^\alpha), \tag{9.3.18}$$

for some $\alpha \in ]0,1]$ (which would automatically be the case if $\kappa$ had been larger than 1/2 instead of being equal to 1/2), and we can then choose $r = \widetilde{\epsilon}^{1/(1+\alpha)}$, so that the right hand side of (9.3.17) becomes $\leq C\widetilde{\epsilon}^{\frac{\alpha}{1+\alpha}}h^{-n}$.

As in [97, 98] we also observe that if $\Gamma$ belongs to a family $\mathcal{G}$ of domains satisfying the assumptions of the Proposition uniformly, then with probability

$$\geq 1 - \frac{C\epsilon_0(h)}{r^2 h^{n+\max(n(M+1),N_4+\widetilde{M})}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} - Ce^{-\frac{1}{Ch}}, \tag{9.3.19}$$

the estimate (9.3.17) holds uniformly and simultaneously for all $\Gamma \in \mathcal{G}$.

## 9.4. End of the proof

Let $\theta_1, \theta_2, N_0$ be as in Theorem 9.1, so that $P(\theta_1, N_0)$ and $P(\theta_2, N_0)$ hold. Combining the propositions 9.3, 9.4, 9.5, we see that (9.3.18) holds with $\alpha = 1/N_0$ when $\Gamma = \Gamma^g_{\theta_1,\theta_2;1,\lambda}$, $\lambda > 0$ fixed, and Proposition 9.7 gives:

**Proposition 9.8.** *With the parameters as in Proposition 9.7 and for every* $\alpha \in ]0, \frac{1}{N_0}[$, *we have with probability*

$$\geq 1 - \frac{C\epsilon_0(h)}{\widetilde{\epsilon}^{\frac{N_0}{1+N_0}} h^{n+\max(n(M+1),N_4+\widetilde{M})}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} - Ce^{-\frac{1}{Ch}} \tag{9.4.1}$$

*that*

$$|\#(\sigma(h^m P_\omega) \cap \Gamma^g_{\theta_1,\theta_2;1,\lambda}) - \frac{1}{(2\pi h)^n} \mathrm{vol}\,(p^{-1}(\Gamma^g_{\theta_1,\theta_2;1,\lambda}))| \leq C \frac{\widetilde{\epsilon}^{\frac{1}{1+N_0}}}{h^n}. \tag{9.4.2}$$

*Moreover, the conclusion (9.4.2) is valid simultaneously for all* $\lambda \in [1, 2]$ *and all* $(\theta_1, \theta_2)$ *in a set where* $P(\theta_1, N_0)$, $P(\theta_2, N_0)$ *hold uniformly, with probability*

$$\geq 1 - \frac{C\epsilon_0(h)}{\widetilde{\epsilon}^{\frac{2N_0}{1+N_0}} h^{n+\max(n(M+1),N_4+\widetilde{M})}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} - Ce^{-\frac{1}{Ch}}. \tag{9.4.3}$$

For $0 < \delta \ll 1$, choose $\widetilde{\epsilon} = h^{-\delta}\epsilon_0 \leq Ch^{\frac{1}{2}-\beta-\delta}(\ln \frac{1}{h})^2$, so that $\widetilde{\epsilon}/\epsilon_0 = h^{-\delta}$. Then for some $N_5$ we have for every $\alpha \in ]0, 1/N_0[$ that

$$|\#(\sigma(h^m P_\omega) \cap \Gamma^g_{\theta_1,\theta_2;1,\lambda}) - \frac{1}{(2\pi h)^n} \mathrm{vol}\,(p^{-1}(\Gamma^g_{\theta_1,\theta_2;1,\lambda}))| \leq \frac{C_\alpha}{h^n} (h^{\frac{1}{2}-\delta-\beta}(\ln \frac{1}{h})^2)^{\frac{1}{1+N_0}}, \tag{9.4.4}$$

simultaneously for $1 \leq \lambda \leq 2$ and all $(\theta_1, \theta_2)$ in a set where $P(\theta_1, N_0)$, $P(\theta_2, N_0)$ hold uniformly, with probability

$$\geq 1 - \frac{C}{h^{N_5}} e^{-\frac{1}{Ch^\delta}}. \tag{9.4.5}$$

The upper bound in (9.4.4) can be replaced by

$$\frac{C_\delta}{h^n} h^{(\frac{1}{2}-\beta-2\delta)/(N_0+1)}.$$

Assuming $P(\theta_1, N_0)$, $P(\theta_2, N_0)$, we want to count the number of eigenvalues of $P_\omega$ in

$$\Gamma_{1,\lambda} = \Gamma^g_{\theta_1,\theta_2;1,\lambda}$$

when $\lambda \to \infty$. Let $k(\lambda)$ be he largest integer $k$ for which $2^k \leq \lambda$ and decompose

$$\Gamma_{1,\lambda} = (\bigcup_0^{k(\lambda)-1} \Gamma_{2^k,2^{k+1}}) \cup \Gamma_{2^{k(\lambda)},\lambda}.$$

In order to count the eigenvalues of $P_\omega^0$ in $\Gamma_{2^k,2^{k+1}}$ we define $h$ by $h^m 2^k = 1$, $h = 2^{-k/m}$, so that

$$\begin{aligned}\#(\sigma(P_\omega^0)\cap\Gamma_{2^k,2^{k+1}}) &= \#(\sigma(h^m P_\omega^0)\cap\Gamma_{1,2}),\\ \frac{1}{(2\pi)^n}\mathrm{vol}\,(p^{-1}(\Gamma_{2^k,2^{k+1}})) &= \frac{1}{(2\pi h)^n}\mathrm{vol}\,(p^{-1}(\Gamma_{1,2})).\end{aligned}$$

Thus, with probability $\geq 1 - C2^{\frac{N_5 k}{m}}e^{-2^{\frac{\delta k}{m}}/C}$ we have

$$|\#(\sigma(P_\omega^0)\cap\Gamma_{2^k,2^{k+1}}) - \frac{1}{(2\pi)^n}\mathrm{vol}\,p^{-1}(\Gamma_{2^k,2^{k+1}})| \leq C_\delta 2^{\frac{kn}{m}} 2^{-\frac{k}{m}(\frac{1}{2}-\beta-2\delta)\frac{1}{N_0+1}}. \quad (9.4.6)$$

Similarly, with probability $\geq 1 - C2^{N_5 k(\lambda)/m}e^{-2^{\delta k(\lambda)/m}/C}$, we have

$$|\#(\sigma(P_\omega^0)\cap\Gamma_{2^{k(\lambda)},\widetilde{\lambda}}) - \frac{1}{(2\pi)^n}\mathrm{vol}\,p^{-1}(\Gamma_{2^{k(\lambda)},\widetilde{\lambda}})| \leq C_\delta \lambda^{\frac{n}{m}}\lambda^{-\frac{1}{m}(\frac{1}{2}-\beta-2\delta)\frac{1}{N_0+1}}, \quad (9.4.7)$$

simultaneously for all $\widetilde{\lambda}\in[\lambda,2\lambda[$.

Now, we proceed as in [10], using essentially the Borel–Cantelli lemma. Use that

$$\begin{aligned}\sum_{\ell}^{\infty} 2^{N_5\frac{k}{m}}e^{-2^{\delta\frac{k}{m}}/C} &= \mathcal{O}(1)2^{N_5\frac{\ell}{m}}e^{-2^{\delta\frac{\ell}{m}}/C},\\ \sum_{2^k\leq\lambda} 2^{k\frac{n}{m}}2^{-\frac{k}{m}(\frac{1}{2}-\beta-2\delta)\frac{1}{N_0+1}} &= \mathcal{O}(1)\lambda^{\frac{n}{m}-\frac{1}{m}(\frac{1}{2}-\beta-2\delta)\frac{1}{N_0+1}},\end{aligned}$$

to conclude that with probability $\geq 1 - C2^{N_5\frac{\ell}{m}}e^{-2^{\delta\frac{\ell}{m}}/C}$, we have

$$|\#(\sigma(P_\omega^0)\cap\Gamma_{2^\ell,\lambda}) - \frac{1}{(2\pi)^n}\mathrm{vol}\,p^{-1}(\Gamma_{2^\ell,\lambda})| \leq C_\delta\lambda^{\frac{n}{m}-\frac{1}{m}(\frac{1}{2}-\beta-2\delta)\frac{1}{N_0+1}}$$

for all $\lambda\geq 2^\ell$. This statement implies Theorem 9.1. □

**Proof** of Theorem 9.2. This is just a minor modification of the proof of Theorem 9.1. Indeed, we already used the second part of Proposition 9.7, to get (9.4.7) with the probability indicated there. In that estimate we are free to vary $(g,\theta_1,\theta_2)$ in $\mathcal{G}$ and the same holds for the estimate (9.4.6). With these modifications, the same proof gives Theorem 9.2. □

## 10. Some open problems.

- The distribution of *resonances* or scattering poles for certain self-adjoint operators like the Schrödinger operator is a very intriguing and difficult problem where many basic questions remain unanswered.

  Resonances can be viewed as eigenvalues of a non-self-adjoint operator, obtained from the original self-adjoint one by changing the Hilbert space. If we take a Schrödinger operator and add a random perturbation to the potential with support in some fixed compact set, we may ask wether with probability close to one the resonances obey some kind of Weyl asymptotics. In dimension 1 there is a classical result of Zworski [112], saying that we do have Weyl asymptotics without any random perturbations, but assuming a non-flatness condition near the end points of the convex hull of the support of the potential. In higher dimensions there are results by T. Christiansen [16, 17] and Christiansen and P. Hislop [18] saying that in the "generic case" there is (roughly) a lower bound on the number of scattering poles in a sequence of large discs which is of the same order of magnitude as would be prescribed by a reasonable Weyl law. The methods of Christiansen and Hislop use the analysis of several complex variables and it would be interesting to understand the relation with the methods developed in our lectures.

- The *damped wave equation* in its stationary version is an example of a non-self-adjoint operator which is close to a self-adjoint one. The eigenvalues are confined to a band parallel to the real axis and since the work of Marcus-Matseev [72] we know that the real parts obey the Weyl law with a good remainder estimate. As for the distribution of imaginary parts many results are known, G. Lebeau [66], Sjöstrand [96], N. Anantharaman [6], S. Nonnenmacher and E. Schenk [88], M. Hitrik, Sjöstrand, S. Vũ Ngọc, [53], [52].

  The following problem seems to be open: Suppose we add some randomness to the damping term, and that the underlying geodesic flow is not ergodic (assuming that we work on a compact manifold without boundary for simplicity). Then with probability close to 1, do the imaginary parts of the eigenvalues distribute according to a Weyl law formulated with the help of the time averages of the damping term?

- What about *statistical properties of eigenvalues*? Can it be true, for instance in the simplest one dimensional situations with Gaussian random variables in the perturbation as in [39], that we have Poisson distribution of the eigenvalues? What about correlations between eigenvalues? Can we have results similar to the ones that are known for the zeros of random polynomials? (Cf [90, 9].) A step in this direction may be the recent preprint of T. Christiansen and Zworski [19] where the authors study the expectation value of the number of eigenvalues in a domain for certain one-dimensional pseudodifferential operators with doubly periodic symbol.

- It would be of great interest to have the sharpest possible *bounds on the norm of the resolvent*. The general theory of non-self-adjoint operators only gives very weak upper bounds which can be sharpened near the boundary of the range of the symbol (like for instantance in Section 4). In the proofs of the various results on Weyl asymptotics for randomly perturbed operators it is quite clear that the random perturbation has the effect of improving the upper bounds on the resolvent and sometimes in dimension one we can even get a polynomial upper bound, as was observed by Bordeaux Montrieux [10]. It is currently not clear how far these improvements go in higher dimensions. Notice that for random matrices there are results showing that we can have a polynomial bound in terms of the size of the matrix. See M. Rudelson [87].

- Sometimes it is natural to have additional *symmetries*. For instance in the case of the Kramers-Fokker-Planck operator one would like to restrict the random perturbations to the class of such operators. Another such class is that of PT-symmetric operators where the question of reality of the spectrum seems to be of great importance. I recently showed that for elliptic PT symmetric operators with PT symmetric random perturbations, we have Weyl asymptotics with probabality close to 1. In particular *most PT symmetric operators have most of their eigenvalues away from the real axis*.

- A long term project may also be to apply the theory to *non-linear problems*.

PART III

# From resolvent bounds to semigroup bounds


**Abstract**

The purpose of this part is to revisit the proof of the Gearhardt-Prüss-Hwang-Greiner theorem for a semigroup $S(t)$, following the general idea of the proofs that we have seen in the literature and to get an explicit estimate on $\|S(t)\|$ in terms of bounds on the resolvent of the generator.


## 11. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $[0,+\infty[\ni t \mapsto S(t) \in \mathcal{L}(\mathcal{H},\mathcal{H})$ be a strongly continuous semigroup with $S(0) = I$. Recall that by the Banach-Steinhaus theorem, $\sup_J \|S(t)\| =: m(J)$ is bounded for every compact interval $J \subset [0,+\infty[$. Using the semigroup property it follows easily that there exist $M \geq 1$ and $\omega_0 \in \mathbb{R}$ such that $S(t)$ has the property

$$P(M,\omega_0): \quad \|S(t)\| \leq Me^{\omega_0 t},\ t \geq 0. \tag{11.0.1}$$

In fact, we have this for $0 \leq t < 1$ and for larger values of $t$, write $t = [t]+r$, $[t] \in \mathbb{N}$, $0 \leq r < 1$, and $S(t) = S(1)^{[t]}S(r)$.

Let $A$ be the generator of the semigroup (so that formally $S(t) = \exp tA$) and recall (cf. [30], Chapter II or [81]) that $A$ is closed and densely defined. We also recall ([30], Theorem II.1.10) that

$$(z-A)^{-1} = \int_0^\infty S(t)e^{-tz}dt, \quad \|(z-A)^{-1}\| \leq \frac{M}{\Re z - \omega_0,} \tag{11.0.2}$$

when $P(M,\omega_0)$ holds and $z$ belongs to the open half-plane $\Re z > \omega_0$.

Recall the Hille-Yoshida theorem ([30], Th. II.3.5) according to which the following three statements are equivalent when $\omega \in \mathbb{R}$:

- $P(1,\omega)$ holds.
- $\|(z-A)^{-1}\| \leq (\Re z - \omega)^{-1}$, when $z \in \mathbb{C}$ and $\Re z > \omega$.
- $\|(\lambda - A)^{-1}\| \leq (\lambda - \omega)^{-1}$, when $\lambda \in ]\omega,+\infty[$.

Here we may notice that we get from the special case $\omega = 0$ to general $\omega$ by passing from $S(t)$ to $\widetilde{S}(t) = e^{-\omega t}S(t)$.

Also recall that there is a similar characterization of the property $P(M,\omega)$ when $M > 1$, in terms of the norms of all powers of the resolvent. This is the Feller-Miyadera-Phillips theorem ([30], Th. II.3.8). Since we need all powers of the resolvent, the practical usefulness of that result is less evident.

We next recall the Gearhardt-Prüss-Hwang-Greiner theorem, [30], Theorem V.I.11, [106], Theorem 19.1. See also [109], [24].

**Theorem 11.1.**
*(a) Assume that* $\|(z-A)^{-1}\|$ *is uniformly bounded in the half-plane* $\Re z \geq \omega$. *Then there exists a constant* $M > 0$ *such that* $P(M, \omega)$ *holds.*
*(b) If* $P(M, \omega)$ *holds, then for every* $\alpha > \omega$, $\|(z-A)^{-1}\|$ *is uniformly bounded in the half-plane* $\Re z \geq \alpha$.

The part (b) follows from (11.0.2) with $\omega_0$ replaced by $\omega$.

**The purpose of this part is to revisit the proof of (a), following the general idea of the proofs that we have seen in the literature and to get an explicit $t$ dependent estimate on $e^{-\omega t}\|S(t)\|$, implying explicit bounds on $M$.**

This idea is essentially to use that the resolvent and the inhomogeneous equation $(\partial_t - A)u = w$ in exponentially weighted spaces are related via Fourier-Laplace transform and we can use Plancherel's formula. Variants of this simple idea have also been used in more concrete situations. See [13, 32, 49, 89].

Note that we can improve a little the conclusion of (a). If the property (a) is true for some $\omega$ then it is automatically true for some $\omega' < \omega$. We recall indeed the following

**Lemma 11.2.**
*If for some* $r(\omega) > 0$, $\|(z-A)^{-1}\| \leq \frac{1}{r(\omega)}$ *for* $\Re z > \omega$, *then for every* $\omega' \in ]\omega - r(\omega), \omega]$ *we have*

$$\|(z-A)^{-1}\| \leq \frac{1}{r(\omega) - (\omega - \omega')},\ \Re z > \omega'.$$

**Proof.** Let $\tilde{z} \in \mathbb{C}$, $\Re \tilde{z} > \omega$. Then $\|(\tilde{z}-A)^{-1}\| \leq \frac{1}{r(\omega)}$. For $z \in \mathbb{C}$ with $|z - \tilde{z}| < r(\omega)$, we have

$(z-A)(\tilde{z}-A)^{-1} = 1 + (z-\tilde{z})(\tilde{z}-A)^{-1}$, where $\|(z-\tilde{z})(\tilde{z}-A)^{-1}\| \leq |z-\tilde{z}|/r(\omega) < 1$,

so $1 + (z-\tilde{z})(\tilde{z}-A)^{-1}$ is invertible and

$$\|(1 + (z-\tilde{z})(\tilde{z}-A)^{-1})^{-1}\| \leq \frac{1}{1 - |z-\tilde{z}|/r(\omega)}.$$

Hence $z$ belongs to the resolvent set of $A$ and

$$(z-A)^{-1} = (\tilde{z}-A)^{-1}(1 + (z-\tilde{z})(\tilde{z}-A)^{-1})^{-1},\ \|(z-A)^{-1}\| \leq \frac{1}{r(\omega) - |z-\tilde{z}|}.$$

Now, if $z \in \mathbb{C}$ and $\Re z > \omega'$, we can find $\tilde{z} \in \mathbb{C}$ with $\Re z > \omega$, $|z - \tilde{z}| < \omega - \omega'$ and the lemma follows. □

**Remark 11.3.**
*Let*

$$\omega_0 = \inf\{\omega \in \mathbb{R}\, \{z \in \mathbb{C}; \Re z > \omega\} \subset \rho(A) \text{ and } \sup_{\Re z > \omega} \|(z-A)^{-1}\| < \infty\}.$$

*For* $\omega > \omega_0$, *we may define* $r(\omega)$ *by*

$$\frac{1}{r(\omega)} = \sup_{\Re z > \omega} \|(z-A)^{-1}\|.$$

*Then $r(\omega)$ is an increasing function of $\omega$; for every $\omega \in ]\omega_0, \infty[$, we have $\omega - r(\omega) \geq \omega_0$ and for $\omega' \in [\omega - r(\omega), \omega]$ we have*

$$r(\omega') \geq r(\omega) - (\omega - \omega').$$

*We may state all this more elegantly by saying that $r$ is a Lipschitz function on $]\omega_0, +\infty[$ satisfying*

$$0 \leq \frac{dr}{d\omega} \leq 1\,.$$

*Moreover, if $\omega_0 > -\infty$, then $r(\omega) \to 0$ when $\omega \searrow \omega_0$.*

**Remark 11.4.**

*Notice that by (11.0.1), (11.0.2), we already know that $\|(z - A)^{-1}\|$ is uniformly bounded in the half-plane $\Re z \geq \beta$, if $\beta > \omega_0$. If $\alpha \leq \omega_0$, we see that $\|(z - A)^{-1}\|$ is uniformly bounded in the half-plane $\Re z \geq \alpha$, provided that*

- *we have this uniform boundedness on the line $\Re z = \alpha$,*
- *$A$ has no spectrum in the half-plane $\Re z \geq \alpha$,*
- *$\|(z - A)^{-1}\|$ does not grow too wildly in the strip*
  $$\alpha \leq \Re z \leq \beta : \|(z - A)^{-1}\| \leq \mathcal{O}(1) \exp(\mathcal{O}(1) \exp(k|\Im z|)),$$
  *where $k < \pi/(\beta - \alpha)$.*

*We then also have*

$$\sup_{\Re z \geq \alpha} \|(z - A)^{-1}\| = \sup_{\Re z = \alpha} \|(z - A)^{-1}\|. \qquad (11.0.3)$$

*This follows from the subharmonicity of $\ln ||(z - A)^{-1}||$, Hadamard's theorem (or Phragmén-Lindelöf in exponential coordinates) and the maximum principle.*

Our main result is:

**Theorem 11.5.**

*We make the assumptions of Theorem 11.1, (a) and define $r(\omega) > 0$ by*

$$\frac{1}{r(\omega)} = \sup_{\Re z \geq \omega} \|(z - A)^{-1}\|.$$

*Let $m(t) \geq \|S(t)\|$ be a continuous positive function. Then for all $t, a, \widetilde{a} > 0$, such that $t = a + \widetilde{a}$, we have*

$$\|S(t)\| \leq \frac{e^{\omega t}}{r(\omega) \|\frac{1}{m}\|_{e^{-\omega \cdot} L^2([0,a])} \|\frac{1}{m}\|_{e^{-\omega \cdot} L^2([0,\widetilde{a}])}}. \qquad (11.0.4)$$

Here the norms are always the natural ones obtained from $\mathcal{H}$, $L^2$, thus for instance $\|S(t)\| = \|S(t)\|_{\mathcal{L}(\mathcal{H},\mathcal{H})}$, if $u$ is a function on $\mathbb{R}$ with values in $\mathbb{C}$ or in $\mathcal{H}$, $\|u\|$ denotes the natural $L^2$ norm, when the norm is taken over a subset $J$ of $\mathbb{R}$, this is indicated with a "$L^2(J)$". In (11.0.4) we also have the natural norm in the exponentially weighted space $e^{-\omega \cdot} L^2([0, a])$ and similarly with $\widetilde{a}$ instead of $a$; $\|f\|_{e^{-\omega \cdot} L^2([0,a])} = \|e^{\omega \cdot} f(\cdot)\|_{L^2([0,a])}$.

As we shall see in the next section, under the assumption of the theorem, we have $P(M, \omega)$ with an explicit $M$. See also the last section.

We also have the following variant of the main result that can be useful in problems of return to equilibrium.

**Theorem 11.6.**
*We make the assumptions of Theorem 11.5, so that (11.0.4) holds. Let $\widetilde{\omega} < \omega$ and assume that $A$ has no spectrum on the line $\Re z = \widetilde{\omega}$ and that the spectrum of $A$ in the half-plane $\Re z > \widetilde{\omega}$ is compact (and included in the strip $\widetilde{\omega} < \Re z < \omega$). Assume that $\|(z-A)^{-1}\|$ is uniformly bounded on $\{z \in \mathbb{C};\, \Re z \geq \widetilde{\omega}\} \setminus U$, where $U$ is any neighborhood of $\sigma_+(A) := \{z \in \sigma(A);\, \Re z > \widetilde{\omega}\}$ and define $r(\widetilde{\omega})$ by*

$$\frac{1}{r(\widetilde{\omega})} = \sup_{\Re z = \widetilde{\omega}} \|(z-A)^{-1}\|.$$

*Then for every $t > 0$,*

$$S(t) = S(t)\Pi_+ + R(t) = S(t)\Pi_+ + S(t)(1-\Pi_+),$$

*where for all $a, \tilde{a} > 0$ with $a + \tilde{a} = t$,*

$$\|R(t)\| \leq \frac{e^{\widetilde{\omega} t}}{r(\widetilde{\omega}) \|\frac{1}{m}\|_{e^{-\widetilde{\omega}\cdot} L^2([0,a])} \|\frac{1}{m}\|_{e^{-\widetilde{\omega}\cdot} L^2([0,\widetilde{a}])}} \|I - \Pi_+\|. \tag{11.0.5}$$

*Here $\Pi_+$ denotes the spectral projection associated to $\sigma_+(A)$:*

$$\Pi_+ = \frac{1}{2\pi i} \int_{\partial V} (z-A)^{-1} dz,$$

*where $V$ is any compact neighborhood of $\sigma_+(A)$ with $C^1$ boundary, disjoint from $\sigma(A) \setminus \sigma_+(A)$.*

## 12. Applications : Explicit bounds in the abstract framework

Theorem 11.5 has two ingredients: the existence of some initial control by $m(t)$ and the additional information on the resolvent.

### 12.1. A quantitative Gearhardt-Prüss statement

As observed in the introduction (see (11.0.1)), we have at least an estimate with $m(t) = \widehat{M} \exp \widehat{\omega} t$, for some $\widehat{\omega} \geq \omega$. We apply Theorem 11.5 with this $m(t)$ and $a = \tilde{a} = \frac{t}{2}$. The term appearing in the denominator of (11.0.4) becomes

$$\|\frac{1}{m}\|_{e^{-\omega\cdot} L^2([0,a])} \|\frac{1}{m}\|_{e^{-\omega\cdot} L^2([0,\widetilde{a}])} = \frac{1}{2} \widehat{M}^{-2}\, t\,, \tag{12.1.1}$$

if $\widehat{\omega} = \omega$, and

$$= \frac{1}{2\widehat{M}^2(\widehat{\omega} - \omega)} \left[1 - \exp((\omega - \widehat{\omega})t)\right]\,, \tag{12.1.2}$$

if $\widehat{\omega} > \omega$.

Hence we obtain the estimate with a new $m^{new}(t)$, with

$$m^{new}(t) = \frac{2\widehat{M}^2(\widehat{\omega} - \omega)}{r(\omega)[1 - \exp((\omega - \widehat{\omega})t)]}\, \exp \omega t.$$

This gives in particular that $S(t)$ satisfies $P(M, \omega)$, with

$$M = \sup_t \left(\exp -\omega t\, \min(\widehat{M} \exp \widehat{\omega} t, m^{new}(t))\right)\,.$$

We will see how to optimize over $\omega$ in Section 12.3.

Let us push the computation. Without loss of generality, we can assume $\widehat{\omega} = 0$ and we make the assumption in Theorem 11.5 for some $\omega < 0$. Combining Theorem 11.5 and the trivial estimate

$$||S(t)|| \leq \widehat{M} = \widehat{M} \exp -\omega t \, \exp \omega t$$

we obtain that we have $P(M, \omega)$ with

$$M = \widehat{M} \sup_t \left( \min(\exp -\omega t, \frac{2\widehat{M}|\omega|}{r(\omega)(1-\exp \omega t)}) \right).$$

This can be rewritten in the form:

$$M = \widehat{M} \sup_{u \in ]0,1[} \left( \min(\frac{1}{u}, \frac{2\widehat{M}|\omega|}{r(\omega)(1-u)} \right) = 1 + 2\frac{\widehat{M}|\omega|}{r(\omega)}.$$

**Proposition 12.1.**
*Let $S(t)$ be a continuous semigroup such that $P(\widehat{M}, \widehat{\omega})$ is satisfied for some pair $(\widehat{M}, \widehat{\omega})$ and such that $r(\omega) > 0$ for some $\omega < \widehat{\omega}$. Then:*

$$||S(t)|| \leq \widehat{M} \left( 1 + \frac{2\widehat{M}(\widehat{\omega} - \omega)}{r(\omega)} \right) \exp \omega t. \qquad (12.1.3)$$

## 12.2. Estimate with exponential gain.

In the same spirit, and combining with Lemma 11.2, we get the following extension of (12.1.3) (with $\widehat{\omega} = 0$)

$$||S(t)|| \leq \widehat{M} \left( \frac{(1-s)r(\omega) + 2\widehat{M}(\widehat{\omega} - \omega + sr(\omega))}{(1-s)r(\omega)} \right) \exp(\omega - sr(\omega))t, \ \forall s \in [0,1[. \qquad (12.2.1)$$

Taking $s = \frac{t}{1+t}$ gives a rather optimal decay at $\infty$ in $\mathcal{O}(t) \exp(\omega - r(\omega))t$.

If we assume now instead the control of the norm of the resolvent on $\Re z \geq 0$, hence if we are in the case $\omega = \widehat{\omega} = 0$, we get

$$||S(t)|| \leq \frac{2\widehat{M}}{r(0)t},$$

and using the semi-group property $\leq \left( \frac{2\widehat{M}N}{r(0)t} \right)^N$, for any $N \geq 1$. Hence we can get an explicit control of the decay of $S(t)$, by optimizing over $N$. As in the theory of analytic symbols, we can take $N = E(\alpha t)$ where $E(s)$ denotes the integer part of $s$ and $\alpha$ such that $\alpha < r(0)/(2\widehat{M})$, we get an exponential decay of $S(t)$.
Alternately, we can use the extension of the resolvent on $\Re z > -sr(0)$ and this leads to :

$$||S(t)|| \leq \widehat{M} \left( \frac{(1-s) + 2\widehat{M}s}{(1-s)} \right) \exp(-sr(0))t, \ \forall s \in [0,1[. \qquad (12.2.2)$$

## 12.3. The limit $\omega \searrow \omega_0$

Consider the situation of Theorem 11.5 and let $\omega_0$ be as in Remark 11.3. Assume that $\omega_0 > -\infty$ so that $r(\omega) \to 0$, when $\omega \to \omega_0$. For $t \geq 1$, $\omega > \omega_0$, we get from (11.0.4):

$$e^{-\omega_0 t}\|S(t)\| \leq \frac{e^{t(\omega-\omega_0)}}{r(\omega)\int_0^{1/2} m(s)^{-2}e^{2\omega_0 s}ds} \leq \mathcal{O}(1)\frac{e^{t(\omega-\omega_0)}}{r(\omega)}. \tag{12.3.1}$$

Optimizing over $\omega \in ]\omega_0, \omega_0+\epsilon_0]$, we get the existence of $C$ such that

$$e^{-\omega_0 t}\|S(t)\| \leq C\exp\Phi(t)\,, \tag{12.3.2}$$

with

$$\Phi(t) = \inf_{\omega\in]\omega_0,\omega_0+\epsilon_0]} t(\omega-\omega_0) - \ln r(\omega)\,.$$

It is clear that $\lim_{t\to+\infty}\Phi(t)/t = 0$, but to have a more quantitative version, we need some information on the behavior of $r(\omega)$ as $\omega \searrow \omega_0$. Let us treat two examples.

If

$$r(\omega) \geq \frac{(\omega-\omega_0)^k}{C},\ \text{when } 0<\omega-\omega_0 \ll 1,$$

for some constants $C, k > 0$, then choosing $\omega - \omega_0 = k/t$ in (12.3.1), we get

$$e^{-\omega_0 t}\|S(t)\| \leq \mathcal{O}(1)t^k,\ t \geq 1.$$

On the other hand, if

$$r(\omega) \geq \exp -\frac{(\omega-\omega_0)^{-\alpha}}{C\alpha},\ \text{when } 0<\omega-\omega_0 \ll 1,$$

for some constants $C, \alpha > 0$, then

$$\frac{e^{t(\omega-\omega_0)}}{r(\omega)} \leq \exp\left(t(\omega-\omega_0) + \frac{(\omega-\omega_0)^{-\alpha}}{C\alpha}\right),$$

and choosing $\omega-\omega_0 = (Ct)^{-\frac{1}{\alpha+1}}$, we get the existence of a constant $\widehat{C}$ such that

$$e^{-\omega_0 t}\|S(t)\| \leq e^{\widehat{C}t^{\frac{\alpha}{\alpha+1}}},\ t \geq 1.$$

# 13. Applications to concrete examples

## 13.1. The complex Airy operator on the half-line

Let us consider (as in [5]) the Dirichlet realization $P^D$ of the Airy operator on $\mathbb{R}^+$ : $D_x^2 + ix$ and $P$ the realization of $D_x^2 + ix$ in $\mathbb{R}$. One can determine explicitly its spectrum (using Sibuya's theory or Combes-Thomas's trick) as

$$\sigma(P^D) := \{\lambda_j\, e^{i\frac{\pi}{3}}\,,\, j \in \mathbb{N}^*\}$$

where the $\lambda_j$'s are the eigenvalues (immediately related to the zeroes of the Airy function) of the Dirichlet realization in $\mathbb{R}^+$ of $D_x^2 + x$.

It was shown in [40], that $||(P^D-z)^{-1}||$ is as $\Re z > 0$ and $\Im z \mapsto +\infty$ asymptotically equivalent to $||(P-\Re z)^{-1}||$ and that $||(P^D-z)^{-1}||$ tends to 0 as $\Re z > 0$ and $\Im z \mapsto -\infty$. The standard Gearhardt-Prüss theorem, applied to $A := -P^D$, permits to show that, for any $\omega > -\lambda_1 \cos\frac{\pi}{3}$, we have

$$||S(t)|| \leq M_\omega \exp(\omega t)\,.$$

Theorem 11.6 permits the following improvment :

$$S(t) = \exp\left(-e^{i\frac{\pi}{3}}\,\lambda_1\,t\right)\;\Pi_+ + R(t)\,,$$

with

$$||R(t)|| \leq M_{\tilde{\omega}}\exp(\tilde{\omega} t)\,,$$

for any $\tilde{\omega} > -\lambda_2\,\cos\frac{\pi}{3}$ .
Here $\Pi_+$ is the projector associated with the eigenfunction of $P^D$ associated with $\lambda_1\,e^{i\frac{\pi}{3}}$ . Hence we get a much better control of the semi-group.

## 13.2. The case of the Kramers-Fokker-Planck operator

Inspired by the work by F. Hérau and F. Nier [47], F. Hérau, J. Sjöstrand and C. Stolk [48] studied the Kramers-Fokker-Planck operator

$$P = y\cdot h\partial_x - V'(x)\cdot h\partial_x + \frac{\gamma}{2}(y-h\partial_y)(y+h\partial_y) \tag{13.2.1}$$

on $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_y$, where $\gamma > 0$ is fixed and we let $h \to 0$. We assume that $V \in C^\infty(\mathbb{R}^n;\mathbb{R})$ with $\partial^\alpha V = \mathcal{O}(1)$ for every $\alpha \in \mathbb{N}^n$ of length $\geq 2$ and we also assume that $V$ is a Morse function such that $|\nabla V(x)| \geq 1/C$ when $|x| \geq C$ for some constant $C > 0$. Then we know from [47] and under much weaker assumptions from B. Helffer, F. Nier [41] that $P$ is maximally accretive with $\Re P \geq 0$, so that $P$ generates a semi-group of contractions $e^{-tP/h}$, $t \geq 0$. In particular the spectrum of $P$ is contained in the closed right half plane. In [48] it was shown that for every fixed $C > 0$ and for $h > 0$ small enough, the spectrum of $P$ in the strip $0 \leq \Re z \leq Ch$ is discrete and the eigenvalues are of the form

$$E_j = \lambda_j h + o(h), \quad \Re\lambda_j \leq Ch, \tag{13.2.2}$$

where $\lambda_j$ are eigenvalues of the different quadratic approximations of $P_{h=1}$ at the various points $(x_k, 0)$ where $V'(x_k) = 0$. Here the points $E_j$ all belong to a sector $|\Im\lambda| \leq \mathcal{O}(\Re\lambda)$, so the eigenvalues in (13.2.2) are all confined to a disc $D(0,\widetilde{C}h)$.

It was also shown in [48] that if $\widetilde{\omega} \geq 0$ and $\Re\lambda_j \neq \widetilde{\omega}$ for all the eigenvalues $\lambda_j$, then $\|(P-z)^{-1}\| = \mathcal{O}(1/h)$ uniformly on the line $\Re z = h\widetilde{\omega}$. The same estimate holds when $0 \leq \Re z \leq Ch$ and $|z| \geq \widetilde{C}h$. Actually, using a form of semi-classical sub-ellipticity (closely related in spirit to the one established in [47] and further studied in [41]) it was also shown that this estimate holds in a larger parabolic neighborhood of $i\mathbb{R}$ away from the disc $D(0,\widetilde{C}h)$, and using this stronger result and a contour deformation in a standard integral representation of $e^{-tP/h}$ (again in the spirit of [47]) it was established in [48] that

$$e^{-tP/h} = e^{-tP/h}\Pi_+ + R(t), \tag{13.2.3}$$

where $\Pi_+$ is the spectral projection associated with $\{z \in \sigma(P);\, 0 \leq \Re z \leq \widetilde{\omega}\}$, and $\|R(t)\| \leq \text{Const.}\, e^{-t\widetilde{\omega}}$. Now this result becomes a direct application of Theorem 11.6 to $A := -P/h$ and we do not need any bounds on the resolvent in the region $\Re z > h\widetilde{\omega}$.

In [45, 46] similar results were obtained for more general operators, for which we do not necessarily have any bound on the resolvent beyond a strip, and the proof was to use microlocal coercivity outside a compact set in slightly weighted $L^2$-spaces. Again Theorem 11.6 would give some simplifications.

## 13.3. The complex harmonic oscillator

The complex harmonic oscillator

$$P := D_x^2 + ix^2$$

on the line was studied by E.B. Davies [21, 22], L. Boulton, [12] and M. Zworski [113] in connection with the analysis of the pseudospectra. As for the complex Airy operator, it is easy to determine the spectrum which is given by $e^{i\frac{\pi}{4}}(2j+1)\,,\, j\in\mathbb{N}$. This operator is maximally accretive and we can apply Theorem 11.6 with $A=-P$. From these works as well as those of K. Pravda-Starov [83] and Dencker-Sjöstrand-Zworski [26], we know that for fixed $\Re z$ as $\Im z\to+\infty$,

$$\lim_{\Im z\to+\infty} ||(P-z)^{-1}|| = 0\,.$$

More precisely, for any compact interval $K$, there exists $C>0$ such that

$$||(P-z)^{-1}|| \leq C\,|\Im z|^{-\frac{1}{3}}\,,\ \text{ for } \Im z\geq C, \Re z\in K\,.$$

This follows from [83, 26], notice here that the results in [26] are given in the semi-classical limit for the spectral parameter in a compact set, but there is a simple scaling argument, allowing to pass to the limit of high frequency. See for example [99] and Section 4.3. As $\Im z\to-\infty$ we have by more elementary estimates:

$$||(P-z)^{-1}|| \leq\ |\Im z|^{-1}\,,\ \text{ for } \Im z<0\,.$$

We can therefore apply Theorem 11.6 and get

$$S(t)=\exp\left(-e^{i\frac{\pi}{4}}\,t\right)\ \Pi_+ + R(t)\,,$$

with

$$||R(t)|| \leq M_{\tilde\omega}\exp(\tilde\omega t)\,,$$

for any $\tilde\omega > -3\,\cos\frac{\pi}{4}$ . Here $\Pi_+$ is the spectral projection associated with the eigenvalue $e^{i\frac{\pi}{4}}$ of $P$.

Hence we get again a much better control of the semi-group.

# 14. Proofs of the main statements

## 14.1. Proof of Theorem 11.5

As already mentioned, we shall use the inhomogeneous equation

$$(\partial_t - A)u = w \text{ on } \mathbb{R}. \tag{14.1.1}$$

Recall that if $v\in\mathcal{H}$, then $S(t)v\in C^0([0,\infty[;\mathcal{H})$, while if $v\in\mathcal{D}(A)$, then $S(t)v\in C^1([0,\infty[;\mathcal{H})\cap C^0([0,\infty[;\mathcal{D}(A))$ and

$$AS(t)v=S(t)Av,\quad (\partial_t - A)S(t)v=0\,. \tag{14.1.2}$$

Let $C_+^0(\mathcal{H})$ denote the subspace of all $v\in C^0(\mathbb{R};\mathcal{H})$ that vanish near $-\infty$. For $k\in\mathbb{N}$, we define $C_+^k(\mathcal{H})$ and $C_+^k(\mathcal{D}(A))$ similarly. For $w\in C_+^0(\mathcal{H})$, we define $Ew\in C_+^0(\mathcal{H})$ by

$$Ew(t)=\int_{-\infty}^t S(t-s)w(s)ds. \tag{14.1.3}$$

It is easy to see that $E$ is continuous: $C_+^k(\mathcal{H}) \to C_+^k(\mathcal{H})$, $C_+^k(\mathcal{D}(A)) \to C_+^k(\mathcal{D}(A))$ and if $w \in C_+^1(\mathcal{H}) \cap C_+^0(\mathcal{D}(A))$, then $u = Ew$ is the unique solution in the same space of (14.1.1). More precisely, we have

$$(\partial_t - A)Ew = w, \quad E(\partial_t - A)u = u, \tag{14.1.4}$$

for all $u, w \in C_+^1(\mathcal{H}) \cap C_+^0(\mathcal{D}(A))$

Now recall that we have $P(M, \omega_0)$ in (11.0.1) for some $M, \omega_0$. If $\omega_1 > \omega_0$ and $w \in C_+^0(\mathcal{H}) \cap e^{\omega_1 \cdot} L^2(\mathbb{R}; \mathcal{H})$ (by which we only mean that $w \in C_+^0(\mathcal{H})$ and that $\|w\|_{e^{\omega_1 \cdot} L^2(\mathbb{R};\mathcal{H})} < \infty$, avoiding to define the larger space $e^{\omega_1 \cdot} L^2(\mathbb{R}; \mathcal{H})$), then $Ew$ belongs to the same space and

$$\begin{aligned} \|Ew\|_{e^{\omega_1 \cdot} L^2(\mathbb{R};\mathcal{H})} &\leq \left( \int_0^\infty e^{-\omega_1 t} \|S(t)\| \, dt \right) \|w\|_{e^{\omega_1 \cdot} L^2(\mathbb{R};\mathcal{H})} \\ &\leq \frac{M}{\omega_1 - \omega_0} \|w\|_{e^{\omega_1 \cdot} L^2(\mathbb{R};\mathcal{H})}. \end{aligned}$$

Now we consider Laplace transforms. If $u \in e^{\omega \cdot} \mathcal{S}(\mathbb{R}; \mathcal{H})$, then the Laplace transform

$$\widehat{u}(\tau) = \int_{-\infty}^{+\infty} e^{-t\tau} u(t) dt$$

is well-defined in $\mathcal{S}(\Gamma_\omega; \mathcal{H})$, where

$$\Gamma_\omega = \{\tau \in \mathbb{C}; \Re\tau = \omega\}$$

and we have Parseval's identity

$$\frac{1}{2\pi} \|\widehat{u}\|_{L^2(\Gamma_\omega)}^2 = \|u\|_{e^{\omega \cdot} L^2}. \tag{14.1.5}$$

Now we make the assumptions in Theorem 11.5, define $\omega$ and $r(\omega)$ as there, and let $M, \omega_0$ be as above. Let $w \in e^{\omega \cdot} \mathcal{S}_+(\mathcal{D}(A))$, where $\mathcal{S}_+(\mathcal{D}(A))$ by definition is the space of all $u \in \mathcal{S}(\mathbb{R}; \mathcal{D}(A))$, vanishing near $-\infty$. Then $w \in e^{\omega_1 \cdot} \mathcal{S}_+(\mathcal{D}(A))$ for all $\omega_1 \geq \omega$. If $\omega_1 > \omega_0$ then $u := Ew$ belongs to $e^{\omega_1 \cdot} \mathcal{S}_+(\mathcal{D}(A))$ and solves (14.1.1). Laplace transforming that equation, we get

$$(\tau - A)\widehat{u}(\tau) = \widehat{w}(\tau), \tag{14.1.6}$$

for $\Re\tau > \omega_0$. Notice here that $\widehat{w}(\tau)$ is continuous in the half-plane $\Re\tau \geq \omega$, holomorphic in $\Re\tau > \omega$, and $\widehat{w}_{|\Gamma_{\widetilde{\omega}}} \in \mathcal{S}(\Gamma_{\widetilde{\omega}})$ for every $\widetilde{\omega} \geq \omega$. We use the assumption in the theorem to write

$$\widehat{u}(\tau) = (\tau - A)^{-1} \widehat{w}(\tau), \tag{14.1.7}$$

and to see that $\widehat{u}(\tau)$ can be extended to the half-plane $\Re\tau \geq \omega$ with the same properties as $\widehat{w}(\tau)$. By Laplace (Fourier) inversion from $\Gamma_\omega$ we conclude that $u \in e^{\omega \cdot} \mathcal{S}_+(\mathcal{D}(A))$. Moreover, since

$$\|\widehat{u}(\tau)\|_{\mathcal{H}} \leq \frac{1}{r(\omega)} \|\widehat{w}(\tau)\|_{\mathcal{H}}, \ \tau \in \Gamma_\omega,$$

we get from Parseval's identity that

$$\|u\|_{e^{\omega \cdot} L^2} \leq \frac{1}{r(\omega)} \|w\|_{e^{\omega \cdot} L^2}. \tag{14.1.8}$$

Using the density of $\mathcal{D}(A)$ in $\mathcal{H}$ together with standard cutoff and regularization arguments, we see that (14.1.8) extends to the case when $w \in e^{\omega \cdot} L^2(\mathbb{R}; \mathcal{H}) \cap C_+^0(\mathcal{H})$, leading to the fact that $u := Ew$ belongs to the same space and satisfies (14.1.8).

Consider $u(t) = S(t)v$, for $v \in D(A)$, solving the Cauchy problem

$$\begin{aligned}&(\partial_t - A)u = 0\,,\ t \geq 0\ ,\\ &u(0) = v\,.\end{aligned}$$

Let $\chi$ be a decreasing Lipschitz function on $\mathbb{R}$, equal to 1 on $]-\infty, 0]$ and vanishing near $+\infty$. Then

$$(\partial_t - A)(1-\chi)u = -\chi'(t)u\,,$$

and

$$\begin{aligned}\|\chi' u\|^2_{e^{\omega\cdot}L^2} &= \int_0^{+\infty} |\chi'(t)|^2 \|u(t)\|^2 e^{-2\omega t}\, dt\\ &\leq \|\chi' m\|^2_{e^{\omega\cdot}L^2}\ \|v\|^2,\end{aligned}$$

where we notice that $\chi' m$ is welldefined on $\mathbb{R}$ since $\operatorname{supp}\chi' \subset [0,\infty[$.

Now $(1-\chi)u$, $\chi' u$ are well-defined on $\mathbb{R}$, so

$$\|(1-\chi)u\|_{e^{\omega\cdot}L^2} \leq r(\omega)^{-1}\|\chi' u\|_{e^{\omega\cdot}L^2} \leq r(\omega)^{-1}\|\chi' m\|_{e^{\omega\cdot}L^2}\ \|v\|\,. \tag{14.1.9}$$

Strictly speaking, in order to apply (14.1.8), we approximate $\chi$ by a sequence of smooth functions. Similarly,

$$\|\chi u\|_{e^{\omega\cdot}L^2(\mathbb{R}_+)} \leq \|\chi m\|_{e^{\omega\cdot}L^2(\mathbb{R}_+)}\ \|v\|\,,$$

so

$$\|u\|_{e^{\omega\cdot}L^2(\mathbb{R}_+)} \leq \Big(r(\omega)^{-1}\|\chi' m\|_{e^{\omega\cdot}L^2} + \|\chi m\|_{e^{\omega\cdot}L^2(\mathbb{R}_+)}\Big)\ \|v\|\,.$$

Let us now go from $L^2$ to $L^\infty$. For $t > 0$, let $\chi_+(s) = \widetilde{\chi}(t-s)$ with $\widetilde{\chi}$ as $\chi$ above and in addition $\operatorname{supp}\widetilde{\chi} \subset ]-\infty, t]$, so that $\chi_+(t) = 1$ and $\operatorname{supp}\chi_+ \subset [0,\infty[$. Then

$$(\partial_s - A)\left(\chi_+(s)u(s)\right) = \chi_+'(s)u(s)\,,$$

and

$$\chi_+ u(t) = \int_{-\infty}^t S(t-s)\,\chi_+'(s)\,u(s)\,ds\,.$$

Hence, we obtain

$$\begin{aligned}e^{-\omega t}\|u(t)\| &= e^{-\omega t}\|\chi_+(t)u(t)\|\\ &\leq \textstyle\int_{-\infty}^t e^{-\omega t}\, m(t-s)|\widetilde{\chi}'(t-s)|\|u(s)\|\, ds\\ &\leq \textstyle\int_{-\infty}^t e^{-\omega(t-s)}\, m(t-s)\,|\widetilde{\chi}'(t-s)|\ e^{-\omega s}\|u(s)\|\, ds\\ &\leq \|m\widetilde{\chi}'\|_{e^{\omega\cdot}L^2}\ \|u\|_{e^{\omega\cdot}L^2(\operatorname{supp}\chi_+)}\,.\end{aligned} \tag{14.1.10}$$

Assume that

$$\chi = 0 \text{ on } \operatorname{supp}\chi_+\,. \tag{14.1.11}$$

Then $u$ can be replaced by $(1-\chi)u$ in the last line in (14.1.10) and using (14.1.9) we get

$$e^{-\omega t}\|u(t)\| \leq r(\omega)^{-1}\|m\chi'\|_{e^{\omega\cdot}L^2}\|m\widetilde{\chi}'\|_{e^{\omega\cdot}L^2}\|v\|\,. \tag{14.1.12}$$

Let

$$\operatorname{supp}\chi \subset ]-\infty, a]\,,\ \operatorname{supp}\widetilde{\chi} \subset ]-\infty, \widetilde{a}], a + \widetilde{a} = t\,, \tag{14.1.13}$$

so that (14.1.11) holds.

For a given $a > 0$, we look for $\chi$ in (14.1.13) such that $\|m\chi'\|_{e^{\omega\cdot}L^2}$ is as small as possible. By the Cauchy-Schwarz inequality,

$$1 = \int_0^a |\chi'(s)|ds \leq \|\chi' m\|_{e^{\omega\cdot}L^2}\|\frac{1}{m}\|_{e^{-\omega\cdot}L^2(]0,a[)}\,, \tag{14.1.14}$$

so

$$\|\chi' m\|_{e^{\omega\cdot}L^2} \geq \frac{1}{\|\frac{1}{m}\|_{e^{-\omega\cdot}L^2(]0,a[)}}. \tag{14.1.15}$$

We get equality in (14.1.15) if for some constant $C$,

$$|\chi'(s)|m(s)e^{-\omega s} = C\frac{1}{m(s)}e^{\omega s}, \text{ on } [0,a],$$

i.e.

$$\chi'(s)m(s)e^{-\omega s} = -C\frac{1}{m(s)}e^{\omega s}, \text{ on } [0,a],$$

where $C$ is determined by the condition $1 = \int_0^a |\chi'(s)|ds$.
We get

$$C = \frac{1}{\|\frac{1}{m}\|^2_{e^{-\omega\cdot}L^2(]0,a[)}},$$

Here $\chi(s) = 1$ for $s \leq 0$, $\chi(s) = 0$ for $s \geq a$,

$$\chi(s) = C\int_s^a \frac{1}{m(\sigma)^2}e^{2\omega\sigma}d\sigma, \ 0 \leq s \leq a.$$

With the similar optimal choice of $\widetilde{\chi}$, for which

$$\|\widetilde{\chi}' m\|_{e^{\omega\cdot}L^2} = \frac{1}{\|\frac{1}{m}\|_{e^{-\omega\cdot}L^2([0,\widetilde{a}])}},$$

we get from (14.1.12):

$$e^{-\omega t}\|u(t)\| \leq \frac{\|v\|}{r(\omega)\|\frac{1}{m}\|_{e^{-\omega\cdot}L^2([0,a])}\|\frac{1}{m}\|_{e^{-\omega\cdot}L^2([0,\widetilde{a}])}}, \tag{14.1.16}$$

provided that $a, \widetilde{a} > 0$, $a + \widetilde{a} = t$, for any $v \in D(A)$. Observing that $D(A)$ is dense in $\mathcal{H}$, this completes the proof of Theorem 11.5.

## 14.2. Proof of Theorem 11.6

We can apply Theorem 11.5 to the restriction $\widetilde{S}(t)$ of $S(t)$ to the range $\mathcal{R}(1 - \Pi_+)$ of $1 - \Pi_+$. The generator is the restriction $\widetilde{A}$ of $A$ so we get

$$\|\widetilde{S}(t)\| \leq \frac{e^{\widetilde{\omega}t}}{r(\widetilde{\omega})\|\frac{1}{m}\|_{e^{-\widetilde{\omega}\cdot}L^2([0,a])}\|\frac{1}{m}\|_{e^{-\widetilde{\omega}\cdot}L^2([0,\widetilde{a}])}}. \tag{14.2.1}$$

Then (11.0.5) follows from the fact that $R(t) = \widetilde{S}(t)(1 - \Pi_+)$.

# 15. An iterative improvement of Theorem 11.5

Working entirely on the semi-group side and applying Theorem 11.5 repeatedly, we shall see how to gain an extra decay $\mathcal{O}(1)\exp(-t^{1/2}/C)$ for some $C > 0$. It is not clear that this result is of practical use, especially in view of Lemma 11.2, but the computations are amusing.

Recall that under the assumptions of Theorem 11.5 we have the estimate (11.0.4). Here we may have $m$ bounded continuous for $0 \leq t < T$ and equal to $+\infty$ for $t \geq T$, where $T > 0$.

Write $m(t) = \widetilde{m}(t)e^{\omega t}$. Then (11.0.4) shows that $\|S(t)\| \le \widehat{m}(t)e^{\omega t}$, where

$$\widehat{m}(t) \le \frac{1}{r(\omega)\|\frac{1}{\widetilde{m}}\|_{[0,a]}\|\frac{1}{\widetilde{m}}\|_{[0,\widetilde{a}]}},\ a + \widetilde{a} = t. \tag{15.0.1}$$

Take $a = \widetilde{a} = t/2$ and divide the previous inequality by $r(\omega)$:

$$\frac{\widehat{m}(t)}{r(\omega)} \le \frac{1}{\int_0^{t/2}(\frac{r(\omega)}{\widetilde{m}(s)})^2\,ds},$$

which we can also write

$$\widehat{f}(t) \ge \int_0^{t/2} \widetilde{f}(s)^2\,ds\,,\ \widetilde{f}(t) := \frac{r(\omega)}{\widetilde{m}(t)},\ \widehat{f}(t) := \frac{r(\omega)}{\widehat{m}(t)}\,.$$

Now assume that $e^{-\omega t}\|S(t)\| \le \widetilde{m}(t) \le \mathcal{O}(1)$ for $0 \le t < T$. Then we extend $\widetilde{m}$ to $[0, +\infty[$, by defining

$$\frac{\widetilde{m}(t)}{r(\omega)} = \frac{1}{\int_0^{t/2}(\frac{r(\omega)}{\widetilde{m}(s)})^2\,ds}, \tag{15.0.2}$$

first for $T \le t < 2T$, then for $2T \le t < 4T$ and so on. Correspondingly, we have

$$\widetilde{f}(t) = \int_0^{t/2} \widetilde{f}(s)^2\,ds,\ t \ge T. \tag{15.0.3}$$

Theorem 11.5 now shows that $e^{-\omega t}\|S(t)\| \le \widetilde{m}(t) \le \mathcal{O}(1)$ for all $t \ge 0$. By construction we see that $\widetilde{m}(t)$ is decreasing on $[T, +\infty[$, so we have

$$e^{-\omega t}\|S(t)\| \le M,\ M = \max(\sup_{[0,T[}\widetilde{m}, \frac{1}{r(\omega)\int_0^{T/2}\widetilde{m}(s)^{-2}\,ds}). \tag{15.0.4}$$

Notice that $\widetilde{f}$ is increasing on $[T, +\infty[$. We look for upper bounds on $\widetilde{m}$ or equivalently for lower bounds on $\widetilde{f}$. For $k \ge 1$, put $I_k = [T2^{k-1}, T2^k[$, so that the length of $I_k$ is $|I_k| = T2^{k-1}$. Put

$$F(k) = \inf_{I_k}\widetilde{f} = \widetilde{f}(T2^{k-1}) \text{ when } k \ge 1, \quad F(0) = \inf_{[0,T[}\widetilde{f}(t)\,.$$

Then, $F(1) = \int_0^{T/2}\widetilde{f}(t)^2\,dt \ge \frac{T}{2}F(0)^2$, which we write

$$TF(1) \ge \frac{1}{2}(TF(0))^2\,.$$

For $k \ge 1$, we get

$$F(k+1) \ge \int_0^{T2^{k-1}} \widetilde{f}(t)^2dt \ge TF(0)^2 + TF(1)^2 + 2TF(2)^2 + ... + 2^{k-2}TF(k-1)^2,$$

which we write

$$TF(k+1) \ge (TF(0))^2 + (TF(1))^2 + 2(TF(2))^2 + ... + 2^{k-2}(TF(k-1))^2. \tag{15.0.5}$$

Since $\widetilde{f}$ is increasing on $[T, +\infty[$, we have

$$F(1) \le F(2) \le F(3) \le ...$$

Thus for $k \ge 2$,

$$TF(k+1) \ge 2^{k-2}(TF(k-1))^2 \ge 2^{k-2}(TF(1))^2 \ge 2^{k-4}(TF(0))^4\,,$$

which we write

$$TF(k) \ge 2^{k-5}(TF(0))^4\,,\ k \ge 3.$$

Let $k_0$ be the smallest integer $k \geq 3$ such that

$$2^{k-5}(TF(0))^4 \geq 2,$$

so that $TF(k) \geq 2$ for $k \geq k_0$.

Now return to (15.0.5) which implies that

$$TF(k+1) \geq 2^{k-2}(TF(k-1))^2,\ k \geq 1.$$

We get

$$TF(k+2) \geq 2^{k-1}(TF(k))^2,\ k \geq 1,$$

implying,

$$T(F(k+2)) \geq (TF(k))^2, \quad \ln(TF(k+2)) \geq 2\ln(TF(k)).$$

In particular,

$$\ln(TF(k_0+2\nu)) \geq 2^\nu \ln(TF(k_0)) \geq 2^\nu \ln 2,\ \nu \in \mathbb{N}.$$

We conclude that

$$T\widetilde{f}(t) \geq 2^{2^\nu},\ 2^{k_0+2\nu-1} \leq t/T < 2^{k_0+2\nu}.$$

The last inequality for $t$ implies that $2^\nu > (2^{-k_0}t/T)^{1/2}$, so we get

$$T\widetilde{f}(t) \geq 2^{(2^{-k_0}t/T)^{1/2}},\ t/T \geq 2^{k_0-1}, \tag{15.0.6}$$

or equivalently,

$$\frac{\widetilde{m}(t)}{r(\omega)T} \leq 2^{-(2^{-k_0}t/T)^{1/2}},\ t/T \geq 2^{k_0-1}, \tag{15.0.7}$$

where we recall that $k_0$ is the smallest integer such that

$$2^{k_0} \geq \max(\frac{2^6}{(TF(0))^4}, 8). \tag{15.0.8}$$

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