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Équations elliptiques non linéaires monotones avec un deuxième membre L^1 ou mesure

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Renormalized solutions of nonlinear elliptic equations with measure data

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Résumé

On considère le problème :

$$\begin{cases} -\operatorname{div} a(x, Du) = f & \text{dans}\, \Omega, \\ u = 0 \quad \text{sur}\, \partial\Omega, \end{cases}$$

où Ω est un ouvert borné de $\mathbf{R}^N$, où $a(x, \xi)$ est une fonction de Carathéodory, monotone en ξ, coercive, qui définit un opérateur dans $W_0^{1,p}(\Omega)$ (avec $1 < p \leq N$), et où f appartient à $L^1(\Omega)$ ou est une mesure bornée sur Ω. On introduit une nouvelle définition de la solution de ce problème, la notion de solution renormalisée (ou entropique), et on montre l'existence d'une telle solution et sa continuité par rapport à f. Quand f appartient à $L^1(\Omega)$, on montre en outre que cette solution est unique.

Consider the nonlinear elliptic problem

$$\text{(P)} \qquad \begin{cases} -\operatorname{div} a(x, \nabla u) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where the operator $u \to \operatorname{div} a(x, \nabla u)$ is a classical monotone operator from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$, and where μ belongs to $\mathcal{M}_b(\Omega)$, the space of bounded Radon measures on Ω. In this lecture I will present the definition of renormalized solution and recent results taken from our joint work with Gianni Dal Maso, Luigi Orsina and Alain Prignet.

In the case in which μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$ (which can be proved to be equivalent to the fact that μ does not charge the sets of zero p-capacity) the notion of "renormalized solution" of problem (P) has been introduced by P.-L. Lions & F. Murat, and independently, in an equivalent setting under the name of "entropy solution", by P. Benilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre & J.-L. Vazquez. These settings are successful since they allow one to prove the existence and uniqueness of the renormalized (or entropy) solution of problem (P), as well as its continuity with respect to μ (in a sense to be specified), if μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$.

In our joint work with Gianni Dal Maso, Luigi Orsina and Alain Prignet we extend the definition of renormalized (or entropy) solution to the general case of an arbitrary measure of $\mathcal{M}_b(\Omega)$, and we prove the existence of a renormalized solution as well as its continuity with respect to μ (in a sense to be specified). We also prove some results concerning uniqueness.

For the remaining of this abstract I will mainly concentrate on the case of an arbitrary measure, since the case where μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$ appears as a particular case of this setting. It can indeed be proved that every measure $\mu \in \mathcal{M}_b(\Omega)$ can be written (in a unique way as far as μ_0 and λ are concerned) as

$$\mu = \mu_0 + \lambda = f - \operatorname{div} g + \lambda^+ - \lambda^-$$

where $\mu_0 \in \mathcal{M}_b(\Omega)$ does not charge the sets of zero p-capacity (and can thus be written as $\mu_0 = f - \operatorname{div} g$, with $f \in L^1(\Omega)$ and $g \in (L^{p'}(\Omega))^N$), and where $\lambda \in \mathcal{M}_b(\Omega)$ is concentrated on a set of zero p-capacity.

Definition. *A function u is a renormalized solution of problem (P) if:*
(1) the function $u : \Omega \to \overline{\mathbf{R}}$ is a Lebesgue measurable function which is finite almost everywhere;
(2) one has

$$T_k(u) \in W_0^{1,p}(\Omega) \quad \text{for every} \quad k > 0$$

where $T_k : \mathbf{R} \to \mathbf{R}$ denotes the truncation at height k defined by $T_k(s) = s$ if $|s| \leq k$, and $T_k(s) = ks/|s|$ if $|s| \geq k$;
(3) one has

$$|\nabla u|^{p-1} \in L^q(\Omega) \quad \text{for every } q \quad \text{such that} \quad 1 < q < \frac{N}{N-1}$$

where ∇u denotes the approximate gradient of u, which is defined as the unique Lebesgue measurable function $\nabla u : \Omega \to \overline{\mathbf{R}}^N$ such that one has for every $k > 0$

$$\nabla T_k(u) = \chi_{\{|u(x)|<k\}} \nabla u \quad \text{almost everywhere in } \Omega\,;$$

(4) for every function $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that there exist $k > 0$, $\varphi_{+\infty} \in C^1(\overline{\Omega})$, and $\varphi_{-\infty} \in C^1(\overline{\Omega})$ such that

$$\begin{cases} w(x) = \varphi_{+\infty}(x) & \operatorname{cap}_p\text{-quasi everywhere in } \{x : u(x) > +k\} \\ w(x) = \varphi_{-\infty}(x) & \operatorname{cap}_p\text{-quasi everywhere in } \{x : u(x) < -k\} \end{cases}$$

one has

$$\int_\Omega a(x, \nabla u) \nabla w \, dx = \int_\Omega f\, w \, dx + \int_\Omega g \, \nabla w \, dx + \int_\Omega \varphi_{+\infty} \, d\lambda^+ - \int_\Omega \varphi_{-\infty} \, d\lambda^- .$$

Let us make some remarks about this definition.

(i) Properties (1), (2) and (3) describe the "space" where the solution u lies. Note that in (3) the approximate gradient ∇u is not in general the gradient of u in distributional sense.

(ii) The test functions which are used in the equation solved by u are described in (4). These test functions can depend on the solution u, and form a "space" which is greater than the space $C_c^\infty(\Omega)$ which is used if the equation is only understood in distributional sense; it is indeed clear that any $w \in C_c^\infty(\Omega)$ can be chosen as test function in (4).

A possible choice for the test functions w which appear in (4) is $w = \varphi S(u)$, where φ belongs to $C_c^\infty(\Omega)$ and $S : \mathbf{R} \to \mathbf{R}$ belongs to $C^1(\mathbf{R})$, with S' having compact support; when S itself has compact support, these functions are the test functions which are used in the definition of renormalized solutions for $L^1(\Omega) + W^{-1,p'}(\Omega)$ data. Another possible choice for the test functions w in (4) is $w = T_k(u - \varphi)$, with φ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$; these functions are the test functions which are used in the definition of entropy solutions for $L^1(\Omega) + W^{-1,p'}(\Omega)$ data.

(iii) In (4), the term $\int_\Omega a(x, \nabla u)\nabla w\, dx$ is equal to

$$\int_{\{|u|<k\}} a(x, \nabla T_k(u))\nabla w\, dx + \int_{\{u>+k\}} a(x, \nabla u)\nabla \varphi_{+\infty} dx + \int_{\{u<-k\}} a(x, \nabla u)\nabla \varphi_{-\infty} dx$$

where every term makes sense in view of (2) and of the fact that $a(x, \nabla u) \in (L^q(\Omega))^N$ for every q such that $1 < q < \frac{N}{N-1}$: this follows from (3), and from the growth hypothesis $|a(x,\xi)| \leq \beta|\xi|^{p-1} + b(x)$ assumed on a.

(iv) If we take in (4) the test function $w = T_k(u)\varphi$ with $\varphi \in C_c^\infty(\Omega)$ (which satisfies the hypothesis with $\varphi_{+\infty} = +k\varphi$ and $\varphi_{-\infty} = -k\varphi$), and if we compare the equation in (4) with the result obtained by the formal computation in which we use in (P) the same test function, we are led to interpret the term " $\int_\Omega T_k(u)\varphi(\lambda^+ - \lambda^-)$" of the formal computation (this term has no meaning) as $\int_\Omega k\varphi\, d\lambda^+ - \int_\Omega (-k)\varphi\, d\lambda^-$ (which makes sense). Since this is valid for every k, the equation in (4) in a certain sense states that $u(x) = +\infty$ on the set where λ^+ is concentrated, while $u(x) = -\infty$ on the set where λ^- is concentrated.

As said before, we prove in our joint work with Gianni Dal Maso, Luigi Orsina and Alain Prignet the existence of at least one renormalized solution of problem (P) when μ is an arbitrary measure of $\mathcal{M}_b(\Omega)$. In order to obtain this existence result, the key point is to prove the strong convergence in $W_0^{1,p}(\Omega)$ of the truncations at every fixed height k of the solutions of problem (P) corresponding to some (special but fairly general) approximations of μ. (This is actually the result of continuity with respect to μ to which we made allusions above.) This continuity result is proved by means of a careful study of the energies of the truncations of these solutions "far" and "near" the set where the measure λ is concentrated.

We also prove results concerning uniqueness, which in particular allow us to recover the uniqueness of the renormalized (or entropy) solution of problem (P) in the particular case where μ belongs to $L^1(\Omega)+W^{-1,p'}(\Omega)$. In the case of an arbitrary measure of $\mathcal{M}_b(\Omega)$, one of our uniqueness results is the following one: let u and $\hat{u}$ be two renormalized solutions of problem (P); if $u-\hat{u}$ belongs to $L^\infty(\Omega)$ (this condition can be replaced by weaker ones), then $u=\hat{u}$.

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