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Abstract

I will explain basic concepts/problems of complex analysis in infinite dimensions, and survey the few approaches that are available to solve those problems.

1. Introduction.

Our subject matter being complex analysis in infinite dimensions, we shall start out by explaining why do analysis, and of the complex variety at that, in infinite dimensions. There is a very general reason for pursuing infinite dimensional analysis: systems with infinitely many degrees of freedom give rise to problems in infinite dimensions, and such systems do occur in theoretical physics, notably in field theories. Of course one could adopt a conservative point of view and maintain that infinitely many degrees of freedom appear but in idealized situations, and only systems with finitely many degrees of freedom are of practical interest. — Oddly, a radical might also be drawn to the same conclusion by positing that space is discrete rather than continuous, and bounded, which would then imply that fields have finitely many degrees of freedom. — Yet even the practically minded conservative will not deny that in many ways, a system with 6 degrees of freedom has much less in common with a system with $6 \cdot 10^{23}$ degrees of freedom, than a system with infinitely many degrees of freedom has. Ergo, finite, but huge dimensional systems should be thought of as perturbations of infinite dimensional systems, and the first step in their study should be understanding infinite dimensional systems.[1] If the need arises for better approximations, one can subsequently try to find terms of an entire perturbation series about the value $d = \infty$ of the dimension.

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[1]This is routinely done in statistical physics. However, not all huge or infinite dimensional problems are amenable to a statistical approach. In our subject, the infinite dimensional theory of the Cauchy–Riemann equations, the first two significant results were found using tools from statistics: Wiener measures on Hilbert spaces, see [H, R]. In certain ways these results were nevertheless unsatisfactory, and now much stronger theorems follow from the recent resp. imminent [L2, L3], which do not rely on statistics, cf. Section 5.

This then justifies doing infinite dimensional analysis. One is led to the complex brand of this analysis because a host of examples of infinite dimensional manifolds that arise in physics, but also in geometry and representation theory, carry natural complex structures. Now most fundamental questions pertaining to complex manifolds can be answered in terms of Dolbeault cohomologies, and this brings us to our subject proper: the inhomogeneous Cauchy–Riemann equations in infinite dimensions.

2. The Cauchy–Riemann equations in a Banach space.

Let us quickly review calculus, real and complex, in a Banach space V over the complex numbers. (For a treatment of general locally convex spaces, see [L1, D].) Suppose $\Omega \subset V$ is open and $u : \Omega \to \mathbb{C}$ is a function. If the directional derivatives

$$du(z;\xi) = \lim(u(z+t\xi) - u(z))/t, \qquad \mathbb{R} \ni t \to 0 \tag{2.1}$$

exist for all $z \in \Omega$, $\xi \in V$, and $du : \Omega \times V \to \mathbb{C}$ is continuous, we write $u \in C^1(\Omega)$. If $du \in C^1(\Omega \times V)$ we write $u \in C^2(\Omega)$ etc.

A 1-form on Ω is a function $f : \Omega \times V \to \mathbb{C}$, $\mathbb{R}$-linear in V. If it is also $\mathbb{C}$-linear resp. $\mathbb{C}$-antilinear, f is called a $(1,0)$ resp. $(0,1)$ form. General (p,q) forms are defined analogously (see [L1]) but no formal definition will be given here since our focus is on $(0,1)$ forms. A 1-form f is said to be r times continuously differentiable on Ω, $r = 0,1,\dots$, if $f \in C^r(\Omega \times V)$, and the space of r-times continuously differentiable $(0,1)$ forms on Ω is denoted $C^r_{0,1}(\Omega)$, $C^0_{0,1}(\Omega) = C_{0,1}(\Omega)$. A continuous 1-form f on Ω is Lipschitz continuous if

$$|f|_1 = \sup\{|f(z;\xi) - f(w;\xi)|/\|z-w\| : z \neq w \in \Omega, \|\xi\| \leq 1\} < \infty.$$

If $u \in C^1(\Omega)$ then du is a continuous 1-form, which can be uniquely decomposed into the sum $\partial u + \overline{\partial} u$ of a $(1,0)$ and a $(0,1)$ form. This defines the operator $\overline{\partial} : C^1(\Omega) \to C_{0,1}(\Omega)$, and allows us to formulate the $\overline{\partial}$, or Cauchy–Riemann equation

$$\overline{\partial} u = f \tag{2.2}$$

for a given $f \in C_{0,1}(\Omega)$.

A distinctive feature of the Cauchy–Riemann equation is that it is hereditary: if (2.2) holds and $W \subset V$ is a subspace (or submanifold) then $\overline{\partial}(u|_W) = f|_W$. Hence there is a strong link between the finite and infinite dimensional theories; in particular regularity and uniqueness results for (2.2) in a Banach space follow directly from the corresponding finite or even one dimensional results. The principal problem remaining is thus whether (2.2) can be solved.

Just like in $\mathbb{C}^n$, $n > 1$, in order that (2.2) be solvable a compatibility condition must be imposed on f. When $\dim V < \infty$, this condition is $\overline{\partial} f = 0$ (in the sense of distribution theory, since we are only assuming f is continuous, see [Hö]). In general, an $f \in C_{0,1}(\Omega)$ is said to be closed if $\overline{\partial}(f|_W) = 0$ for all finite dimensional subspaces $W \subset V$. Clearly f must be closed if (2.2) is to have a solution, and then the problem becomes finding reasonable conditions on V, Ω, and the closed $f \in C_{0,1}(\Omega)$ that ensure (2.2) has a solution $u \in C^1(\Omega)$.

When $\dim V < \infty$ and Ω is pseudoconvex, (in particular, if it is convex) (2.2) is known to be solvable for any closed $f \in C_{0,1}(\Omega)$, see e.g. [Hö]. The infinite dimensional theory is not nearly as complete. So far three approaches have been proposed to solve (2.2) that we are going to discuss presently. They all have serious limitations; on the other hand a family of examples shows that in infinite dimensions (2.2) will not be solvable in sweeping generality, see Theorem 4.4.

As in $\mathbb{C}^n$, pseudoconvexity will play an important role in the theory of the $\overline{\partial}$ equation. Various equivalent definitions are known. For example, $\Omega \subset V$ is pseudoconvex if $-\log \operatorname{dist}(z, \partial\Omega)$ is a plurisubharmonic function (with the understanding that V itself is pseudoconvex). Here a continuous function $\varphi : \Omega \to \mathbb{R}$ is called plurisubharmonic if its restrictions to one dimensional affine subspaces are subharmonic. Alternatively, Ω is pseudoconvex if its intersections with finite (or even only two) dimensional affine subspaces are. One should keep in mind that all convex domains are pseudoconvex.

3. Early work.

The most obvious attempt to solve the equation

$$\overline{\partial} u = f, \qquad f \in C_{0,1}(\Omega), \quad \text{closed} \tag{3.1}$$

in infinite dimensions is to solve (3.1) on Ω intersected with finite dimensional subspaces $W \subset V$, prove pointwise or preferably uniform estimates for the solutions, and see what happens when $\dim W \to \infty$. If the estimates are independent of $\dim W$, it is reasonable to expect that from the finite dimensional solutions one can construct a solution of the infinite dimensional equation. Unfortunately this approach fails because the uniform estimates that the usual methods yield in $\mathbb{C}^N$ tend to blow up as $N \to \infty$.

Henrich, and subsequently in a much improved manner Raboin, noticed that instead of uniform estimates one can use L^2 estimates of Hörmander, see [H, R], which are independent of the dimension. This does not rid us of all difficulties, though. Indeed, even if the constants in the estimates do not blow up as $\dim W \to \infty$, Lebesgue measure on W does. Accordingly, Henrich and Raboin are able to treat (3.1) when V is a Hilbert space, Ω is pseudoconvex (and $f \in C^1_{0,1}(\Omega)$), but they cannot solve (3.1) on the whole of Ω; instead, only on $\Omega \cap X$ where $X \subset V$ is a "not too large" linear subspace. We shall not precisely formulate their theorems here, as much stronger results are now seen to follow from Theorem 5.1 to be discussed further down.

4. The second method.

Another approach to connect (3.1) with finite dimensional $\overline{\partial}$ problems is to consider one dimensional affine subspaces only, i.e. lines $L \subset V$, and for each L solve

$$\overline{\partial} u_L = f|_{L\cap\Omega}. \tag{4.1}$$

In this approach one can estimate various norms of u_L, independently of L, but the question is then how to construct a solution of (3.1) out of the functions u_L. Suppose we select a family $\mathcal{L}$ of lines that covers Ω simply. Then $\{u_L\}_{L\in\mathcal{L}}$ patch together to give a function $u : \Omega \to \mathbb{C}$

$$u(z) = u_L(z) \qquad \text{if} z \in L \cap \Omega, \quad L \in \mathcal{L}. \tag{4.2}$$

Is it reasonable to expect that this u will solve (3.1)? Clearly no. The problem is that (4.1) has many solutions u_L, and carelessly choosing u_L for varying L will result in a function u that is not even continuous. Even if the u_L are chosen to depend somewhat regularly on L so that the resulting function u is C^1, one should not expect u to solve (3.1). Indeed, for each $z \in \Omega$ (3.1) requires infinitely many relations to be satisfied among the directional derivatives $du(z;\cdot)$, while (4.1) takes care only of derivatives along the one $L \in \mathcal{L}$ that passes through z.

Thus the main issue is whether an intelligent choice of the solution u_L of (4.1) is possible that leads, through (4.2), to a solution of (3.1). In certain situations it indeed is possible to make such a choice. The most interesting instance of this is when Ω is not an open set in a Banach space but a complex Banach manifold: projectivized Banach space $\mathbb{P}V$, i.e. the space of one dimensional subspaces of V, endowed with a complex structure pretty much in the same way as is done for $\mathbb{P}_n = \mathbb{P}\mathbb{C}^{n+1}$. In this way one obtains

Theorem 4.1 *For any Banach space V and any closed $f \in C^\infty_{0,1}(\mathbb{P}V)$ the equation $\overline{\partial} u = f$ is solvable.*

This is but a special case of a more general result for forms of higher degree, except that in general one requires that V should admit smooth cut off functions.

Theorem 4.2 *Suppose there exists a not identically 0 function $\varphi \in C^\infty(V)$ with bounded support. Then for any closed $f \in C^\infty_{0,q}(\mathbb{P}V)$, $1 \le q < \dim \mathbb{P}V$ the equation $\overline{\partial} u = f$ admits a solution $u \in C^\infty_{0,q-1}(\mathbb{P}V)$.*

For example Hilbert spaces and L^p spaces with p an even integer satisfy the condition of the theorem. — One can further generalize Theorem 4.2 to (p,q) forms, with values in holomorphic vector bundles of finite rank over $\mathbb{P}V$; for all this, consult [L1].

Another situation where the method introduced above works is when Ω is the whole Banach space, but then a growth condition has to be imposed on f. We shall say that $f \in C_{0,1}(V)$ is of order $d > 0$ if with some $A \in \mathbb{R}$

$$|f(z;\xi)| \le A(1 + \|z\|^d), \quad z, \xi \in V, \quad \|\xi\| \le 1.$$

Theorem 4.3 *(cf. [L1].) Suppose $r \in \mathbb{N}$, $f \in C^r_{0,1}(V)$ is closed and of order $r - \epsilon$ with some $\epsilon > 0$. Then the equation $\overline{\partial} u = f$ has a solution $u \in C^r(V)$.*

In the proof of both Theorems 4.1 and 4.3 the family $\mathcal{L}$ of lines one works with consists of all lines through a given point of $\mathbb{P}V$ resp. V. These lines do cover

$\mathbb{P}V$ resp. V simply, except for the base point. Accordingly one is first able to solve $\overline{\partial}u = f$ on $\mathbb{P}V$ resp. V minus a point, and then one shows that the isolated singularity of u is removable.

Perhaps surprisingly, Theorem 4.3 is sharp in that for general Banach spaces the statement would not be true without the ϵ. In [L2] we prove

Theorem 4.4 *For $p = 1, 2, \ldots$ there is a closed $f \in C^{p-1}_{0,1}(l^p)$, of order $p-1$, such that on no open set $\Omega \neq 0$ is the equation $\overline{\partial}u = f$ solvable.*

The case $p = 2$ was first found in the late seventies by Coeuré, see [C, M]. The general case requires but small modifications of the original construction.

5. The $\overline{\partial}$ equation in l^1.

There is now a third method, the only one that is capable of solving the $\overline{\partial}$-equation on domains in a Banach space; however, so far it has brought fruit only in the space l^1:

Theorem 5.1 *(cf. [L2, L3]) If $\Omega \subset l^1$ is pseudoconvex, $f \in C_{0,1}(\Omega)$ is Lipschitz continuous and closed then the equation*

$$\overline{\partial}u = f \tag{5.1}$$

has a solution $u \in C^1(\Omega)$.

Note that by Theorem 4.4 mere continuity of f, as opposed to Lipschitz continuity, would not guarantee that (5.1) is solvable, not even locally. This is in marked contrast with the finite dimensional theory.

The proof consists of three steps. In the first step one solves (5.1) when $\Omega = B(R) = \{z \in l^1 : \|z\| < R\}$. This step uses the symmetry of the ball $B(R)$ as follows. One considers the circle $\mathbb{R}/\mathbb{Z}$ and the infinite dimensional torus $T = \prod_1^\infty \mathbb{R}/\mathbb{Z}$. This is a compact group, which acts continuously on l^1 and $B(R)$ by

$$\rho_t(z) = (e^{2\pi i t_\nu} z_\nu)_{\nu=1}^\infty, \quad t = (t_\nu) \in T, \quad z = (z_\nu) \in l^1.$$

The action ρ induces a Fourier decomposition of (5.1). Indeed, the given form f can be expanded into a series

$$f \sim \sum_k f_k, \qquad f_k = \int_T e^{-2\pi i k t} \rho_t^* f \, dt. \tag{5.2}$$

Here $k = (k_1, k_2, \ldots) \in \mathbb{Z}^\infty$ is a multiindex such that only finitely many $k_\nu \neq 0$, $kt = \sum_\nu k_\nu t_\nu$, and dt stands for the Haar probability measure on T. The terms in the series (5.2) satisfy

$$\rho_t^* f_k = e^{2\pi i k t} f_k. \tag{5.3}$$

It is not hard to solve the equations

$$\overline{\partial}u_k = f_k \tag{5.4}$$

with u_k that themselves transform as the f_k do in (5.3). Indeed, using (5.3), (5.4) reduces to an equation involving de Rham's d operator rather than the Dolbeault $\overline{\partial}$ operator (plus a potential term, when $k \neq 0$). The question then is whether the u_k add up to a solution u of (5.1). Again, (5.4) does not uniquely determine u_k for each $k = (k_\nu)$, notably when all $k_\nu \geq 0$; but even in the ambiguous cases an optimal choice of u_k is possible, to ensure that $\sum u_k$ does converge (in some sense) to a solution u of (5.1). All this is explained in [L2].

The second step is a slight generalization. Instead of a ball in l^1 one considers an $\Omega \subset \mathbb{C}^N \oplus l^1$ that fibers into balls. More precisely, if $M \subset \mathbb{C}^N$ is a domain and $R : M \to (0, \infty)$ is a continuous function, the domain

$$\Omega = \{(\zeta, z) \in M \times l^1 : \|z\| < R(\zeta)\} \subset \mathbb{C}^N \times l^1$$

will be called a domain of type (B). Combining the result (and method) of Step 1 with Hörmander's theorem on solving $\overline{\partial}$ in $\mathbb{C}^N$, see [Hö, Theorem 4.2], one can show that Theorem 5.1 holds whenever Ω is a pseudoconvex domain of type (B).

The third, and final step is to consider a general pseudoconvex domain $\Omega \neq l^1$, and observe that it can be represented as an increasing union of pseudoconvex domains Ω_N, all of type (B). Indeed, given N, consider the two complementary projections $\pi, \sigma : l^1 \to l^1$,

$$\pi(z) = (z_1, \ldots, z_N, 0, \ldots), \quad \sigma(z) = (0, \ldots, 0, z_{N+1}, z_{N+2}, \ldots),$$

and

$$\Omega_N = \{z \in \Omega : \|\sigma(z)\| < \operatorname{dist}(\pi(z), \partial\Omega)\},$$

which is a domain of type (B) in $l^1 = \pi(l^1) \oplus \sigma(l^1) \approx \mathbb{C}^N \oplus l^1$. It is also pseudoconvex, so that by Step 2 (5.1) has a solution $u = u^N$ on Ω_N. The proof is concluded by constructing a solution u on Ω out of the solutions u^N; this involves a Runge–type approximation theorem, see [L4].

6. Concluding remarks.

The reader will notice that as far as the infinite dimensional theory of the $\overline{\partial}$ equation is concerned, in spite of the existence of some results, our knowledge is dwarfed by our ignorance. In this last section we would like to point out what we consider to be the most urgent questions in the field that should be answered.

First and foremost one should see whether some variant of Theorem 5.1 is true in the Hilbert space l^2 rather than in l^1. The space l^1 is quite awkward to do analysis in, for example it does not admit smooth (or even C^1) partitions of unity. By constrast, in Hilbert spaces many tools that one is accustomed to in finite dimensions are available. However, the proof of Theorem 5.1 would break down in l^2 — even if the role of l^1 may not be apparent from the sketch given in section 5. Of course, Theorem 4.4 shows that Lipschitz continuity of f is not sufficient even for the local solvability of the $\overline{\partial}$ equation in l^2. Instead we offer the conjecture that on pseudoconvex domains Ω in l^p the $\overline{\partial}$ equation is solvable if $f \in C_{0,1}(\Omega)$ is Hölder continuous of order p.

Second, it would be important to clarify whether the *global* condition of Lipschitz continuity in Theorem 5.1 can be replaced by a *local* condition such as $f \in C^1_{0,1}(\Omega)$.

In finite dimensions the distinction between local and global conditions is much less important, since any domain can be exhausted by relatively compact subdomains, and local conditions have global consequences on these subdomains. However, this is not so in infinite dimensions, and typically it is very hard if not impossible to bridge the gap between local and global.

The assumption of Lipschitz continuity in Theorem 5.1 is also unsatisfactory because it is not holomorphically invariant. In other words, if $\Phi : \Omega' \to \Omega$ is a biholomorphism, then $f \in C_{0,1}(\Omega)$ may be Lipschitz continuous while $\Phi^* f$ may not be. Short of proving Theorem 5.1 for general $f \in C^1_{0,1}(\Omega)$ one would at least like to replace Lipschitz continuity by a holomorphically invariant condition.

Finally, most available results on the $\overline{\partial}$ equation pertain to $(0,1)$ forms f, except for those in section 4. Many problems in geometry lead to the $\overline{\partial}$ equation for $(0,2)$ forms. For example, an extension of the Newlander-Nirenberg theorem on integrating almost complex structures (see [NN]) to infinite dimensions seems to require a solution of $\overline{\partial} u = f$ with f a closed $(0,2)$ form. It would therefore be important to extend Theorem 5.1 to general $(0,q)$-forms f.

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