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Cartesian Currents
in the
Calculus of Variations*†

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The aim of this lecture is to illustrate the notion of Cartesian currents and some of its uses in the Calculus of Variations. Cartesian currents were introduced in [?] [?] and a comprehensive account can be found in [?].

Definition 1. *Let Ω be an open bounded domain in $\mathbf{R}^n$. A* Cartesian current *in $\Omega \times \mathbf{R}^N$, $N \geq 1$, is a current T satisfying*

(i) *T is a rectifiable current with integral multiplicity, $T = \tau(\mathcal{M}, \theta, \vec{T})$*

(ii) *T has finite mass and finite L^1-norm*
$\mathbf{M}(T) := \sup\{T(\omega) \mid \omega \text{ } n\text{-form with compact support in } \Omega \times \mathbf{R}^N,\ |\omega| \leq 1\} < \infty$
$\| T \|_1 := \sup\{T(|y|\varphi(x,y)dx) \mid \phi \in C_c^1(\Omega \times \mathbf{R}^N),\ |\phi| \leq 1\} < \infty$

(iii) *$T \llcorner dx^1 \wedge \ldots \wedge dx^n$ is a non-negative measure in $\Omega \times \mathbf{R}^N$ and $\pi_\# T = [\![\Omega]\!]$, π denoting the linear projection $\pi : \mathbf{R}^n \times \mathbf{R}^N \to \mathbf{R}^n$, $\pi(x,y) = x$*

(iv) $\partial T \llcorner \Omega \times \mathbf{R}^N = 0$

The class of Cartesian currents in $\Omega \times \mathbf{R}^N$ is denoted by

$$\mathrm{cart}(\Omega \times \mathbf{R}^N)$$

and we define the "norm" of $T \in \mathrm{cart}(\Omega \times \mathbf{R}^N)$ by

$$\| T \|_{\mathrm{cart}} := \mathbf{M}(T) + \| T \|_1 .$$

Taking into account *Federer's support theorem* the previous definition extends immediately to the case in which Ω is a bounded domain of an oriented Riemannian manifold $\mathcal{X}$ and $\mathbf{R}^N$ is replaced by a submanifold $\mathcal{Y}$ of $\mathbf{R}^N$, as

$$\mathrm{cart}(\Omega \times \mathcal{Y}) := \mathrm{cart}(\Omega \times \mathbf{R}^N) \cap \{T \in \mathcal{D}_n(\mathcal{X} \times \mathbf{R}^N) \mid \mathrm{spt}\, T \subset \overline{\Omega} \times \mathcal{Y}\}.$$

Theorem 1. *We have*

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(i) *The class* $\mathrm{cart}(\Omega \times \mathcal{Y})$ *is weakly closed with respect to the weak convergence of* $\| \ \|_{\mathrm{cart}}$-*equibounded sequences*

(ii) *For every* $T \in \mathrm{cart}(\Omega \times \mathcal{Y})$ *there exists an* a.e.-*approximately differentiable map* $u_T : \Omega \to \mathcal{Y}$ *such that*

$$T = G_{u_T} + S_T$$

where G_{u_T} *denotes the current carried by the rectifiable graph of* u_T *and* S_T *is a vertical current.*

In particular, if $T = \tau(\mathcal{M}, \theta, \vec{T})$, the set $\mathcal{M}_+$ of points in $\mathcal{M}$ at which the tangent space to $\mathcal{M}$ contains no completely vertical direction has a one-to-one projection of full measure into Ω, T has density 1 in $\mathcal{M}_+$

$$G_{u_T} = \tau(\mathcal{M}_+, 1, \vec{T}) \qquad S_T = \tau(\mathcal{M} \setminus \mathcal{M}_+, \theta, \vec{T}),$$

and points in $\mathcal{M} \setminus \mathcal{M}_+$ have tangent plane with at least one completely vertical direction. Finally we emphasize that, while the density of T in $\mathcal{M}_+$ is 1, it can be any integer in $\mathcal{M} \setminus \mathcal{M}_+$.

Besides the weak closure of $\mathrm{cart}(\Omega \times \mathcal{Y})$, and consequently the weak compactness of bounded sets in $\mathrm{cart}(\Omega \times \mathcal{Y})$, which make this class very convenient in the calculus of variations, it turns out that to every Cartesian current it is naturally associated a *homology map*

$$T_* : H_k(\mathcal{X}, \mathbf{Z}) \to H_k(\mathcal{Y}, \mathbf{Z}) \qquad k = 0, 1, 2, \ldots, n.$$

Assuming for the sake of simplicity that $\mathcal{X}$ and $\mathcal{Y}$ be compact and without boundary and for instance $\mathcal{Y}$ is torsionless, T can be described in terms of *periods*, from which one easily sees that the map

$$T \longrightarrow T_*$$

is continuous with respect to the weak convergence in cart. This allows us of course to treat easily variational problems with homological constraints in the setting of Cartesian currents.

The motivations, or at least our motivations, to introduce the notion of Cartesian currents comes from a few specific problems in the calculus of Variation

(i) nonlinear elasticity, which requires to identify a geometric setting in which one defines elastic deformations (the kinematics) and among elastic deformations then prove existence of stable equilibrium configurations (dynamics)

(ii) energy minimizing harmonic mappings between manifold, where one is forced to confront himself with so-called gap phenomena

(iii) the non-parametric area problem which preliminarily might require to understand the notion of Lebesgue area.

A common feature is the following: we have an *enerjy* $\mathcal{F}(u, \Omega)$ which is well defined among *smooth* maps or fields, we have a sequence of smooth maps or fields u_k with

$$\sup_k \mathcal{F}(u_k, \Omega) < \infty$$

and we ask ourselves to identify the *weak limits of the* u_k's

Example 1 Consider the sequence of smooth maps $u_k : (-\pi,\pi) \simeq S^1 \to S^1 \subset \mathbf{R}^2$:

$$u_k(\theta) := \begin{cases} (1,0) & \text{for } \theta < 0 \text{ or } \theta > 2\pi/k \\ (\cos k\theta, \sin k\theta) & \text{for } 0 < \theta \leq 2\pi/k. \end{cases}$$

Clearly each map u_k has degree one and

$$\int_{-\pi}^{\pi} |\dot{u}_k| \, d\theta = 2\pi.$$

The u_k's converge weakly in BV to the constant map $u_\infty(\theta) := (1,0)$ of degree zero and zero energy. Notice that for every smooth map $u : S^1 \to S^1$ of degree one we have

$$\int_{-\pi}^{\pi} |\dot{u}_k| \, d\theta \geq 2\pi.$$

If G_{u_k} denotes the current carried by the graph of u_k we instead have

$$G_{u_k} \rightharpoonup T := G_{u_\infty} + [\![\{0\} \times S^1]\!], \qquad \text{in } \mathcal{D}_1(S^1 \times S^1).$$

Example 2 A similar phenomenon occurs for the Dirichlet integrals for mappings from $\mathbf{R}^2$ or S^2 into S^2. Denoting by σ the stereographic projection from S^2 into $\mathbf{R}^2$, and setting $u_k(x) := \sigma^{-1}(k\sigma(x))$ we find

$$\frac{1}{2}\int_{S^2} |du_k|^2 = 4\pi, \;\; u_k \rightharpoonup u_\infty := \text{south pole}, \; \deg u_k = 1, \;\; \deg u_\infty = 0.$$

This is the phenomenon of bubbling off of spheres well-known after for instance Sacks and Uhlenbeck [?]. Notice that instead we have

$$G_{u_k} \rightharpoonup G_{u_\infty} \times [\![\{0\} \times S^2]\!] \quad \text{in } \mathcal{D}_2(S^2 \times S^2).$$

Example 3 In the Sobolev space approach to variational problems for vector valued maps a typical feature is the occurrence of a *gap phenomenon* at least if the target manifold has a "nontrivial" homology. This was first pointed out by Hardt and Lin [?]. One has for smooth maps $\varphi : \partial B^3 \simeq S^2 \to S^2$ even of degree zero

$$\begin{aligned} &\inf \left\{ \frac{1}{2}\int_{B^3} |Du|^2 \, dx \;\mid\; u \in W^{1,2}(B^3, S^2), \; u = \varphi \text{ on } \partial B^3 \right\} \\ < \; &\inf \left\{ \frac{1}{2}\int_{B^3} |Du|^2 \, dx \;\mid\; u \in C^1(B^3, S^2) \cap C^0(\overline{B}^3, S^2), \; u = \varphi \text{ on } \partial B^3 \right\}. \end{aligned}$$

We can say that Cartesian currents naturally arise when considering "limits" of graphs with equibounded masses and the vertical parts S_T appear as necessary to keep the homological information $\partial G_{u_k} \llcorner \omega_{S^2} \times \mathbf{R}^N = 0$ contained in the graphs of smooth maps. For instance, it turns out that reasonable smooth approximations u_k of the map $\frac{x}{|x|} : B^3 \to \mathbf{R}^3$ have graphs G_{u_k} converging either to

$$G_{x/|x|} + \{\{0\} \times B(0,1)\}$$

or to

$$G_{x/|x|} + L \times [\![S^2]\!],$$

L being a 1-dimensional integer rectifiable current in B^2 with $\partial L \llcorner B^3 = -\delta_0$.

A natural question is whether the converse holds: is every Cartesian current the weak limit of graphs with equibounded masses? The answer is in general no, as one can easily infer observing that for the Cartesian current

$$T := G_0 + [\![\partial B(0,r) \times \partial B(y_0,1)]\!] \quad \text{in } \mathbf{R}^2 \times \mathbf{R}^2$$

we have

$$\mathbf{M}(G_{u_k}) \geq \frac{\pi r}{4} \operatorname{dist}(\partial B(y_0,1),(0,0))$$

whenever u_k are smooth maps such that

$$G_{u_k} \rightharpoonup T.$$

But facts are even more subtle especially if one thinks of a non-parametric notion of area.

Lebesgue's area. Let Ω be a bounded domain in $\mathbf{R}^n$, $n \geq 2$, and $u : \Omega \to \mathbf{R}^N$ be a smooth map. The area of the graph of u is given by

$$A(u,\Omega) := \int_\Omega |M(Du)|\, dx$$

where

$$M(Du) = (e_1 + D_1 u^i \varepsilon_i) \wedge \ldots \wedge (e_n + D_n u^i \varepsilon_i),$$

$(e_1, \ldots, e_n)$, $(\varepsilon_1, \ldots, \varepsilon_N)$ being standard bases in $\mathbf{R}^n$ and $\mathbf{R}^N$, respectively. In the same spirit as Lebesgue's area of graphs of continuous functions, the *relaxed area* of the graph of an L^1 map $u : \Omega \to \mathbf{R}^N$ can be defined by

$$\overline{A}(u,\Omega) := \inf\{\liminf_{k\to+\infty} A(u_k,\Omega) \mid u_k \in C^1(\Omega,\mathbf{R}^N),\ u_k \to u \text{ in } L^1\}.$$

Denote by $\mathcal{A}^1(\Omega,\mathbf{R}^N)$ the class of maps $u \in W^{1,1}(\Omega,\mathbf{R}^N)$ such that all Jacobian minors of Du are summable in Ω. For $u \in \mathcal{A}^1(\Omega,\mathbf{R}^N)$ $A(u,\Omega)$ is well defined and it turns out that

$$A(u,\Omega) \leq \overline{A}(u,\Omega)$$

and by considering the rectifiable 1-graph of u, compare e.g. [?], that

$$A(u,\Omega) = \mathbf{M}(G_u).$$

The homological condition $\partial G_u \llcorner \Omega \times \mathbf{R}^N = 0$ is clearly a necessary condition for the existence of a smooth maps u_k such that

$$G_{u_k} \rightharpoonup G_u.$$

Consider then the "subclass" of Cartesian currents

$$\mathrm{cart}^1(\Omega, \mathbf{R}^N) := \{u \in \mathcal{A}^1(\Omega, \mathbf{R}^N) \mid \partial G_u \llcorner \Omega \times \mathbf{R}^N = 0\}.$$

It turns out, compare [?] [?] th it *in general for* $u \in \mathrm{cart}^1(\Omega, \mathbf{R}^N)$ *we may have*

$$A(u, \Omega) = \mathbf{M}(G_u) < \overline{A}(u, \Omega).$$

This suggests to define the *relaxed area for graphs* of maps in $\mathrm{cart}^1(\Omega, \mathbf{R}^N)$ as

$$\begin{aligned}\widetilde{A}(u, \Omega) := \inf \quad \{ & \liminf_{k\to\infty} A(u_k, \Omega) \mid u_k \in C^1(\Omega, \mathbf{R}^N), \\ & G_{u_k} \rightharpoonup G_u \text{ weakly in } \mathcal{D}_n(\Omega \times \mathbf{R}^N)\}\end{aligned}$$

so that

$$A(u, \Omega) \leq \overline{A}(u, \Omega) \leq \widetilde{A}(u, \Omega);$$

in fact, if $\widetilde{A}(T)$, $T \in \mathrm{cart}(\Omega \times \mathbf{R}^N)$, is defined similarly, $T = G_{u_T} + S_T$, one also proves that

$$\overline{A}(u, \Omega) = \inf\{\widetilde{A}(T) \mid T \in \mathrm{cart}(\Omega \times \mathbf{R}^N),\ u_T = u\}$$

so that in general $\overline{A}(u, \Omega)$ is a *non local functional.*

It is an open question to characterize Cartesian currents, and even Cartesian maps $u \in \mathcal{A}^1(\Omega, \mathbf{R}^N)$ for which

$$A(u, \Omega) = \widetilde{A}(G_u, \Omega).$$

However the following extension of *Lebesgue's theorem* concerning the area of continuous graphs holds.

Theorem 2. (Mucci, [?], [?]) *Let* $u \in \mathrm{cart}^1(\Omega, \mathbf{R}^N) \cap C^0(\Omega, \mathbf{R}^N)$. *Then there exists a sequence of smooth maps* $u_k : \Omega \to \mathbf{R}^N$ *such that*

$$G_{u_k} \rightharpoonup G_u \quad \textit{and} \quad \mathbf{M}(G_{u_k}) \to \mathbf{M}(G_u).$$

The proof of Theorem 2. is quite complicated and uses ideas from [?].

Convergence of minors. Here I shall not discuss nonlinear elasticity in the context of Cartesian currents for which I refer the interest reader to [?], [?], and I confine myself to present some remarks concerning the convergence of minors.

Let $\{u_k\}$ be a sequence of maps in $W^{1,1}(\Omega, \mathbf{R}^N)$ with $M^\beta_{\bar{\alpha}}(Du) \in L^1$, where for multiindices α, β, $|\beta| = k$, $|\alpha| = n - k$, $0 < k < \min(n, N)$, $M^\beta_{\bar{\alpha}}(Du)$ denotes the determinant of the $k \times k$-submatrix of G with rows $(\beta_1, \ldots, \beta_k)$ and columns $(\bar{\alpha}_1, \ldots, \bar{\alpha}_k)$. Suppose that

$$u_k \rightharpoonup u,\ Du \rightharpoonup Du,\ M^\beta_{\bar{\alpha}}(Du) \rightharpoonup v^\beta_{\bar{\alpha}} \text{ weakly in } L^1;$$

is that true that $v^\beta_{\bar\alpha} = M^\beta_{\bar\alpha}(Du)$? Of course the answer is in general no. By introducing the current $S_{u,v}$ with components

$$S^{\bar{0}0}_{u,v}(\phi) := \int \phi(x,u)\, dx$$
$$S^{\alpha\beta}_{u,v}(\phi) := \sigma(\alpha,\bar\alpha) \int \phi(x,u(x)) v^\beta_{\bar\alpha}(x)\, dx$$

one proves

Theorem 3. *The following claims are equivalent*

(i) $v^\beta_{\bar\alpha}(x) = M^\beta_{\bar\alpha}(Du(x))$

(ii) $u \in \mathcal{A}^1$ *and* $S_{u,v} = G_u$

(iii) S *is an integer rectifiable current.*

Consequently an answer to our question is:

$$\textit{If } \partial G_{u_k} \llcorner \Omega \times \mathbf{R}^N = 0, \textit{ then } v^\beta_\alpha = M^\beta_\alpha(Du_k)$$

this extends and unifies results e.g. by Reshetwyek, Bell, Šverak concerning the continuity of determinants.

The regular Dirichlet integral. Of course Cartesian currents are a reasonable setting to work in only when dealing with energies $\mathcal{F}(u,\Omega)$ which are *coercive* with respect to the area of graphs

$$\mathcal{F}(u,\Omega) \geq \mathbf{M}(G_u).$$

We call *regular* those energies. According to this, the Dirichlet energy for mappings $u : \mathcal{X}^n \to \mathcal{Y}^n$ is regular only if $\min(n,m) \leq 2$. This should not be a surprise as if $\min(n,m) > 2$ equiboundedness of the Dirichlet energy does not ensure *any* control on completely vertical components of the tangent space to the graph of u. Of course there are several interesting regular energies, for instance: Oseen-Franck energy for liquid crystals, Skyrme type integral, conformally invariant integral; but here I shall discuss only a few facts related to the Dirichlet energy.

As for the area, given a regular energy $\mathcal{F}(u,\Omega)$, one would like and should work with the *relaxed* energy

$$\mathcal{F}(T,\Omega\times\mathbf{R}^N) := \inf\{\liminf_{k\to\infty} \mathcal{F}(u_k,\Omega) \mid u_k \text{ smooth}, G_{u_k} \rightharpoonup T\}.$$

But, as in general we are unable to work directly with this extension, we extend instead of $\mathcal{F}$ its integrand. This is done as follows: we can read the integrand of $\mathcal{F}$ as a map F defined on the unit n-vectors orienting graph u at the point $(x,u(x))$, i.e.

$$\mathcal{F}(u,\Omega) := \int_{\text{graph } u} F\Big(z, \frac{M(Du)}{|M(Du)|}\Big)\, dH^n,$$

and then we extend F (by using convex analysis) to all n-vectors. This way we can extend the energy $\mathcal{F}$ to all Cartesian currents $T = \tau(\mathcal{M}, \theta, \vec{T})$ by setting

$$\widehat{\mathcal{F}}(T, \Omega \times \mathbf{R}^N) := \int_{\mathcal{M}} F(z, \vec{T}(z))\theta(z)\, dH^n(z).$$

Returning to the Dirichlet integral from now on for maps u from B^3 into S^2, we then find the following theorem which gives an explicit expression for the relaxed energy and characterizes completely weak limits of sequences of smooth maps with equibounded Dirichlet integral.

Theorem 4. *We have*

$$\begin{aligned}
&\{T \in \operatorname{cart}(B^3 \times S^2) \mid \mathcal{D}(T, B^2 \times S^2) < \infty\}\\
&= \{T \in \operatorname{cart}(B^3 \times S^2) \mid \widehat{\mathcal{D}}(T, B^2 \times S^2) < \infty\}\\
&= \{T \in \operatorname{cart}(B^3 \times S^2) \mid \exists u_k : B^2 \to S^2 \text{ smooth}, \sup \mathcal{D}(u_k, B) < \infty\\
&\qquad G_{u_k} \rightharpoonup T\}\\
&= \{T \in \operatorname{cart}(B^3 \times S^2) \mid T = G_{u_T} + L_T \times [\![S^2]\!],\ u_T \in W^{1,2}(B^3 \times S^2),\\
&\qquad L \text{ is a 1-dim. integer rectifiable current in } B\}.
\end{aligned}$$

Also $\mathcal{D} = \widehat{\mathcal{D}}$ *and*

$$\mathcal{D}(T, B^3 \times S^2) = \frac{1}{2}\int_{B^3} |du_T|^2 + 4\pi \mathbf{M}(L_T).$$

A few more remarks can be of interest, compare also [?]. Set

$$\operatorname{cart}^{2,1}(B^3 \times S^2) := \operatorname{cart}(B^3 \times S^2) \cap \{T \mid \mathcal{D}(T, B^3 \times S^2) < \infty\},$$

one sees that

$$G_u \in \operatorname{cart}^{2,1}(B^3 \times S^2) \Leftrightarrow u \in W^{1,2}(\Omega, S^2),\ \partial \mathbf{D}(u) = 0$$

where

$$\mathbf{D}(u) = \frac{1}{4\pi}\pi_{\#}(G_u \llcorner \omega_{S^2}),$$

and that

$$\mathbf{D}(u)(\gamma) = \frac{1}{4\pi}\int_{B^3} <\gamma, D(u)>$$

where $D(u)$ is the vector field

$$D(u) := (u \cdot u_{x^2} \times u_{x^3}, u \cdot u_{x^3} \times u_{x^1}, u \cdot u_{x^1} \times u_{x^2}).$$

In other words

$$\partial G_u \llcorner B^3 \times S^2 = 0 \Leftrightarrow \operatorname{div} D(u) = 0.$$

If $\{u_k\}$ is a sequence of smooth maps with equibounded Dirichlet integral and $G_{u_k} \rightharpoonup T = G_u + L \times [\![S^2]\!]$ we have then

$$\operatorname{div} D(u_k) = 0,\ \mathbf{D}(u_k) \rightharpoonup \mathbf{D}(T),\ \partial \mathbf{D}(T) = 0$$

while

$$\operatorname{div} D(u) \neq 0$$

$\mathbf{D}(T)$ can be identified with a divergence free field which is the sum of the fields $D(u)$ distributed in Ω and $\mathbf{D}(L)$ concentrated on a rectifiable set $\mathcal{L}$ which are generated by topological charges on ∂L.

Energy minimizing harmonic maps. Theorem 4. allows us to find easily minimizers of the relaxed Dirichlet energy $\mathcal{D}(T)$

$$\mathcal{D}(T) \to \min \quad T \in \operatorname{cart}(\Omega, S^2), \quad \partial T = \partial G_\varphi$$

and the minimizers will in general have the form

$$T = G_{u_T} + L_T \times [\![S^2]\!].$$

Let $\overline{u}$ be the minimizer of

$$\mathcal{D}(u, \Omega) \to \min \quad u \in W^{1,2}_\varphi(\Omega, S^2).$$

It is easily seen that u_T is always different from $\overline{u}$, except $\overline{u}$ is smooth. A reasonable unsolved question is whether L_T is in general non-empty. In this respect we have only partial answers [?], [?].

(i) u_T is smooth in an open set Ω_0 and $\mathcal{H}^1(\Omega \setminus \Omega_0) < \infty$

(ii) The tangent map of a minimizer at a singular point is again a minimizer which is radial.

Apart for the trivial "cones" $T = G_{c_0} + k[\![(-y_0, y_0)]\!]$, c_0 =cost$\in S^2$ $y_0 \in S^2$, are there other non trivial minimizing cones? We do not know the answer, it can be reduced to proving or disproving that there exist *minimizing* currents of the type $G_u + L \times [\![S^2]\!]$ where

(a) $u(x) = g(x/|x|)$, $\deg(g, S^2) \geq 2$, g harmonic

(b) $L = [\![(y_1, 0)]\!] + [\![(y_2, 0)]\!]$, $y_1 \neq y_2$.

We remark that one can construct examples of critical points which are moreover *stationary* and satisfy (a) and (b), see [?].

Finally, I refer to [?] for a discussion of the Dirichlet integral in the non regular case and to [?] for further information, and conclude mentioning the following result: limit currents of sequence of smooth graphs $u : \mathcal{X} \to \mathcal{Y}$ with equibounded Dirichlet energy will in general have the form $T = G_{u_T} + L_T^{(i)} \times C_i + S_T$ where S_T is completely vertical, in general quite wild, but homologically trivial, while the C_i are non trivial homological 2-cycles, but only of the type S^2.

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