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Abstract

This conference is a report on a joint work with A. Aftalion and X. Blanc. A detailed paper is available on HAL: *Fast rotating condensates in an asymmetric trap* (`http://hal.archives-ouvertes.fr/hal-00342278/fr/`).

1. Introduction

1.1. Bose - Einstein condensates

Bose-Einstein Condensation in a gas: it is a new form of matter at the coldest temperatures in the universe... Predicted 1924... by Albert Einstein and Satyendra Nath Bose ...Created 1995 by Eric Cornell, Wolfgang Ketterle, Carl Wieman.

When the temperature is close enough to the absolute zero, ($\sim$ 1 nanokelvin), identical atoms with integer spin, trapped in a potential well are all described by the same wave function, linked to the ground state of the well. Fast rotating BEC: the rotation is triggering the creation of vortices, i.e. lines where the density $|\psi|^2$ is vanishing.

1.2. The Gross-Pitaevskii energy

The starting point is

$$E_{GP}(\psi) = \frac{1}{2}\langle q^w_{\omega,\nu,\epsilon}\psi, \psi\rangle_{L^2(\mathbb{R}^2)} + \frac{g}{2}\int_{\mathbb{R}^2} |\psi|^4 dx, \tag{1.1}$$

where $q_{\omega,\nu,\epsilon}$ is a quadratic form depending on the real parameters ω, ν, ϵ such that

$$\omega^2 + \nu^2 + \epsilon^2 = 1.$$

Here q^w is the operator with Weyl symbol

$$q(x_1, x_2, \xi_1, \xi_2) = \xi_1^2 + \xi_2^2 + (1-\nu^2)x_1^2 + (1+\nu^2)x_2^2 - 2\omega(x_1\xi_2 - x_2\xi_1),$$

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i.e. $q^w = D_1^2 + D_2^2 + (1-\nu^2)x_1^2 + (1+\nu^2)x_2^2 - 2\omega(x_1 D_2 - x_2 D_1)$. We would like to minimize the energy $E_{GP}(\psi)$ under the constraint $\|\psi\|_{L^2} = 1$ and understand what is happening when $\epsilon \to 0$.

1.3. The isotropic Lowest Landau Level

When the harmonic trap is isotropic, i.e. when $\nu = 0$, it turns out that

$$q = (\xi_1 + \omega x_2)^2 + (\xi_2 - \omega x_1)^2 + \epsilon^2(x_1^2 + x_2^2)$$

so that

$$2E_{GP}(\psi) = \|(D_1 + \omega x_2)\psi + i(D_2 - \omega x_1)\psi\|^2 + \frac{\omega}{\pi}\|\psi\|^2 + \epsilon^2\||x|\psi\|^2 + g\int |\psi|^4 dx.$$

We note that, with $z = x_1 + ix_2$,

$$D_1 + \omega x_2 + i(D_2 - \omega x_1) = \frac{1}{i\pi}\bar{\partial} - i\omega z = \frac{1}{i\pi}(\bar{\partial} + \pi\omega z)$$

and defining $LLL_\omega = \{\psi \in L^2, \psi = f(z)e^{-\pi\omega|z|^2}\} = \ker(\bar{\partial} + \pi\omega z) \cap L^2$, we get for $\psi \in LLL, \|\psi\|_{L^2} = 1$,

$$E_{GP}(\psi) = \frac{1}{2}\|\underbrace{(D_1 + \omega x_2)\psi + i(D_2 - \omega x_1)\psi}_{(i\pi)^{-1}(\bar{\partial}+\pi\omega z)\psi=0}\|^2 + \frac{\omega}{2\pi} + \frac{\epsilon^2}{2}\||x|\psi\|^2 + \frac{g}{2}\int |\psi|^4 dx,$$

and with $\psi(x) = u\Big((\omega\epsilon)^{1/2}x\Big)(\omega\epsilon)^{1/2}$ (unitary change in $L^2(\mathbb{R}^2)$),

$$E_{GP}(\psi) = \frac{\omega}{2\pi} + \frac{\epsilon}{2\omega}\left(\int |y|^2|u(y)|^2 dy + \omega^2 g\int |u(y)|^4 dy\right).$$

The minimization problem of $E_{GP}(\psi)$ in the LLL_ω is thus reduced to study

$$E_{LLL}(u) = \||x|u\|_{L^2}^2 + \omega^2 g\|u\|_{L^4}^4, \quad u \in LLL_{\epsilon^{-1}}$$

i.e. with $z = x_1 + ix_2$, $u(x_1, x_2) = f(z)e^{-\pi\epsilon^{-1}|z|^2}$, f entire (and $u \in L^2$).

$$E_{LLL}(u) = \int |x|^2|u(x)|^2 dx + \omega^2 g\int |u(x)|^4 dx, \ L^2 \ni u = f(z)e^{-\pi\epsilon^{-1}|z|^2}, \ f \text{ entire}.$$

This program has been successively carried out in a paper by A. Aftalion, X. Blanc, F. Nier. A key point is the fact that, in this isotropic case, the diagonalisation of the quadratic hamiltonian is rather simple:

$$q = \overbrace{(\frac{1-\omega}{2})(\xi_1 - x_2)^2}^{\eta_1^2} + \overbrace{(\frac{1-\omega}{2})(\xi_2 + x_1)^2}^{\mu_1^2 y_1^2} + \underbrace{(\frac{1+\omega}{2})(\xi_1 + x_2)^2}_{\eta_2^2} + \underbrace{(\frac{1+\omega}{2})(\xi_2 - x_1)^2}_{\mu_2^2 y_2^2}$$

Since we want to tackle the non-isotropic case where $\nu \neq 0$ in

$$q = \xi_1^2 + \xi_2^2 + (1-\nu^2)x_1^2 + (1+\nu^2)x_2^2 - 2\omega(x_1\xi_2 - x_2\xi_1),$$

one would like to determine the anisotropic LLL. The symplectic diagonalization of q requires some particular attention.

2. Quadratic Hamiltonians

2.1. On positive definite quadratic forms on symplectic spaces

If Q is a positive definite quadratic form on the symplectic $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$, one can find some symplectic linear coordinates (y, η) such that

$$Q = \sum_{1 \le j \le n} \mu_j (\eta_j^2 + y_j^2), \quad \mu_j > 0.$$

The $2n$ eigenvalues of the fundamental matrix

$$F = \sigma Q, \quad \sigma = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

(σ is the matrix of the symplectic form) are the $\{\pm i\mu_j\}_{1 \le j \le n}$, related to the $2n$ eigenvectors $\{e_j \pm i\epsilon_j\}_{1 \le j \le n}$. The $\{e_j, \epsilon_j\}_{1 \le j \le n}$ make a symplectic basis of $\mathbb{R}^{2n}$:

$$\sigma(\epsilon_j, e_k) = \delta_{j,k}, \quad \sigma(\epsilon_j, \epsilon_k) = \sigma(e_j, e_k) = 0,$$

and the symplectic planes $\Pi_j = \mathbb{R}e_j \oplus \mathbb{R}\epsilon_j$ are orthogonal for Q.

A one-line-proof of these classical facts: on $\mathbb{C}^{2n}$ equipped with the dot-product Q, diagonalize the sesquilinear Hermitian form $i\sigma$. The symplectic group $Sp(n)$ (a subgroup of $Sl(2n, \mathbb{R})$), is defined by the equation

$$\chi^* \sigma \chi = \sigma, \qquad \text{i.e. } \forall X, Y \in \mathbb{R}^{2n},\ \langle \sigma \chi X, \chi Y \rangle = \langle \sigma X, Y \rangle.$$

Lemma. Let $B \in GL(n, \mathbb{R})$ and let A, C be $n \times n$ real symmetric matrices. Then the matrix Ξ, given by $n \times n$ blocks

$$\Xi_{A,B,C} = \begin{pmatrix} B^{-1} & -B^{-1}C \\ AB^{-1} & B^* - AB^{-1}C \end{pmatrix} = \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} I & -C \\ 0 & I \end{pmatrix}$$

belongs to $Sp(n)$. Any element of $Sp(n)$ can be written as a product $\Xi_{A_1,B_1,C_1}\Xi_{A_2,B_2,C_2}$.

We shall use that lemma to diagonalize our Hamiltonian.

2.2. Effective diagonalization and the symplectic group

We want $q = \eta_1^2 + \mu_1^2 y_1^2 + \eta_2^2 + \mu_2^2 y_2^2$. We find (easily) the eigenvalues

$$0 \le \mu_1^2 = 1 + \omega^2 - \alpha \le \mu_2^2 = 1 + \omega^2 + \alpha, \quad \alpha = \sqrt{\nu^4 + 4\omega^2}.$$

We want to determine the symplectic mapping reducing q. It gets more complicated: we introduce the parameters

$$\beta_1 = \frac{2\omega\mu_1}{\alpha - 2\omega^2 + \nu^2} = \frac{\alpha - 2\omega^2 - \nu^2}{2\omega\mu_1} \quad \text{since } (\alpha-2\omega^2)^2 - \nu^4 = 4\omega^2 + 4\omega^4 - 4\omega^2\alpha = 4\omega^2\mu_1^2\ ,$$
$$\beta_2 = \frac{2\omega\mu_2}{\alpha + 2\omega^2 + \nu^2} = \frac{\alpha + 2\omega^2 - \nu^2}{2\omega\mu_2} \quad \text{since } (\alpha+2\omega^2)^2 - \nu^4 = 4\omega^2 + 4\omega^4 + 4\omega^2\alpha = 4\omega^2\mu_2^2\ ,$$
$$\gamma = \frac{2\alpha}{\omega},$$
$$\lambda_1^2 = \frac{\mu_1}{\mu_1 + \beta_1\beta_2\mu_2} = \frac{1}{1 + \frac{\beta_1\beta_2\mu_2}{\mu_1}} = \frac{1}{1 + \frac{\alpha+2\omega^2-\nu^2}{\alpha-2\omega^2+\nu^2}} = \frac{\alpha - 2\omega^2 + \nu^2}{2\alpha},$$
$$\lambda_2^2 = \frac{\mu_2}{\mu_2 + \beta_1\beta_2\mu_1} = \frac{1}{1 + \frac{\beta_1\beta_2\mu_1}{\mu_2}} = \frac{1}{1 + \frac{\alpha-2\omega^2-\nu^2}{\alpha+2\omega^2+\nu^2}} = \frac{\alpha + 2\omega^2 + \nu^2}{2\alpha},$$
$$\text{and we have} \quad \lambda_1^2 + \lambda_2^2 = 1 + \frac{\nu^2}{\alpha}, \qquad \lambda_1^2\lambda_2^2 = \frac{(\alpha+\nu^2)^2 - 4\omega^4}{4\alpha^2},$$
$$d = \frac{\gamma\lambda_1\lambda_2}{2}, \quad c = \frac{\lambda_1^2 + \lambda_2^2}{2\lambda_1\lambda_2} \quad \text{which gives} \quad cd = \frac{2\alpha(1+\nu^2/\alpha)}{4\omega} = \frac{\alpha+\nu^2}{2\omega}.$$

We define the symplectic mapping χ:

$$\begin{pmatrix} x_1 \\ x_2 \\ \xi_1 \\ \xi_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda_1 & 0 & 0 & -\frac{\lambda_1}{d} \\ 0 & \lambda_2 & -\frac{\lambda_2}{d} & 0 \\ 0 & \frac{d}{\lambda_1} - \lambda_2 cd & c\lambda_2 & 0 \\ \frac{d}{\lambda_2} - \lambda_1 cd & 0 & 0 & c\lambda_1 \end{pmatrix}}_{=\chi} \begin{pmatrix} y_1 \\ y_2 \\ \eta_1 \\ \eta_2 \end{pmatrix}$$

and we have

$$\begin{pmatrix} y_1 \\ y_2 \\ \eta_1 \\ \eta_2 \end{pmatrix} = \underbrace{\begin{pmatrix} c\lambda_2 & 0 & 0 & \frac{\lambda_2}{d} \\ 0 & c\lambda_1 & \frac{\lambda_1}{d} & 0 \\ 0 & -\frac{d}{\lambda_2} + \lambda_1 cd & \lambda_1 & 0 \\ -\frac{d}{\lambda_1} + \lambda_2 cd & 0 & 0 & \lambda_2 \end{pmatrix}}_{=\chi^{-1}} \begin{pmatrix} x_1 \\ x_2 \\ \xi_1 \\ \xi_2 \end{pmatrix}.$$

The matrix $\chi = \Xi_{A,B,C} = \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} I & -C \\ 0 & I \end{pmatrix}$ with

$$B = \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & d^{-1} \\ d^{-1} & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \frac{d}{\lambda_1\lambda_2} - cd \\ \frac{d}{\lambda_1\lambda_2} - cd & 0 \end{pmatrix}.$$

Anyhow, is is possible to find explicitly a symplectic transformation diagonalizing our quadratic form.

$$\begin{aligned} q(x,\xi) &= \xi_1^2 + \xi_2^2 + (1-\nu^2)x_1^2 + (1+\nu^2)x_2^2 - 2\omega(x_1\xi_2 - x_2\xi_1) \\ &= (q \circ \chi)(y,\eta) = \eta_1^2 + \mu_1^2 y_1^2 + \eta_2^2 + \mu_2^2 y_2^2. \end{aligned}$$

Moreover the particular form $\Xi_{A,B,C}$ above is quite helpful.

2.3. Generating functions

We define on $\mathbb{R}^n \times \mathbb{R}^n$ the *generating function* S of the symplectic mapping

$$\Xi_{A,B,C} = \begin{pmatrix} B^{-1} & -B^{-1}C \\ AB^{-1} & B^* - AB^{-1}C \end{pmatrix} = \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} I & -C \\ 0 & I \end{pmatrix}$$

by the identity

$$S(x,\eta) = \frac{1}{2}\Big(\langle Ax, x\rangle + 2\langle Bx, \eta\rangle + \langle C\eta, \eta\rangle\Big) \quad \text{so that} \quad \Xi\Big(\frac{\partial S}{\partial \eta} \oplus \eta\Big) = x \oplus \frac{\partial S}{\partial x}.$$

Note: for a symplectic mapping Ξ, to be of the form above is equivalent to the assumption that the mapping $x \mapsto pr_1\Xi(x\oplus 0)$ is invertible from $\mathbb{R}^n$ to $\mathbb{R}^n$; moreover, if this mapping is not invertible, the symplectic mapping Ξ is the product of two mappings of the type $\Xi_{A,B,C}$. The (symplectic) mapping $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ has no generating function but is the product

$$2^{-1/2}\begin{pmatrix} I & I \\ -I & I \end{pmatrix} 2^{-1/2}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}.$$

3. Quantization

3.1. Irving E. Segal formula

Let a be defined on $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$. Its Weyl quantization is the operator

$$(a^w u)(x) = \iint e^{2i\pi(x-x')\xi} a(\frac{x+x'}{2}, \xi) u(x') dx' d\xi.$$

Note that $\xi_j^w u = \frac{1}{2i\pi}\frac{\partial u}{\partial x_j} = D_j u$, $\quad x_j^w u = x_j u$, $\quad (x_j\xi_j)^w = \frac{1}{2}\Big(x_j D_j + D_j x_j\Big)$. Let χ be a linear symplectic transformation $\chi(y,\eta) = (x,\xi)$. Segal's formula : there exists a unitary transformation M of $L^2(\mathbb{R}^n)$ such that

$$(a \circ \chi)^w = M^* a^w M, \quad \text{i.e.} \quad \langle (a\circ\chi)^w v_1, v_2\rangle_{L^2(\mathbb{R}^n_y)} = \langle a^w M v_1, M v_2\rangle_{L^2(\mathbb{R}^n_x)}.$$

$$\begin{array}{ccc} L^2(\mathbb{R}^n_x) & \xrightarrow{a^w} & L^2(\mathbb{R}^n_x) \\ {\scriptstyle M}\uparrow & & \downarrow{\scriptstyle M^*} \\ L^2(\mathbb{R}^n_y) & \xrightarrow[(a\circ\chi)^w]{} & L^2(\mathbb{R}^n_y) \end{array}$$

3.2. The metaplectic group and the generating functions

For a given χ, how can we determine M ? Of course M is not unique, since it can be multiplied by a complex number with modulus 1 (we shall see that it is the only latitude). We shall not need here the rich algebraic structure of the two-fold covering $Mp(n)$ (the metaplectic group in which live the transformations M) of the symplectic group $Sp(n)$. We are glad to know a generating function for χ since, in that case

$$(Mv)(x) = \int e^{2i\pi S(x,\eta)} \hat{v}(\eta) d\eta |\det B|^{1/2},$$

with $S(x,\eta) = \frac{1}{2}\Big(\langle Ax, x\rangle + 2\langle Bx, \eta\rangle + \langle C\eta, \eta\rangle\Big)$.

3.3. Explicit expression for M

We have

$$\begin{aligned} q(x,\xi) &= \xi_1^2 + \xi_2^2 + (1-\nu^2)x_1^2 + (1+\nu^2)x_2^2 - 2\omega(x_1\xi_2 - x_2\xi_1) \\ &= (q\circ\chi)(y,\eta) = \eta_1^2 + \mu_1^2 y_1^2 + \eta_2^2 + \mu_2^2 y_2^2, \\ (q\circ\chi)^w &= M^* q^w M, \quad (Mv)(x) = \int e^{2i\pi S(x,\eta)}\hat{v}(\eta)d\eta|\det B|^{1/2}, \\ S(x,\eta) &= \frac{1}{2}\Big(\langle Ax, x\rangle + 2\langle Bx, \eta\rangle + \langle C\eta, \eta\rangle. \end{aligned}$$

We get readily

$$\begin{aligned} (Mv)(x_1, x_2) =& (\lambda_1\lambda_2)^{-1/2} e^{2i\pi d\left((\lambda_1\lambda_2)^{-1} - c\right)x_1x_2} \\ & \times \iint e^{2i\pi d^{-1}\eta_1\eta_2}\hat{v}(\eta_1, \eta_2)e^{2i\pi(\lambda_1^{-1}x_1\eta_1 + \lambda_2^{-1}x_2\eta_2)}d\eta_1 d\eta_2, \\ (Mv)(x_1, x_2) =& (\lambda_1\lambda_2)^{-1/2} e^{2i\pi d\left((\lambda_1\lambda_2)^{-1} - c\right)x_1x_2}\big(e^{2i\pi d^{-1}D_1D_2}v\big)(\lambda_1^{-1}x_1, \lambda_2^{-1}x_2). \end{aligned}$$

After some algebraic complications, χ and M are known explicitely in terms of ν, ω. Because of this complexity, it does not seem possible to avoid using some algebraic structure as a compass in the computations in the anisotropic case.

4. The Fock-Bargmann space and the anisotropic LLL

4.1. Nonnegative quantization and entire functions

The isotropic lowest Landau level was defined earlier as

$$LLL = \{u \in L^2, u = f(z)e^{-\pi|z|^2}, \text{ with } f \text{ entire}\} = \ker(\bar{\partial} + \pi z) \cap L^2.$$

That space plays an important rôle in our problem of minimisation, since it is the space in which we minimize the Gross-Pitaevskii energy, leading to the problem, for $L^2(\mathbb{R}^2) \ni u = f(z)e^{-\pi\epsilon^{-1}|z|^2}$, f entire, $\|u\|_{L^2} = 1$,

$$E_{LLL}(u) = \int |x|^2|u(x)|^2 dx + \omega^2 g\int |u(x)|^4 dx, \quad \omega^2 + \epsilon^2 = 1.$$

As already said, we wish to determine the anisotropic analogue of that space and that energy, and we notice that it has many links with well-known objects and notions, such as the Coherent States Method, anti-Wick symbols...

For $X, Y \in \mathbb{R}^{2n}$ we set

$$\Pi_H(X, Y) = e^{-\frac{\pi}{2}|X-Y|^2}e^{-i\pi[X,Y]}$$

where $[X, Y]$ is the symplectic form (if $X = (x, \xi), Y = (y, \eta)$, $[X, Y] = \xi\cdot y - \eta\cdot x$). The operator Π_H with kernel $\Pi_H(X, Y)$ is the orthogonal projection in $L^2(\mathbb{R}^{2n})$ on a proper closed subspace H, canonically isomorphic to $L^2(\mathbb{R}^n)$. In fact, one may define $W : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$ by the formula

$$(Wu)(y,\eta) = \langle u, \varphi_{y,\eta}\rangle_{L^2(\mathbb{R}^n)}, \quad \varphi_{y,\eta}(x) = 2^{n/4}e^{-\pi(x-y)^2}e^{2i\pi(x-\frac{y}{2})\eta}.$$

It is standard and easy to see that

$$
\begin{aligned}
&W^*W = \mathrm{Id}_{L^2(\mathbb{R}^n)} \quad \text{(reconstruction formula } u(x) = \int_{\mathbb{R}^{2n}} Wu(Y)\varphi_Y(x)dY),\\
&WW^* = \Pi_H, \quad W \text{ is an isomorphism from } L^2(\mathbb{R}^n) \text{ onto } H,\\
&\quad H =\{u \in L^2(\mathbb{R}^{2n}_{y,\eta}) \text{ such that } u = f(z)e^{-\frac{\pi}{2}|z|^2},\ z = \eta + iy\ ,\ f \text{ entire}\},\\
&\qquad \text{which is the isotropic } LLL, \text{ up to some normalization constant.}
\end{aligned}
$$

For a Hamiltonian a, we define $a^{\mathrm{Wick}} = W^*aW$:

$$
\begin{array}{ccc}
L^2(\mathbb{R}^{2n}) & \xrightarrow[\text{(multiplication by } a)]{a} & L^2(\mathbb{R}^{2n}) \\
W \big\uparrow & & \big\downarrow W^* \\
L^2(\mathbb{R}^n) & \xrightarrow[a^{\mathrm{Wick}}]{} & L^2(\mathbb{R}^n)
\end{array}
$$

Note that $a(x,\xi) \geq 0 \Longrightarrow a^{\mathrm{Wick}} = W^*aW \geq 0,$ as an operator. The relationship between the Weyl quantization and that Wick formula is interesting, but for our problem here it is enough to note that, if q is a quadratic form,

$$
q^{\mathrm{Wick}} = q^w + \frac{1}{4\pi}\operatorname{trace} q,
$$

in such a way that, up to some unimportant shift in the value of the energy, we may choose whatever quantization. A consequence of the above formula is also that, for $\Phi = Wu \in H$, $\|\Phi\|_{L^2(\mathbb{R}^{2n})} = 1 = \|u\|_{L^2(\mathbb{R}^n)}$,

$$
\iint y_j^2|\Phi(y,\eta)|^2 dyd\eta = \ \langle y_j^2 Wu, Wu\rangle_{L^2(\mathbb{R}^{2n})} = \langle y_j^2 u, u\rangle_{L^2(\mathbb{R}^n)} + \frac{1}{4\pi}\|u\|^2_{L^2(\mathbb{R}^n)} \geq 1/(4\pi).
$$

In other words: the lowerbound of the multiplication by x_1^2 in the LLL is bounded below by a positive constant.

4.2. The anisotropic LLL

We recall:

$$
\begin{aligned}
q(x,\xi) &= \xi_1^2 + \xi_2^2 + (1-\nu^2)x_1^2 + (1+\nu^2)x_2^2 - 2\omega(x_1\xi_2 - x_2\xi_1)\\
&= (q\circ\chi)(y,\eta) = \eta_1^2 + \mu_1^2 y_1^2 + \eta_2^2 + \mu_2^2 y_2^2.\\
(q\circ\chi)^w &= M^* q^w M, \quad (Mv)(x) = \int e^{2i\pi S(x,\eta)}\hat{v}(\eta)d\eta|\det B|^{1/2},
\end{aligned}
$$

S is a generating function of χ, $\mu_2 \sim 1$, $\mu_1 \sim \nu\epsilon$,

$$
\begin{aligned}
q^w = &\overbrace{\Big((\lambda_1 cd - d\lambda_2^{-1})x_2 + \lambda_1 D_{x_1}\Big)^2}^{\eta_1^2} + \overbrace{\mu_1^2\Big(\lambda_2 d^{-1}D_{x_2} + c\lambda_2 x_1\Big)^2}^{\mu_1^2 y_1^2}\\
&+ \underbrace{\Big((\lambda_2 cd - d\lambda_1^{-1})x_1 + \lambda_2 D_{x_2}\Big)^2}_{\eta_2^2} + \underbrace{\mu_2^2\Big(\lambda_1 d^{-1}D_{x_1} + c\lambda_1 x_2\Big)^2}_{\mu_2^2 y_2^2}.
\end{aligned}
$$

In the (y,η) coordinates, we want to consider the orthogonal projection onto the space of the $v_1(y_1)\otimes 2^{1/4}\mu_2^{1/4}e^{-\pi\mu_2 y_2^2}$, where $v_1\in L^2(\mathbb{R})$. In fact we examine

$$q_0^w=\Big(\mu_2\lambda_1 d^{-1}D_{x_1}+\mu_2 c\lambda_1 x_2\Big)^2+\Big(\lambda_2 D_{x_2}-(d\lambda_1^{-1}-\lambda_2 cd)x_1\Big)^2$$

and we note that

$$\langle q_0^w u,u\rangle=\|\Big(\mu_2\lambda_1 d^{-1}D_{x_1}+\mu_2 c\lambda_1 x_2\Big)u+i\Big(\lambda_2 D_{x_2}-(d\lambda_1^{-1}-\lambda_2 cd)x_1\Big)u\|^2+\frac{\mu_2}{2\pi}\|u\|^2.$$

The equation defining the LLL is

$$\overbrace{\Big(\mu_2\lambda_1 d^{-1}D_{x_1}+\mu_2 c\lambda_1 x_2\Big)}^{\mu_2 y_2}+i\overbrace{\Big(\lambda_2 D_{x_2}-(d\lambda_1^{-1}-\lambda_2 cd)x_1\Big)}^{\eta_2}$$
$$=\frac{1}{2i\pi}\Big(\mu_2\lambda_1 d^{-1}\partial_1+i\lambda_2\partial_2+2i\pi\mu_2 c\lambda_1 x_2+2\pi(d\lambda_1^{-1}-\lambda_2 cd)x_1\Big)$$
$$=\frac{1}{i\pi}\Big(\frac12\mu_2\lambda_1 d^{-1}\partial_1+i\frac12\lambda_2\partial_2+i\pi\mu_2 c\lambda_1 x_2+\pi(d\lambda_1^{-1}-\lambda_2 cd)x_1\Big).$$

We set $t_1=\mu_2^{-1}\lambda_1^{-1}dx_1,\quad t_2=\lambda_2^{-1}x_2$ and we get for $z=t_1+it_2$

$$\frac{\partial}{\partial\bar z}+i\pi\mu_2 c\lambda_1\lambda_2 t_2+\pi(d\lambda_1^{-1}-\lambda_2 cd)\mu_2\lambda_1 d^{-1}t_1$$
$$=\frac{\partial}{\partial\bar z}+i\pi\mu_2 c\lambda_1\lambda_2\frac{z-\bar z}{2i}+\pi(d\lambda_1^{-1}-\lambda_2 cd)\mu_2\lambda_1 d^{-1}\frac{z+\bar z}{2}$$
$$=\frac{\partial}{\partial\bar z}+z\pi\frac{\mu_2}{2}+\bar z\pi\frac{\mu_2}{2}(1-2\lambda_1\lambda_2 c)=\frac{\partial}{\partial\bar z}+z\pi\frac{\mu_2}{2}-\bar z\pi\frac{\mu_2}{2}\nu^2\alpha^{-1}$$
$$=e^{-\pi\frac{\mu_2}{2}z\bar z}e^{\pi\frac{\nu^2\mu_2}{4\alpha}(\bar z)^2}\frac{\partial}{\partial\bar z}e^{\pi\frac{\mu_2}{2}z\bar z}e^{-\pi\frac{\nu^2\mu_2}{4\alpha}(\bar z)^2}$$

Going back to the original coordinates, the anisotropic LLL is thus the subspace of the functions u of $L^2(\mathbb{R}^2)$ such that

$$f\Big(x_1+i\beta_2 x_2\Big)\exp\left(-\frac{\gamma\pi}{4\beta_2}\Big[x_1^2(1-\frac{\nu^2}{2\alpha})+(\beta_2 x_2)^2(1+\frac{\nu^2}{2\alpha})\Big]\right)\exp(-i\frac{\pi\nu^2\gamma}{4\alpha}x_1x_2),$$

where f is entire. The parameters $\beta_2,\dots$ were defined earlier and are explicitely known in terms of ω,ν.

Two remarks.

- First, it seems difficult to guess the above definition without going through the explicit computations on the diagonalization of q.
- The operator M can be used to give an explicit expression for the isomorphism between $L^2(\mathbb{R})$ and the anisotropic LLL and also to express the Gross-Pitaevskii energy in the new coordinates.

4.3. From a Lagrangean plane to a symplectic plane for the LLL

The Fock-Bargmann space has a particular structure: it is a proper closed subspace of $L^2(\mathbb{R}^2)$, canonically isomorphic to $L^2(\mathbb{R})$; we have seen in the isotropic case that

$$LLL=W(L^2(\mathbb{R}))$$

with some explicit isometric transformation W. The same properties can be carried out in similar terms for the anisotropic case, with a different W, say $\widetilde{W}$; since we know that, for quadratic forms, the Wick and the Weyl quantizations are the same, up to some shift, we may decide to choose the quantization given by $\widetilde{W}$ to quantize q, that is to consider the operator

$$(\widetilde{W})^* q \widetilde{W}.$$

The outcome of these computations lead directly to a one-dimensional problem on which we will give some details in the conclusion. The Lagrangean plane $\{\xi_1 = \xi_2 = 0\}$ of $\mathbb{R}^4$ on which the function u is defined, has also a symplectic structure.

5. The various regimes for the energy

5.1. The energy in the anisotropic LLL

In the LLL, one can simplify the energy. We define

$$A_2 = M(\eta_2 - i\mu_2 y_2)^w M^* = \mu_2 \left(\lambda_1 d^{-1} D_{x_1} + c\lambda_1 x_2\right) + i\left(\lambda_2 D_{x_2} - (d\lambda_1^{-1} - \lambda_2 cd)x_1\right),$$

$$A_1 = M(\eta_1 - i\mu_1 y_1)^w M^* = \mu_1 \left(\lambda_2 d^{-1} D_{x_2} + c\lambda_2 x_1\right) + i\left((\lambda_1 cd - d\lambda_2^{-1})x_2 + \lambda_1 D_{x_1}\right).$$

We have proven

$$q^w = A_1^* A_1 + A_2^* A_2 + \frac{\mu_1 + \mu_2}{2\pi} = (\operatorname{Re} A_1)^2 + (\operatorname{Im} A_1)^2 + (\operatorname{Re} A_2)^2 + (\operatorname{Im} A_2)^2$$

and the LLL is defined by the equation $A_2 u = 0$. On the other hand, we have

$$d\mu_1^{-1} \operatorname{Re} A_1 - \operatorname{Im} A_2 = d\lambda_1^{-1} x_1, \quad d\mu_2^{-1} \operatorname{Re} A_2 - \operatorname{Im} A_1 = d\lambda_2^{-1} x_2,$$

and thus for $u \in LLL$, since $A_2 u = 0$, using the commutation relations of the A_j's, one gets

$$\begin{aligned} d^2\lambda_1^{-2} x_1^2 &= d^2\mu_1^{-2} (\operatorname{Re} A_1)^2 + ((A_2 - A_2^*)/2i)^2 + 2d\mu_1^{-1} (\operatorname{Re} A_1)(A_2 - A_2^*)/2i \\ &= d^2\mu_1^{-2} (\operatorname{Re} A_1)^2 + \frac{\mu_2}{4\pi}, \end{aligned}$$

and similarly,

$$\begin{aligned} d^2\lambda_2^{-2} x_2^2 &= d^2\mu_2^{-2} ((A_2 + A_2^*)/2)^2 + (\operatorname{Im} A_1)^2 \\ &= (\operatorname{Im} A_1)^2 + \frac{d^2}{4\pi\mu_2}. \end{aligned}$$

As a result, we get on the LLL,

$$\mu_1^2 \lambda_1^{-2} x_1^2 + d^2 \lambda_2^{-2} x_2^2 = (\operatorname{Re} A_1)^2 + (\operatorname{Im} A_1)^2 + \frac{d^2}{4\pi\mu_2} + \frac{\mu_2 \mu_1^2}{4\pi d^2},$$

and $q^w = \mu_1^2 \lambda_1^{-2} x_1^2 + d^2 \lambda_2^{-2} x_2^2 - \frac{d^2}{4\pi\mu_2} - \frac{\mu_2\mu_1^2}{4\pi d^2} + \frac{\mu_2}{2\pi}$, so that

$$\begin{aligned} 2E_{LLL}(u) = \frac{\gamma}{2} \int_{\mathbb{R}^2} \left(\mu_1 \beta_1 x_1^2 + \frac{\mu_1}{\beta_1} x_2^2\right) |u(x_1, x_2)|^2 dx_1 dx_2 + g \int_{\mathbb{R}^2} |u(x_1, x_2)|^4 dx_1 dx_2 \\ + \frac{\mu_2}{2\pi} - \frac{\mu_1}{4\pi} \left(\beta_1 \beta_2 + \frac{1}{\beta_1 \beta_2}\right). \end{aligned}$$

We note that

$$\frac{\gamma\mu_1\beta_1}{2} = \frac{2\alpha}{\alpha+2\omega^2+\nu^2}\epsilon^2, \text{ (coefficient of } x_1^2) , \qquad \frac{\gamma\mu_1}{2\beta_1} = \frac{2\alpha(2\nu^2+\epsilon^2)}{\alpha-\nu^2+2\omega^2}, \text{ (coefficient of } x_2^2) .$$

A function in the LLL is in $L^2(\mathbb{R}^2)$ and can be written as

$$f\Big(x_1+i\beta_2 x_2\Big)\exp\left(-\frac{\gamma\pi}{4\beta_2}\Big[x_1^2(1-\frac{\nu^2}{2\alpha})+(\beta_2 x_2)^2(1+\frac{\nu^2}{2\alpha})\Big]\right)\exp\big(-i\frac{\pi\nu^2\gamma}{4\alpha}x_1x_2\big),$$

with f entire. We have to deal with

$$\mathcal{E}_{LLL}(u) = \int_{\mathbb{R}^2}\Big(\frac{1}{2}(\epsilon^2x_1^2+\kappa^2x_2^2)|u(x_1,x_2)|^2+\frac{g}{2}|u(x_1,x_2)|^4\Big)dx,$$

with $\kappa^2 \sim \frac{(\nu^2+\epsilon^2/2)(2-\nu^2)}{1-\nu^2}$, $\quad \nu^2+\omega^2+\epsilon^2=1$. The Euler-Lagrange equation without the holomorphic constraint is

$$(\epsilon^2x_1^2+\kappa^2x_2^2)u+2g|u|^2u=\lambda u,$$

so that the Thomas - Fermi profile for the density is

$$\rho=|u|^2=\frac{2}{\pi R_1R_2}\Big(1-\frac{x_1^2}{R_1^2}-\frac{x_2^2}{R_2^2}\Big), \quad R_1^4=4g\kappa\pi^{-1}\epsilon^{-3}, \quad R_2^4=4g\epsilon\pi^{-1}\kappa^{-3}.$$

In the isotropic case: $\nu=0, \kappa=\epsilon, R_1=R_2$, circular shape.

5.2. The weakly anisotropic case

With

$$\rho=|u|^2=\frac{2}{\pi R_1R_2}\Big(1-\frac{x_1^2}{R_1^2}-\frac{x_2^2}{R_2^2}\Big), \quad R_1^4=4g\kappa\pi^{-1}\epsilon^{-3}, \quad R_2^4=4g\epsilon\pi^{-1}\kappa^{-3},$$

R_2 is large when $\epsilon \gg \kappa^3$, i.e. $\nu \ll \epsilon^{1/3}$. We are interested first in this regime, that we call the weakly anisotropic regime. In that case, a truly anisotropic effect is visible since we may have

$$R_1^4 \sim \kappa\epsilon^{-3} \gg R_2^4 \sim \epsilon\kappa^{-3}$$

when $\kappa \gg \epsilon$, i.e. $\epsilon \ll \nu(\ll \epsilon^{1/3})$. The true isotropy ($R_1 \sim R_2$) is lasting until ν reaches the order ϵ, then for $\epsilon \ll \nu \ll \epsilon^{1/3}$, the profile is truly anisotropic with $R_1 \gg R_2$. The projected GP equation is

$$\Pi_{LLL}\Big[\Big(\frac{\epsilon^2}{2}x_1^2+\frac{\kappa^2}{2}x_2^2 g|u|^2-\mu\Big)u\Big]=0,$$

and in the weakly anisotropic regime, we can look at the equation of the Abrikosov problem

$$\Pi_{LLL}\Big[\Big(g|u|^2-\mu\Big)u\Big]=0,$$

whose solution is given by a Theta function,

$$u(x,\zeta)=e^{\frac{\gamma}{8\beta}(z^2-|z|^2)}\Theta\left(z\sqrt{\frac{\gamma\,\mathrm{Im}\,\tau}{4\pi\beta}},\tau\right), \quad z=x_1+i\beta x_2, \ \tau \text{ is the lattice parameter.}$$

The zeroes of the function u lie on the lattice

$$\sqrt{\frac{4\pi\beta}{\gamma\,\mathrm{Im}\,\tau}}(\mathbb{Z}+\tau\mathbb{Z})$$

and $|u|$ is periodic. The optimal lattice minimizing $\fint |u|^4/(\fint |u|^2)^2$ (integrals taken on one period) is the hexagonal one ($\tau = e^{2i\pi/3}$). As in the isotropic case, we can construct an approximate ground state by multiplying the Theta function by a profile ρ as above. To have a solution in the LLL, we define

$$v = \Pi_{LLL}(\rho u_\Theta).$$

We prove then that

$$\mathcal{E}_{LLL}(v) = \int \frac{1}{2}(\epsilon^2 x_1^2 + \kappa^2 x_2^2)\rho dx + \frac{g\mu(\tau)}{2}\int \rho^2 dx + O\Big((\kappa\epsilon)^{1/2}(\kappa^3/\epsilon)^{1/8} = \kappa^{7/8}\epsilon^{3/8}\Big).$$

5.3. The strongly anisotropic case

This is the case $\epsilon^{1/3} \ll \kappa$. In that case, the limit of the projected GP equation is

$$\Pi_{LLL}\Big[(\frac{\kappa^2}{2}x_2^2 + g|u|^2)u\Big] = \mu u.$$

The Θ function is no longer a solution, but a particular solution is given by the Gaussian

$$(\frac{\gamma\beta}{2\pi})^{1/4} e^{-\frac{\gamma\beta}{4}x_2^2 + i\frac{\gamma}{4}x_1 x_2}.$$

In this regime, the ground state behaves like this Gaussian in the x_2 direction, and an inverted parabola in the x_1 direction, without visible vortices. Indeed with

$$p(x_1) = (\frac{3}{4R})^{1/2}(1 - \frac{x_1^2}{R^2})_+^{1/2}, \quad R = \Big(\frac{3g}{4\epsilon^2}(\frac{\gamma\beta}{\pi})^{1/2}\Big)^{1/3}, \quad R \sim \epsilon^{-2/3}.$$

The ground state is approximated by

$$A e^{-\frac{\gamma\beta}{8}x_2^2}\int_{\mathbb{R}} e^{-\frac{\gamma}{8\beta}\Big((x-t)^2 - 2it\beta x_2\Big)} p(t)dt.$$

6. Conclusions and perspectives

6.1. The critical case

This is the regime $\nu = \epsilon^{1/3}$. Open. Both methods (weakly isotropic and strongly anisotropic) break down.

6.2. The full GP energy

We may consider

$$\widetilde{E}_{GP}(v) = \langle (q\circ\chi)^w v, v\rangle + \frac{g}{\lambda_1\lambda_2}\|Nv\|^4_{L^4(\mathbb{R}^2)}, \text{ with } \|v\|_{L^2(\mathbb{R}^2)} = 1,$$

with the Euler-Lagrange equation

$$\Big(D_1^2 + \mu_1^2 y_1^2 + D_2^2 + \mu_2^2 y_2^2\Big)v + \frac{2g}{\lambda_1\lambda_2}N^*\Big(|Nv|^2 Nv\Big) = \lambda v,$$
$$N = \exp 2i\pi d^{-1} D_1 D_2, \quad \mu_2 \sim 1, \quad \mu_1 \sim \nu^2\epsilon^2.$$

6.3. A quasi-ODE

Since the $LLL = W(L^2(\mathbb{R}))$, we are in fact reduced to a one-dimensional problem

$$(D_t^2 + \mu_1^2 t^2)\varphi + \frac{2g}{\lambda_1\lambda_2} W^*\Big(|W\varphi|^2 W\varphi\Big) = \lambda\varphi,$$

where φ is a function of $L^2(\mathbb{R})$. The non-linear term is close to be a local one.

- Is it interesting to use the simplicity of the quadratic terms ?
- Is it possible to handle that quasi-ODE, since W is rather explicit ?
- Maybe find some particular special functions solutions ?

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