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The Signature Theorem for Manifolds with Metric Horns

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1.   The results

The spectral analysis of geometric operators on compact Riemannian manifolds is a highly developed subject. In the last 15 years there has been increasing interest in extending the more successful parts of the theory to spaces with singularities. Usually, one deals with compact metric spaces which contain a Riemannian manifold, M, as a dense open subset. Then M can be decomposed as

$$M = M_1 \cup U \tag{1.1}$$

where M_1 (the *regular part*) is a compact Riemannian manifold with boundary, N, and U (the *singular part*) is open with $N = \partial U$. In the present discussion, the most prominent class of singular spaces are those with conic singularities in which case we assume that, as a manifold

$$U = (0, \varepsilon_0) \times N, \qquad 0 < \varepsilon_0 \le 1, \tag{1.2a}$$

equipped with the metric

$$g = dx^2 \oplus x^2 g_N. \tag{1.2b}$$

Here x is the natural coordinate in $(0, \varepsilon_0)$ and g_N is a smooth metric on N. Contributions to spectral analysis on these spaces may be found eg. in [Ch1], [Ch2], [BS1], [BS2].

Natural examples of singular spaces arise eg. from projective varieties contained in some $\mathbb{C}P^N$, equipped with the Fubini-Study metric. Looking at the metric near isolated singularities one discovers, alas, structures which are far more complicated than conic singularities. To bridge the frustratingly wide gap seems, as of today, still rather hopeless. All we can offer here is a modest first step: we will, instead of (1.2b), allow metrics of the form

$$g_\alpha := dx^2 \oplus x^{2\alpha} g_N, \qquad \alpha \ge 1. \tag{1.2c}$$

In the context at hand, Cheeger [Ch3] was apparently the first to study such spaces, he called them "manifolds with metric horns". The main concern of [Ch3] is the

L^2-Hodge theory, whereas Peyerimhoff and Lesch [P],[LP] studied index problems for geometric operators on these spaces by extendending methods of regular singular analysis as developed by Brüning and Seeley for conic singularities [BS1], [BS2]. They did not succeed, however, in proving a Signature Theorem; this we will do here.

To be more specific, we introduce the signature operator on an oriented manifold, M, with metric horns: let $m = \dim M$ be even and consider on $\Omega_0(M)$, the space of compactly supported smooth forms, the operator

$$D := d + \delta. \tag{1.3}$$

Then D anticommutes with the involution

$$\omega_M := \sqrt{-1}^{[\frac{m+1}{2}]} c(e_1) \dots c(e_m), \tag{1.4}$$

given by the complex volume element of M, where $(e_i)_1^m$ is a local oriented and orthonormal frame for TM and "c" denotes Clifford multiplication. More precisely, if $\flat : TM \to T^*M$ denotes the "musical" isomorphism (with inverse $\#$) then, for $\eta \in \Omega_0(M)$,

$$c(e_i)\eta = e_i^\flat \wedge \eta - e_i \llcorner \eta.$$

Thus, we optain a splitting $\Omega_0(M) = \Omega_0^+(M) \oplus \Omega_0^-(M)$ and

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix},$$

defining $D_S := D_+$ as the *signature operator* on M.

The first question that we encounter concerns the existence and uniqueness of self-adjoint extension of D anticommuting with ω or, equivalently, the existence of closed extensions of D_S.

Lemma 1.1 ([PL, Thm. 42) D_S *has a unique closed extension which is a Fredholm operator.* D *is essentially self-adjoint on* $\Omega_0(M)$.

We identify D and D_+ with their closed extensions. It follows that both $D_+^* D_+$ and $D_+ D_+^*$ coincide with their Friedrichs extensions on $\Omega_0(M)$. Thus, we obtain as usual

$$\text{ind } D_S = \text{tr}_{L^2(\wedge^* M)} \left[\omega e^{-tD^2} \right], \quad t > 0, \tag{1.5}$$

and we can take (1.5) as the basis to prove an index formula for D_+. The main tool in the proof is the asymptotic expansion of the function

$$I_l(t) := \text{tr}_{L^2(\wedge^* M)} \left[\omega^l e^{-tD^2} \right] \tag{1.6}$$

as $t \to 0+$, for $l = 0, 1$. Such an expansion has been announced by Callias [Ca] in the most simple case of the scalar operator, T_α, which is the Friedrichs extension in $L^2(\mathbb{R}_+)$ of

$$-\partial_x^2 + x^{-2\alpha}, \quad \alpha \geq 1, \tag{1.7}$$

with domain $C_0^\infty(0, \infty)$. The complicated details of Callias' proof have not yet appeared, as far as we know. Moreover, his proof does not seem to provide good insight into the mechanism that creates the powers of t and $\log t$ figuring in the asymptotic expansion. We will show that the problem fits nicely into the framework of regular singular analysis. In fact, the asymptotic expansion will result from a natural generalisation of the Singular Asymptotics Lemma (SAL) in [BS3]. Thus, as Callias correctly though somewhat mysteriously remarks, manifolds with metric horns can be well understood from the point of view of "conical" analysis!

The main results of this note read as follows.

Theorem 1.2 *For $l = 0, 1$ there are asymptotic expansions*

$$I_l(t) \underset{t \to 0+}{\sim} \sum_{j \geq 0} a_j^l \, t^{j-m/2} + \sum_{j \geq 0} b_j^l \, t^{(1+j\beta)/2\alpha} + \sum_{\substack{k_1, k_2 \in \mathbb{Z}_+ \\ m/2 - j + 1 = k_1 + k_2 \beta}} c_{j\,k_1 k_2}^l t^{j-m/2} \log t. \tag{1.8}$$

Theorem 1.3

$$\operatorname{ind} D_S = \int_M L(M) - \frac{1}{2}\eta(N). \tag{1.9}$$

Here $L(M)$ denotes the Hirzebruch L-form, and $\eta(N)$ the η-invariant of the "odd signature operator" given by

$$D_N := \omega_N(d_N + \delta_N) \tag{1.10}$$

on $\Omega(N)$.

Both theorems are known in the conic case so we will assume

$$\alpha =: 1 + \beta, \quad \beta > 0, \tag{1.11}$$

in what follows.

2. Proof of Theorem 1.2

The proof of this result can only be sketched here, the full details will appear elsewhere.

We begin with a suitable representation of D on U. It proceeds by separation of variables as in the conic case; the details have been worked out in [LP, Sec. 2].

Lemma 2.1 *On $\Omega_0(N)$, D is unitarily equivalent to the operator*

$$\tilde{D} := \gamma \partial_x + \tau x^{-\alpha} S(x) \tag{2.1}$$

acting on $C_0^\infty((0,\varepsilon_0),\mathbb{C}^2 \otimes \Omega(N))$. Here

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.2}$$

and, with D_N in (1.10),

$$S(x) = D_N + \alpha x^{\alpha-1}\mathrm{diag}\,(b_j) =: D_N + \alpha x^{\alpha-1}S_1,$$

$$b_j = n/2 - j,\ n = \dim N = m - 1.$$

Moreover, under this equivalence, ω_M corresponds to

$$\tilde{\omega} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

From Lemma 2.1, it is easy to compute that

$$
\begin{aligned}
\tilde{D}^2 &= -\partial_x^2 + x^{-2\alpha}\{D_N^2 + x^\beta(\omega\,(\delta_N - d_N) - \alpha\tilde{\omega}D_N) + x^{2\beta}(\alpha^2 S_1^2 + \tilde{\omega}\,\alpha\,(2\alpha - 1)S_1)\} \\
&=: -\partial_x^2 + x^{-2\alpha}\{D_N^2 + x^\beta \tilde{D}_N + x^{2\beta}\tilde{S}_1\} \\
&=: -\partial_x^2 + x^{-2\alpha}A_S(x^\beta).
\end{aligned}
\tag{2.4}
$$

We extend the operator function $A_S(x)$ smoothly to $(0,\infty)$ in such a way that

$$A_S(x) = D_N^2 + f(x)\tilde{D}_N + f(x)^2 \tilde{S}_1,$$

with $f(x) = x$ in $(0,\varepsilon_0]$, $f(x) \geq 0$ and smooth for $x > 0$, and $f(x) = 0$ for $x \geq 2\varepsilon_0$. Thus $\tilde{D}^2$ can be regarded as an operator in $L^2(\mathbb{R}_+,\mathbb{C}^2 \otimes L^2(\wedge^* N)) =: L^2(\mathbb{R}_+,\mathbb{C}^2 \otimes H)$ with domain $C_0^\infty((0,\infty),\mathbb{C}^2 \otimes \Omega(N))$. By abuse of notation, we denote by $\tilde{D}^2$ also the Friedrichs extension of this operator. Then, by Lemma 1.1, the explicit construction of the unitary equivalence in Lemma 2.1, and standard comparison arguments (as

explained in great generality eg. in [B2, Sec.4] we see that for $\phi \in C_0^\infty(-\varepsilon_0, \varepsilon_0)$ with $\phi = 1$ near 0

$$\text{tr}_{L^2(\mathbb{R}_+ \mathcal{C}^2 \otimes H)} \left[\phi \tilde{\omega}^l e^{-t\tilde{D}^2} \right] \underset{t \to 0+}{\sim} \text{tr}_{L^2(\wedge^* M)} \left[\phi \circ \pi_1 \omega_M^l \, e^{-tD^2} \right], \tag{2.5}$$

$l = 0, 1, \pi_1 : U \to (0, \varepsilon_0)$ the canonical projection. Since the expansion in the interior is well known, we have reduced the problem to an expansion problem for operator valued Sturm-Liouville equations of the type (2.4); this is, of course, the heart of the analysis.

Specifically, we will work with the same abstract framework as in [B1]. Thus we assume

$$\text{the family } (A_S(x))_{x \geq 0} \text{ satisfies the assumptions} \tag{2.6}$$
$$\text{(1.2) through (1.6) of [B1].}$$

The expansion result (loc. cit. Thm. 2.1) yields the short time asymptotics for the operator heat kernel in the commutative case. This does not apply to the family $(A_S(x))_{x \geq 0}$ in (2.4); instead we use

$$\text{for some } \gamma \in [0, 2), [A_S(x_1), A_S(x_2)] A_S(x_3)^{-\gamma}$$
$$\text{extends to a bounded operator in } \mathcal{C}^2 \otimes H, \tag{2.7}$$
$$\text{with uniform norm bound for } x, x_2, x_3 \geq 0.$$

Next we conclude by local analysis as in [BS1, Sec.4] that $\phi \, e^{-tT}$ is trace class. Hence the Trace Lemma [BS1, Appendix] implies the existence of an operator heat kernel with values in the trace class of K and the identity

$$\text{tr}_{L^2(\mathbb{R}_+, K)} \left[\phi B e^{-tT} \right] = \int_0^\infty \phi(x) \text{tr}_K \left[B e^{-tT}(x, x) \right] dx, \tag{2.8}$$

for any $B \in \mathcal{L}(K)$. To analyze this, we now use the same scaling as in the conic case i.e. we introduce the unitary operator

$$U_z f(x) := z^{1/2} f(zx), \quad f \in L^2(\mathbb{R}_+, K), \quad x, z > 0. \tag{2.9}$$

U_z maps $C_0^\infty((0, \infty), K_1)$ and $D(T)$ into itself, and we copute

$$z^2 U_z T U_z^* = -\partial_x^2 + z^{-2\beta} x^{-2\alpha} A(zx) =: \tau_z. \tag{2.10}$$

Denoting by T_z the Friedrichs extension of τ_z, we obtain the analogue of (2.10) for T_z hence

$$U_z e^{-tz^2 T} U_z^* = e^{-tT_z}, \tag{2.11a}$$

and for the kernel

$$e^{-tT}(x_1, x_2) = z^{-1}e^{-t/z^2\, T_z}(x_1/z, x_2/z).$$ (2.11b)

Using this in (2.8), we arrive at

$$\text{tr}_{L^2(\mathbb{R}_+, K)}\left[\phi B e^{-tT}\right] = \int_0^\infty \phi(x)\text{tr}_K\left[Be^{-t/x^2\, T x}(1,1)\right]\frac{dx}{x}.$$ (2.12)

The main obstacle to obtain the expansion of this integral by the SAL, as in the conic case, is the singularity $z^{-2\beta}$ in (2.10), created by the scaling.
To avoid it we put $t := \zeta^{-2\alpha}$ and rewrite the integrand in (2.12) as

$$x^{-1}\phi(x)\text{tr}_K\left[Be^{-(t/x^{2\alpha})x^{2\beta}T_x}(1,1)\right]$$

$$= x^{-1}\phi(x)\text{tr}_K\left[Be^{-(\zeta x)^{-1/2\alpha}\, x^{2\beta}\, T_x}(1,1)\right]$$

$$=: \sigma(x, x^\beta, \zeta x).$$ (2.13)

The next technical difficulty consists in the fact that σ is not a smooth function of $x \in \mathbb{R}_+$. But it depends smoothly on the two variables $y_1 = x$ and $y_2 = x^\beta$ in $\mathbb{R}_+^2$ so we can apply the following extension of the SAL.

Theorem 2.2 *Let $\sigma \in C^\infty(\mathbb{R}_+^m \times (0,\infty))$ and*

$$\beta(x) := (x_1^{\beta_1}, x_2^{\beta_2}, \ldots, x_m^{\beta_m}),\ x \in \mathbb{R}_+^m,$$ (2.14)

with $\beta_i > 0$, and put $\overline{\beta} := \sum_{i=1}^m \beta_i$. Assume that σ satisfies the following conditions.
(I) There is a countable familiy $(\sigma_{\mu j})_{\substack{\mu \to -\infty \\ 0 \le j \le j(\mu)}} \subset C^\infty(\mathbb{R}_+^m)$,
such that

$$(1 + |y|)^k\, |\sigma_{\mu j}(y)| \le C_{\mu j k},\quad k \in \mathbb{N}.$$

(II) Writing, for $N \in \mathbb{N}$,

$$R_N(y, \zeta) := \sigma(y, \zeta) - \sum_{\substack{\alpha > -N \\ 0 \le j \le j(\mu)}} \sigma_{\mu j}(y)\, \zeta^\mu \log^j \zeta$$ (2.15)

we have for all N and $\gamma \in \mathbb{Z}_+^m$

$$|\partial_y^\gamma R_N(y, \zeta)| \le C_{N\gamma}\zeta^{-N}\ \ for\ \ \zeta \ge 1,\ y_1 \ldots y_m \le \zeta^{\overline{\beta}}.$$

(III) For all $\gamma \in \mathbb{Z}_+^m$ and $\theta \in [0,1]$ we have the estimate

$$\int_0^1 \int_0^1 \beta(x)^\gamma\, |(\partial_y^\gamma \sigma)(\theta u \beta(x), x)|\, du\, dx \le C_\gamma.$$

Then, the Mellin regularized integral

$$I(\zeta) := \fint_0^\infty \sigma(\beta(x), x\zeta) \, dx$$

exists and admits the following asymptotic expansion as $\zeta \to \infty$:

$$I(\zeta) \underset{\zeta \to \infty}{\sim} \sum_{\mu,j} \fint_0^\infty \sigma_{\mu j}(\beta(x)) \, (\zeta x)^\mu \log^j(\zeta x) \, dx \qquad (2.16a)$$

$$+ \sum_{\gamma \in \mathbb{Z}_+^m} \zeta^{-1-<\beta,\gamma>} \fint_0^\infty \frac{\beta(x)^\gamma}{\gamma!} (\partial_y^\gamma \sigma)(0, \zeta) \, d\zeta \qquad (2.16b)$$

$$+ \sum_{\substack{\mu,j \\ <\beta,\gamma>+\mu=-1}} \zeta^\mu \log^{j+1}\zeta \, (\partial_y^\gamma \sigma_{\mu j}) \, (0) \, / (j+1). \qquad (2.16c)$$

We recall that the Mellin regularized integral of a locally integrable function f on $I\!\!R_+$ is defined as

$$\fint_0^\infty f(x) \, dx = \operatorname{Res}_0 \int_0^1 x^{w-1} f(x) \, dx\Big|_{w=1} + \operatorname{Res}_0 \int_1^\infty x^{w-1} f(x) \, dx\Big|_{w=1} \qquad (2.17)$$

provided both Mellin transforms in (2.17) exist and are meromorphic near $w = 1$.

Accepting for the moment that Theorem 2.2 applies to (2.13), we see that the powers occuring in the expansion depend strongly on the exponents β_i, the dependence becoming entirely transparent through (2.16).

Moreover, it is useful to remark that contributions to the power t^0 can come only from the "interior" terms in (2.16a).

It remains to verify that the σ in (2.13) fulfills indeed the assumptions of Theorem 2.2. To do so, we imitate Hadamard's expansion method for the heat kernel in our setting. This has been done in [B1] in the more special case of a commutative family $(A(x))_{x \geq 0}$ and with only small time estimates. But this approach can be generalized to the present setting to yield the following result.

Theorem 2.3 *Assume the conditions (2.6) and (2.7). Then, for any $N \in \mathbb{Z}_+$, we can write*

$$e^{-ty^T y}(1, 1) = (4\pi t y)^{-1/2} \sum_{j=0}^N (ty)^j \, U_j(y) \, e^{-tA(y)} + S_N(t, y), \qquad (2.18)$$

where U_j is a universal polynomial in the variables y^{-1}, y, and $A^{(l)}(y)$, of degree at most $2j/3$ in the first and third variables. Moreover, with some sequence $(\nu_N) \nearrow \infty$,

$$|| \partial_t^{\gamma_1} \partial_y^{\gamma_2} S_N(t,y) ||_{tr} \le c_{N\gamma_1\gamma_2} \, y^{\nu_N - \gamma_1 - |\gamma_2|} \frac{(t/y)^{\nu_N - \gamma_1 - |\gamma_2|}}{1 + (t/y)^{\nu_N - \gamma_1 - |\gamma_2|}}, \qquad (2.19)$$

for $\gamma_1 \in \mathbb{Z}_+$, $\gamma_2 \in \mathbb{Z}_+^m$, $|y| \le 1$, and $t > 0$.

Two technical adjustments are necessary now to satisfy the assumptions of Theorem 2.2: Firstly, to remove the singularities, x^{-1} in (2.13) and $y^{-1/2}$ in (2.18), we multiply (2.13) by $t^{1/2}$, using

$$t^{1/2}/x = (t/x^{2\alpha})^{1/2} x^\beta.$$

Secondly, to cut down the y-support (as necessary for (2.19)) we choose $\psi \in C_0^\infty(\mathbb{R})$ with $\psi = 1$ in a neighborhood of supp ϕ and multiply (2.13) by $\psi(x^\beta)$. Then it is easy to see that, in view of the conditions (2.6) and (2.7) each term in the sum satisfies the assumptions of Theorem 2.2. Thus, the same is true for the left hand side if we take into account the crucial estimate (2.19). Combining all arguments completes the proof of Thm. 1.2.

3. Proof of Theorem 1.3

Now we use Thm. 1.2 with $l = 1$ to derive an index formula for D_S. We choose $\varepsilon > 0$ and write $\phi_\varepsilon(x) := \phi(x/\varepsilon)$, for ϕ as in (2.5). Then we derive from (1.5), Lemma 2.1, (2.8), and the Local Index Theorem the identity

$$
\begin{aligned}
\operatorname{ind} D_S &= \operatorname{tr}_{L^2(\wedge^* M)}\left[\omega e^{-t\varepsilon^{2\alpha} D^2}\right] \\
&= \operatorname{tr}_{L^2(\mathbb{R}_+, \mathbb{C}^2 \otimes H)}\left[\phi_\varepsilon \tilde{\omega} e^{-t\varepsilon^{2\alpha} \tilde{D}^2}\right] + \int_M (1 - \phi_\varepsilon \circ \pi_1) L(M) \\
&\quad + 0_\varepsilon(t).
\end{aligned}
$$

From (1.2c) we see that U is conformally equivalent to a Riemannian product implying that the Pontrjagin classes and hence $L(M)$ vanish on U. Therefore,

$$\operatorname{ind} D_S =: \lim_{\varepsilon \to 0+} \lim_{t \to 0+} I(\varepsilon, t) + \int_{M_1} L(M). \qquad (3.1)$$

Since the family

$$A(x,y) := y^{-2\beta} A_S(xy)$$

satisfies the assumptions (2.6) and (2.7) we can apply the arguments of Sect.2 to evaluate $I(\varepsilon, t)$:

$$
\begin{aligned}
I(\varepsilon, t) &= \int\limits_0^\infty \phi_\varepsilon(x)\, \mathrm{tr}_{\mathbb{C}^2 \otimes H} \left[\tilde{\omega} e^{-t(\varepsilon/x)^{2\alpha} x^{2\beta} T_x}(1,1) \right] \frac{dx}{x} \\
&= \int\limits_0^\infty \phi(x)\mathrm{tr}_{\mathbb{C}^2 \otimes H} \left[\tilde{\omega} e^{-t x^{-2\alpha}(\varepsilon x)^{2\beta} T_{\varepsilon x}}(1,1) \right] \frac{dx}{x}.
\end{aligned} \tag{3.2}
$$

Next we plug in the expansion (2.18) for the kernel and observe that all terms except the one with $j = 0$ either get multiplied by a positive power of ε or else are $0_\varepsilon(t)$. Thus, it follows from Theorem 2.2 that a nonvanishing contribution to t^0 can at most result from the expansion of

$$
I_0(\varepsilon, t) := \int\limits_0^\infty \phi(x)(4\pi\, t x^{-2\alpha})^{-1/2}(\varepsilon x)^{-\beta}\, \mathrm{tr}_{\mathbb{C}^2 \times H} \left[\tilde{\omega} e^{-(t x^{-2\alpha})A_S(\varepsilon x)} \right] \frac{dx}{x}.
$$

We now expand the exponential around $\varepsilon x = 0$. Arguing as before we see that the $\varepsilon^{-\beta}$ singularity drops out, in view of (2.3), and that only the contribution from the second term can survive. This gives, with (2.4),

$$
\begin{aligned}
\lim_{\varepsilon \to 0+} \lim_{t \to 0+} I(\varepsilon, t) &= \lim_{\varepsilon \to 0+} \lim_{t \to 0+} I_1(\varepsilon, t) \\
&= \lim_{t \to 0+} (4\pi)^{-1/2} \int\limits_0^\infty \phi(x)(t x^{-2\alpha})^{1/2} \mathrm{tr}_{\mathbb{C}^2 \times H} \left[\tilde{\omega} \tilde{D}_N e^{-t x^{-2\alpha} D_N^2} \right] \frac{dx}{x} \\
&= \lim_{t \to 0+} -\alpha\pi^{-1/2} \int\limits_0^\infty \phi(x)(t x^{-2\alpha})^{1/2} \mathrm{tr}_H \left[D_N e^{-t x^{-2\alpha} D_N^2} \right] \frac{dx}{x} \\
&= \lim_{t \to 0+} \left(-\frac{1}{2\sqrt{\pi}} \int\limits_0^\infty \phi((t/u)^{1/2\alpha}) u^{-1/2} \mathrm{tr}_H \left[D_N e^{-u D_N^2} \right] du \right) \\
&= -\frac{1}{2\Gamma(1/2)} \int\limits_0^\infty u^{-1/2} \mathrm{tr}_H \left[D_N e^{-u D_N^2} \right] du \\
&= -\frac{1}{2}\eta(N). \tag{3.3}
\end{aligned}
$$

To carry out the last limit we have used dominated convergence, observing that the integrand admits an asymptotic expansion near $u = 0$ and hence must be integrable since the t-limit exists. This finishes the proof of Theorem 1.3.

It is remarkable that the same argument does not work in the conic case due to the lack of ε-powers. Thus the metric horns are not only amenable to conical

analysis, they turn out to be, in fact, much simpler!

However, in the conical case the same type of argument goes through if we deform the metric (1.2b), conformally on U, to the metric

$$g_\varepsilon := dx^2 \oplus x^2 \varepsilon^2 g_N.$$

This will change neither the index formula nor the interior contribution.

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