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Stable solutions and their spatial structure of the Ginzburg-Landau equation

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Abstract

Recent papers [7], [8] and [10] studies the Ginzburg-Landau equation $\Delta\Phi + \lambda(1 - |\Phi|^2)\Phi = 0, \Phi = u_1 + iu_2$ in a bounded domain $\Omega \subset \mathbb{R}^n$ with the homogeneous Neumann boundary condition. Those works revealed the instability of non-constant solutions in any convex domain and the existance of stable non-constant solutions in topologically non-trivial domains. This report surveys these studies together with introduction of a new result.

1 Introduction

In the field of superconductivity the Ginzburg-Landau (GL) equation has been playing an important role for the understanding of macroscopic superconducting phenomena. This equation was originally proposed in [4], where the magnetic effect casued by the current of superconducting electrons is taken account into the equation. Here we are concerned the simple version with no magnetic effect. Then GL equation with non-dimensional form is written as

$$\Delta\Phi + \lambda(1 - |\Phi|^2)\Phi = 0$$

where Δ denotes the Laplacian, Φ is a complex valued function $\Phi = u_1 + iu_2$ and λ is a positive parameter. Hereafter the set of complex values $\mathbb{C}$ is identified with the one of 2-dimensional vectors $\mathbb{R}^2$, so the above equation is also written in real vector form:

$$\Delta u + \lambda(1 - |u|^2)u = 0, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad |u|^2 = u_1^2 + u_2^2.$$

In the Ginzburg-Landau theory Φ denotes the macrowave function describing a superconducting state and $|\Phi|^2$ is the desnsity of superconducting electrons. Therefore $|\Phi| = 0$ corresponds to the normal state and a solution with zeros physically represents a mixed state of superconducting and normal ones. Then the zero of Φ is called a vortex.

In this paper we are concerned with GL equation in a bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2)$ subject to the homogeneous Neumann boundary condition, that is,

$$\begin{cases} \Delta\Phi + \lambda(1 - |\Phi|^2)\Phi = 0 & \text{in } \Omega \\ \frac{\partial\Phi}{\partial\nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\partial/\partial\nu$ denotes the outer normal derivatives on the smooth boundary $\partial\Omega$. We remark that Equation (1) is an Euler equation of the following energy functional (called the Ginzburg-Landau energy):

$$\mathcal{E}(\Phi) := \frac{1}{2}\int_\Omega \{|\nabla\Phi|^2 + \frac{\lambda}{2}(1 - |\Phi|^2)^2\}dx \tag{2}$$

or the stationary equation of the evolutionary GL equation

$$\begin{cases} \frac{\partial\Phi}{\partial t} = \Delta\Phi + \lambda(1 - |\Phi|^2)\Phi & \text{in } \Omega \\ \frac{\partial\Phi}{\partial\nu} = 0 & \text{on } \partial\Omega \\ \Phi(0, x) = \Phi_0(x) \end{cases} \tag{3}$$

namely a solution to (1) is given by an equlibrium solution to (3). Indeed Equation (3) is the gradient equation for the energy functional (2) and equilibrium solutions are only allowed as the asymptotic state as $t \to \infty$ of (3) (see [5]). Here, as a state space, we take a fuction space $C(\bar{\Omega} : \mathbb{C})$ of continuous functions of $\bar{\Omega}$ into $\mathbb{C}$ with sup-norm, where $\bar{\Omega}$ denotes the closure of Ω.

One easily sees that the functional (2) has a family of global minimizers $\{\Phi(x) \equiv a : |a| = 1\}$ and that those are stable constant equilibrium solutions to (3), so we are interested in the existence of stable non-constant solutions to (1) (or non-constant local minimizers of (2).) Then "stable solutions" to (1) are meant by stable equilibrium solutions to (3) and "stable" is used in the sense of Lyapunov.

We summarize some of main results given in the works [7], [8] and [10] in relevence to the existence of stable non-constant solutions to (1). In the work [7] it was revealed that any convex domain Ω never admits stable non-constant solutions, that is, the constant solutions, $\Phi \equiv a, |a| = 1$, are only stable solutions in all convex domains. On the other hand we see from [7], [8] and [10] that for a domain Ω which is topologically "non-trivial" in some sense, there exist stable non-constant solutions if λ is sufficiently large. Here "non-trivial" means "not simply connected" provided that $n = 2$ or 3 (for the precise definition, see §3).

We remark that all the stable solutions constructed in [7], [8] and [10] don't vanish in the domains. Hence we have the natural query: is there a stable solution with zeros? Or is there a stable non-constant solution in a simply connected domain? Dancer [3] first proved that there is a stable non-constant solution in a simply connecterd domain of $\mathbb{R}^n (n \geq 3)$. One can also find [9] which proves it in a different method from [3]. Moreover [9] proves that for appropriate non-trivial domains there exist stable solutions with zeros.

2 Instability of Non-constant Solutions

The next theorem was given by [7].

Theorem 2.1 *Let Ω be a bounded domain of $\mathbb{R}^n$ with C^3-boundary $\partial\Omega$. Supppose that F is a C^3-fuction of $\mathbb{R}^N$ into $\mathbb{R}$ and consider the following system of parabolic equations:*

$$\partial u_j/\partial t = \Delta u_j + \frac{\partial F}{\partial u_j}(u_1, \cdots, u_N) \quad \text{in} \quad \Omega, \qquad \partial u_j/\partial \nu = 0 \quad \text{on} \quad \partial\Omega \quad (j = 1, \cdots, N). \quad (4)$$

If the domain Ω is convex, then all the non-constant equilibrium solutions to (4) are unstable.

An equilibrium solution to (4) is given by solving the stationary problem

$$\Delta u_j + \frac{\partial F}{\partial u_j}(u_1, \cdots, u_N) = 0 \quad \text{in} \quad \Omega \qquad \partial u_j/\partial \nu = 0 \quad \text{on} \quad \partial\Omega \quad (j = 1, \cdots, N), \qquad (5)$$

This instability theorem is including the case of scalar reaction diffusion equations, that is, $N = 1$ in (4), which is known as the theorem proved by [2] and [11]. Taking $N = 2$ and $F = \lambda(1 - u_1^2 - u_2^2)^2/2$ yields Equation (1). Therefore in convex domains there is no stable non-constant solutions to the Ginzburg-Landau equation (1). We remark that a non-constant solution in a convex domain must have one or more zeros. Hence the above theorem also tells that any solution with zeros in the convex domain is unstable.

3 Stable Non-constant Solutions

We say that a domain Ω is "non-trivial" if there exists a continuous map γ in $C(\bar{\Omega}; S^1)$ which is not homotopic to a constant valued map in $C(\bar{\Omega}; S^1)$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. If a domain is non-trivial, then there are infinitely many homotopy classes in $C(\bar{\Omega}; S^1)$. It is trivial that if a domain is not simply connected, then it is non-trivial. Indeed "not simply connected" and "non-trivial" are equivalent in $\mathbb{R}^n, n = 2, 3$. It, however, is known that there is a counter example in $\mathbb{R}^n, n \geq 4$ which is not simply connected but allows only one homotopy class including a constant map (see [10]).

The existence of stable non-constant solutions to (1) was first proved in [7] for a thin annulus domain

$$\Omega(\epsilon) = \{x \in \mathbb{R}^2 : 1 < |x| < 1 + \epsilon\}$$

Actually, with polar cordinate (r, θ), Equation (1) has the following limit eqaution as $\epsilon \to 0$:

$$\frac{\partial^2 \varphi}{\partial \theta^2} + \lambda(1 - |\varphi|^2)\varphi = 0 \qquad \text{on} \quad S^1 = \mathbb{R}/2\pi\mathbb{Z}$$

This 1-dimensional equation has solutions

$$\varphi = \tilde{a} e^{im\theta}, \qquad \tilde{a} = (1 - m^2/\lambda)^{1/2}, \; quadm = \pm 1, \pm 2, \cdots$$

and those are stable for $\lambda > 3m^2 - 1/2$. From a perturbation argument of the domain we see that for fixed $\lambda > 3m^2 - 1/2$ there is an ϵ_1 such that for each $\epsilon \in (0, \epsilon_1)$ Equation (1) with $\Omega = \Omega(\epsilon)$ has stable solutions

$$\Phi = a_\epsilon(r) e^{im\theta}, \qquad a_\epsilon(r) \to \tilde{a} \quad (\epsilon \to 0)$$

On the other hand in the theorem of [8] it was proved that for sufficiently lage λ there exist stable non-constant solutions $\Phi = W_\lambda e^{im\theta}$ in a rotational domain homeomorphic to a solid torus in $\mathbb{R}^3$, having the asymptotics $|\Phi| = W_\lambda \to 1$ as $\lambda \to \infty$.

The solutions constructed in the both cases have no zeros in the domain. Actually to prove the stablility of the above solutions, it was important that $|\Phi|$ is so close to one. This fact suggests that if a domain Ω is non-trivial, there might be a stable solution Φ_λ such that $|\Phi_\lambda| \to 1$ and $\phi := \Phi_\lambda / |\Phi_\lambda|$ is not homotopic to a constant map in $C(\bar{\Omega}; S^1)$. This was just proved in [10].

Theorem 3.1 *Let Ω be a non-trivial domain with C^3 boundary and let $\gamma \in C(\bar{\Omega}; S^1)$ be not homotopic to a constant map. Then there is a $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ Equation (1) has a stable solution Φ_λ such that $|\Phi_\lambda| \to 1$ and the mapping*

$$x \longmapsto \Phi_\lambda(x)/|\Phi_\lambda(x)|$$

is homotopic to γ.

We remark that as observed in the proof of the paper [10], the solution of Theorem 3.1 approaches a harmonic map of $\bar{\Omega}$ into S^1 as $\lambda \to \infty$.

Finally we introduce a new result of [9] in brief. Let us consider a domain Ω including a non-trivial domain D. Assume that λ is so large that the domain D admits a stable non-constant solution Φ_0 and fix such D and λ. If the volume $vol(\Omega \setminus D)$ is sufficiently small, say $vol(\Omega \setminus D) < \delta$, then there exists a stable solution Φ in Ω satisfying

$$\inf_{\theta} \|\Phi - \Phi_0 e^{i\theta}\|_{L^2(D)} < \eta = \eta(\delta)$$

where $\eta(\delta) \to 0$ as $\delta \to 0$. As an example of the domain, consider Ω consisting of a solid torus D and a very thin sylinder G which is put into the space surrouned by the solid torus. This is a simply connected but non-convex domain which admits a stable non-constant solution. One can also find an example of non-trivial domain which allows stable solutions with zeros (see [9]).

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