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Surface Waves: Propagation and Localization

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1 Introduction

Surface waves were discovered by Rayleigh at the end of the last century [1]. He considered a homogeneous and isotropic elastic halfspace $\mathbf{R}^3_+ = \{(x, \xi),\ x \geq 0,\ \xi \in \mathbf{R}^2\}$, whose boundary surface $x = 0$ is free of traction. He found that there are two types of solutions of the respective boundary value problem:

(i) Solutions which are oscillating and nondecaying at infinity in all variables. They are called the volume (bulk) waves.

(ii) Solutions which are the plane waves in the longitudinal variables ξ and which are exponentially decaying (localized) in the transverse variable x.

These solutions are called the surface (grazing) waves. They propagate only in the longitudinal directions, with the velocity slightly smaller than the velocity of the volume waves.

The similar solutions exist if the plane $x = 0$ is the interface between the two halfspaces with different elastic constants (see [1,2] for references and discussion).

This should be compared to the case when a homogeneous elastic body occupies the whole $\mathbf{R}^3$ and when all the solutions are the plane waves in all variables.

The Rayleigh result is perhaps the first demonstration of a rather general property of solutions of differential and finite-difference equations which can be summarized as follows. If the coefficients of an equations are strongly inhomogeneous (spatially dependent), then the equations may have solutions which are localized near the inhomogeneities, i.e. decay exponentially with the increase of the distance from the inhomogeneity.

Returning to the surface waves, we remark that after Rayleigh the similar solutions for the Maxwell equations were found, at the turn of the century, by the Sommerfeld school in the study of the propagation properties of the radio waves around the earth surface. These are the electromagnetic waves that propagate along the surface of a dielectric subspace or the interface between the two halfspaces with different dielectric constants and localized near the surface (intersurface). These solutions are now known as the surface polaritons or the surface plasmons (the latter correspond to the limiting case $c = \infty$, where c is the velocity of light) [3].

The natural analogue of the Rayleigh problem, in the case of inhomogeneous (and in particular, randomly inhomogeneous) media, is a model of an inhomogeneous elastic halfspace. The common wisdom of spectral theory of the PDE's with random coefficients suggests that in the case of a randomly inhomogeneous elastic medium occupying the whole space $\mathbf{R}^d$, $d \geq 3$, nondecaying at infinity (delocalized or extended) solutions exist for low and high frequencies, and exponentially decaying at infinity (localized) solutions exist for an interval ("window") of intermediate frequencies if the disorder is large enough. If so, then it is natural to expect that in the case of half-infinite random inhomogeneous media the above picture should be complemented by the surface solutions which are delocalized (propagating) with respect to the transverse coordinates, if the inhomogeneity is weak enough or if their frequency is low enough.

The above picture assumes the positive solution of a hard problem, the proof of the existence of delocalized solutions in a randomly inhomogeneous media. We will consider here a class of simpler problems where similar phenomena is expected to emerge. A typical example is the boundary value problem for the Laplace equation:

$$-\Delta_X u = Eu, \quad X = (x, \xi) \in \mathbf{R}^d_+ = \{x \geq 0, \ \xi \in \mathbf{R}^{d-1}\} \tag{1.1}$$

with the boundary condition

$$\frac{\partial u}{\partial x} |_{x=0} = V(\xi) u(0, \xi), \ \ \xi \in \mathbf{R}^{d-1}. \tag{1.2}$$

If $V(\xi)$ is a constant, $V(\xi) \equiv a$, then the eigenvalue problem can be solved by separation of the variables. Its solutions can be explicitly identified as follows.

(i) If $a > 0$, then the solutions are

$$u(X, K) = \left(2^{d-2}\pi^d (k^2 + a^2)\right)^{-1/2} e^{i\varphi\xi} (k \cos kx + a \sin kx) \tag{1.3}$$

where $K = (k, \varphi) \in \mathbf{R}^d$, $E = k^2 + \varphi^2 \geq 0$.

(ii) If $a < 0$ then we have two classes of solutions:

$$u_1(X, K) = \left(2^{d-2}\pi^d (k^2 + a^2)\right)^{-1/2} e^{i\varphi\xi} (k \cos kx + a \sin kx) \tag{1.4}$$

where $K = (k, \varphi) \in \mathbf{R}^d$, $E = k^2 + \varphi^2 > 0$, and

$$u_2(X, \varphi) = \left(2^d \pi^{d-1} |a|\right)^{-1/2} e^{i\varphi\xi - |a|x} \tag{1.5}$$

where $\varphi \in \mathbf{R}^{d-1}$, $E = \varphi^2 - a^2 \geq -a^2$.

Thus for $a < 0$ we have analogues of the volume and the surface waves; the only difference between (1.1)-(1.2) and the Rayleigh problem is that for the former one the spectral parameter can be negative.

In this note we discuss the structure of the solutions of the eigenvalue problem (1.1)-(1.2) in which V is a quasi-periodic or random function and some related problems. In the next section we give a more precise description of problems; the explicitly solvable model $V(\xi) \equiv const$ will serve us as a guide. In the Section 3 we present some results concerning the discrete analog of the boundary value problem (1.1)-(1.2).

2 Generalities

We begin by reformulating the boundary value problem (1.1)-(1.2) in terms of spectral theory.

Let us recall that the spectrum of an abstract selfadjoint operator H consists of the absolutely continuous , singular continuous and pure point components: $\sigma(H) = \sigma_{ac}(H) \bigcup \sigma_{sc}(H) \bigcup \sigma_{pp}(H)$. In our case H is a differential or finite difference operator and it is widely accepted that the generalized eigenfunctions corresponding to $\sigma_{ac}(H)$ describe the propagating waves and particles. A typical example is the Schrödinger operator in $L^2\left(\mathbf{R}^d\right)$, whose potential decays at infinity. If the decay is fast enough, then the absolutely continuous spectrum of this operator is $\mathbf{R}_+$; the respective eigenfunctions are superpositions of the incident plane waves and scattered spherical waves (the Sommerfeld solutions).

The boundary value problem (1.1)-(1.2) defines the selfadjoint operator H_V acting in the space $L^2\left(\mathbf{R}^d_+\right)$. Thus, we can reformulate the results (1.3)-(1.5) for $V(\xi) \equiv a$ as follows:

(i) $a > 0$. The spectrum of H_V is $\mathbf{R}_+$ and is purely absolutely continuous. The respective eigenfunctions are given by (1.3); they are the plane waves with respect to the longitudinal coordinates $\xi \in \mathbf{R}^{d-1}$ and the standing waves with respect to the transverse coordinate $x \geq 0$. This system of eigenfunctions is orthonormal and complete, i.e.

$$\int_{\mathbf{R}^d_+} u(X, K_1) u(X, K_2) dX = \delta(K_1 - K_2)$$

and

$$\int_{\mathbf{R}^d} u(X_1, K)u(X_2, K)dK = \delta(X_1 - X_2).$$

(ii) $a < 0$. The spectrum of H_V is the interval $[-a^2, \infty) \supset \mathbf{R}_+$, and is again purely absolutely continuous. The generalized eigenfunctions are given by (1.4) and (1.5); we call them respectively the volume (bulk) and the surface (grazing) solutions (waves). These eigenfunctions satisfy the relations:

$$\begin{aligned} \int_{\mathbf{R}^d_+} u_1(X, K_1)u_1(X, K_2)dX &= \delta(K_1 - K_2), \\ \int_{\mathbf{R}^d_+} u_2(X, \varphi_1)u_2(X, \varphi_2)dX &= \delta(\varphi_1 - \varphi_2), \\ \int_{\mathbf{R}^d_+} u_1(X, K)u_2(X, \varphi)dX &= 0 \end{aligned}$$

$$\int_{\mathbf{R}^d} u_1(X_1, K)u_1(X_2, K)dK + \int_{\mathbf{R}^{d-1}} u_2(X_1, \varphi)u_2(X_2, \varphi)d\varphi = \delta(X_1 - X_2).$$

Thus, the volume waves $\{u_1(X, K)\}_{K \in \mathbf{R}^d}$ and the surface waves $\{u_2(X, \varphi)\}_{\varphi \in \mathbf{R}^{d-1}}$ generate two orthogonal subspaces. In other words, the spectrum of H_V, for $V(\xi) \equiv a < 0$, consists of two "layers" (channels) $[0, \infty)$ and $[-a^2, \infty)$ corresponding to volume waves (1.4) and surface waves (1.5). There is no scattering between the volume channel and the surface channel.

We now mention a few other problems that have similar structure of spectrum.

The Schrödinger operator in $\mathbf{R}^d$ with a surface potential.

We consider the Schrödinger equation

$$Hu = -\Delta_X u + 2\delta(x)V(\xi)u = Eu \tag{2.6}$$

on $\mathbf{R}^d = \left\{X = (x, \xi) \mid x \in \mathbf{R},\ \xi \in \mathbf{R}^{d-1}\right\}$, where $\delta(x)$ is the Dirac δ - function. We assume for the simplicity that the surface potential $V(\xi)$, $\xi \in \mathbf{R}^{d-1}$, is bounded. Then by using the Green formulae it easy to show that (2.6) is equivalent to (1.1)-(1.2).

Discrete boundary value problem. On the half-space

$$\mathbf{Z}^d_+ = \{X = (x, \xi),\ x \in [0, \infty) = \mathbf{Z}_+,\ \xi \in \mathbf{Z}^{\mathbf{d-1}}\}$$

we consider the spectral problem

$$u(x-1, \xi) + u(x+1, \xi) + (\Delta_{d-1}u)(x, \xi) = Eu,\ x \geq 0 \tag{2.7}$$

$$u(-1, \xi) = V(\xi)u(0, \xi), \tag{2.8}$$

where

$$(\Delta_{d-1}u)(\xi) = \sum_{\eta\in \mathbf{Z}^{d-1}, |\xi-\eta|=1} u(\eta) \tag{2.9}$$

is the discrete Laplacian in $\mathbf{Z}^{d-1}$. This boundary value problem is the natural discrete analogue of (1.1)-(1.2).

Discrete Schrödinger operator with the "subspace" potential. We decompose $\mathbf{Z}^d$ as

$$\mathbf{Z}^d = \mathbf{Z}^{d_1} \times \mathbf{Z}^{d_2} = \{X = (x,\xi),\ x \in \mathbf{Z}^{d_1},\ \xi \in \mathbf{Z}^{d_2}\},$$

and consider the finite-difference equation

$$-\Delta_d u + \delta(x)V(\xi)u = Eu \tag{2.10}$$

where

$$\delta(x) = \prod_{j=1}^{d_1} \delta(x_j)$$

The potential is now concentrated on the subspace $\mathbf{Z}^{d_2}$. If $d = 3$, $d_1 = 1$, $d_2 = 2$ this model can be regarded as a model of the thin film; for $d = 3$, $d_1 = 2$, $d_2 = 1$ it can be regarded as a model of the line inhomogeneity. We call the latter case the polymer problem. The case $d = 3$, $d_1 = 1$, $d_2 = 2$ and V depending only on ξ_1 reduces to $d = 2$, $d_1 = d_2 = 1$ and can be regarded as a model of the grating (the linear interferometer).

It is easy to show that all these problems with $V = \text{const}$ have the surface (subspace) solutions which decay exponentially as $|x| \to \infty$ and which are the plane waves in ξ-variable , i.e. propagate along the subspace $\mathbf{Z}^{d_2}$.

Similar results are also known for the case when $V(\xi)$ is periodic (see [5-7]).

For the rest of this note we will discuss mainly the discrete boundary value problem (2.7) - (2.9). To give the reader the taste of the results we are aiming to, we finish this section with the following simple result. Recalling the property of polynomial boundedness of generalized eigenfunctions of finite-difference operators we define the set S of the surface solutions for the problem (2.7)-(2.9) as

$$S = \left\{ u_E(x,\xi),\ \sup_{\xi\in\mathbf{Z}^{d-1}} (1+|\xi|^a)^{-1} \sum_{x\in\mathbf{Z}_+} |u_E(x,\xi)|^2 < \infty \right\} \tag{2.11}$$

where $a > d - 1/2$ is fixed. We also introduce

$$\sigma_S = \{E :\ u_E \in S\}.$$

Proposition 2.1 *Let $H_0 = \Delta_d$ and let H_V be the selfadjoint operator defined by (2.10). Let $\sigma(H_V)$ and $\sigma(H_0) = \sigma_{ac}(H_0) = [-2d, 2d]$ be their spectra. Then $\sigma(H_V)\backslash\sigma(H_0) \subset \sigma_S$ and the corresponding generalized eigenfunctions decay exponentially as $|x| \to \infty$.*

The proof of this proposition follows from:

(i) the Green formula

$$u_E(x,\xi) = \sum_{\eta\in \mathbf{Z}^{d-1}} g_E(x,\xi-\eta)V(\eta)u_E(0,\eta),$$

(ii) the polynomial bound

$$|u_E(x)| \le C'_\varepsilon(1+|x|^{d/2+\varepsilon}), \qquad \varepsilon > 0,$$

which is valid for almost all E with respect to the spectral measure of H_V [17];

(iii) the exponential decay of the "free" Green function $g_E(X)$, $E \notin \sigma(H_0)$.

3 Discrete boundary value problem

In this section we present some results concerning the structure of solutions of the discrete boundary value problem (2.7)-(2.9) . We would like to emphasize in advance that our understanding of the problem is limited; we have only a few results to announce here. More complete analysis of the problem will be given in [4].

Set $V(\xi) = gv(\xi)$ where $v(\xi)$, $\xi \in Z^{d-1}$, are independent, identically distributed random variables with continuous and bounded probability density $p(v)$. The parameter g measures the strength of the coupling between waves (quantum particles) and the random corrugated surface of the medium. We denote respective operator H_V.

In the case when the random variables $v(\xi)$ are degenerate, $v(\xi) \equiv a > 0$, the spectrum of the operator H_V is absolutely continuous and fills the union of two intervals

$$\sigma(H_V) = [-2d, 2d] \cup [-2(d-1) + E_0(ga), 2(d-1) + E_0(ga)];$$

where $E_0(ga) = ga + [ga]^{-1} > 2$ is the only eigenvalue of the one-dimensional boundary value problem $u(x-1) + u(x+1) = Eu(x)$, $u(-1) = gau(0)$. This eigenvalue exists if $ga > 1$. As in the continuous case, these intervals correspond to two "channels", volume waves and surface waves. If the surface channel exists ($ga > 1$), then it has a "tail" lying outside of $\sigma(H_0)$. The generalized eigenfunction associated to the "volume channel" do not decay in any direction; the ones associated to the the "surface channel" are exponentially decaying in x-variable.

Our goal is to understand how is the structure of spectrum affected after the replacement of the constant potential along the boundary with the random one.

The standard ergodicity argument (see e.g. [9], or [10]), yields that there are closed sets $\Sigma_{ac}, \Sigma_{pp}, \Sigma_{ac} \subset \mathbf{R}$ so that for a.e. V, $\sigma_{ac}(H_V) = \Sigma_{ac}$, $\sigma_{sc}(H_V) = \Sigma_{sc}$, $\sigma_{pp}(H_V) = \Sigma_{pp}$. In particular, for a.e. V

$$\sigma(H_V) = \Sigma_{ac} \cup \Sigma_{sc} \cup \Sigma_{pp} \equiv \Sigma.$$

In fact, it is not too difficult to find the set Σ. If $\mathcal{V}$ is the support of the probability density $p(v)$, then

$$\Sigma = [-2d, 2d] \cup \{[-2(d-1), 2(d-1)] + E_0(g\mathcal{V})\}$$

where $X + Y = \{x + y, x \in X, y \in Y\}$.

It is a characteristic feature of the problem that the spectrum of the operator H_V always has massive enough absolutely continuous component, due to the free propagation along the ξ-variables.

Theorem 3.1 [4] *If* $\int_{\mathbf{R}} |v|p(v)dv < \infty$ *then* $\Sigma_{ac} \supset [-2d, 2d]$.

The basic idea of the proof is to show that there is a dense set of vectors $\mathcal{D} \subset l^2(Z_+^d)$ so that the limit

$$s - \lim_{t\to\infty} \exp(itH_V)\exp(-itH_0)u, \qquad u \in \mathcal{D},$$

exists, see e.g. [16]. By the Cook criterion, it suffices to show that

$$\int_1^\infty \|(H_V - H_0)\exp(-itH_0)u\| dt < \infty \tag{3.12}$$

for each $u \in \mathcal{D}$. Clearly, (3.12) will follow if

$$\sum_{\xi \in Z^{d-1}} |v(\xi)||(\delta_n, \exp(-itH_0)u)| < \infty, \tag{3.13}$$

for almost all V and all $u \in \mathcal{D}$. One establishes (3.13) by passing first to the Fourier variables in the ξ-variables (see the discussion below), and then using integration by parts, judicious choice of the set $\mathcal{D}$, and Borel-Cantelli lemma. The details will be presented in [4].

Further discussion of the spectral properties of H_V is based on the Fourier transformed form of the H_V. The operator H_V is unitarily equivalent to operator $\hat{H}_V$ which acts on $l^2(Z^+) \otimes L^2(T^{d-1})$. We denote the variables on that space by $(x, \phi) = (x, \phi_1, \phi_2, \dots \phi_{d-1})$, and its elements by $\hat{u}(x, \phi)$. Let

$$\Phi(\phi) = 2\sum_{i=1}^{d-1} \cos(\phi_i).$$

The operator $\hat{H}_V$ is given by

$$\begin{aligned} \hat{H}_V \hat{u}(x, \phi) &= \hat{u}(x+1, \phi) + \hat{u}(x-1, \phi) + \Phi(\phi)\hat{u}(x, \phi), \qquad x \geq 1, \\ \hat{H}_V \hat{u}(0, \phi) &= \hat{u}(1, \phi) + g\widehat{[v_V u]}(0, \phi). \end{aligned}$$

Square integrable solutions of the equation

$$\hat{H}_V \hat{u}(x, \phi) = E\hat{u}(x, \phi) \tag{3.14}$$

have the form

$$\hat{u}(x, \phi) = \hat{u}(0, \phi)\Lambda^x_{\phi,E};$$

$$\Lambda_{\phi,E} + \frac{1}{\Lambda_{\phi,E}} + \Phi(\phi) = E \qquad \text{and} \qquad 0 < \Lambda_{\phi,E} < 1.$$

In particular, if $E \in [-2d, 2d]$, we may assume that the function $\hat{u}(0, \phi)$ is supported on the set

$$\{\phi \in T^{d-1} : |\Phi(\phi) - E| > 2\}.$$

It follows from (3.14) that function $\hat{u}(0, \phi)$ satisfies the equation

$$\hat{u}(0, \phi)\Lambda^{-1}_{\phi,E} = -g\widehat{[vu]}(\phi). \tag{3.15}$$

This equation will play the central role in the subsequent discussion. We recall that the set $\mathcal{V}$ is the support of the probability density $p(v)$. Our first application of (3.15) is:

Proposition 3.2 [4] *a) Suppose that* $\mathcal{V} \subset [-a, a]$ *for some* $a > 0$. *If* $|g| < 1/a$ *then* H_V *has no eigenvalues in* $[-2d, 2d]$.
b) Suppose that $\mathcal{V} \subset [a, b]$ *for some constants* $0 < a < b$. *If* $g > 2(2d - 1)/a$ *then* H_V *has no eigenvalues on* $[-2d, 2d]$.

Remark. The above results are in fact deterministic.
Remark. Suppose that $\mathcal{V} = [-a, a]$. Then the part a) yields that there are no eigenvalues on $[-2d, 2d]$ as long as there is no spectrum outside the spectrum of Laplacian. The part b) yields that as long as the spectrum outside $[-2d, 2d]$ is separated from the one in $[-2d, 2d]$ by a sufficiently large gap, then are no embedded eigenvalues in $[-2d, 2d]$.
Proof: The equation (3.14) yields that

$$\int_{T^{d-1}} |\hat{u}(0, \phi)|^2 |\Lambda^{-1}_{\phi,E}|^2 d\phi = |g| \int_{T^{d-1}} |\widehat{[vu]}(0, \phi)|^2 d\phi.$$

Assume that u is normalized as

$$1 = \int_{T^{d-1}} |\hat{u}(0, \phi)|^2 d\phi = \sum_{\xi \in \mathbf{Z}^{d-1}} |u(0, \xi)|^2.$$

Since

$$1 \le \Lambda^{-1}_{\phi,E} \le 2(2d - 1),$$

we get from (3.14) that

$$1 \leq g^2 \sum_{\xi \in Z^{d-1}} |v(\xi)|^2 |u(0,n)|^2 \leq [2(2d-1)]^2.$$

The result is immediate. □

We now turn to the analysis of the structure of the "tail" part of the spectrum of H_V, given by $\Sigma \setminus [-2d, 2d]$. Let $R_V(X, X'; E + i\epsilon)$ be the matrix elements of the resolvent $(H_V - z)^{-1}$ for $z = E + i\epsilon$. Let $X_0 \equiv (0, \xi_0)$ be fixed point on the boundary ∂Z^d_+ and let $|E| > 2d$. Then

$$\hat{R}_V(X_0, (x, \phi); E + i\epsilon) = \hat{R}_V(X_0, (0, \phi); E + i\epsilon)[\Lambda_{\phi, E+i\epsilon}]^x. \tag{3.16}$$

Here $\Lambda_{\phi,E+i\epsilon}$ is the analytic continuation of the function $\Lambda_{\phi,E}$ from the part of the real axis $|E| > 2d$ to the upper half-plane $\{w \mid \mathrm{Im}(w) > 0\}$. For $\xi \in Z^{d-1}$ let

$$D(\xi, E + i\epsilon) \equiv [\widehat{\Lambda_{\phi, E+i\epsilon}}]$$

Obviously, for each $|E| > 2d$ there is $C_E > 0$ and $\gamma(E) > 0$ so that

$$\sup_{\epsilon>0} |D_E(\xi; E + i\epsilon)| < C_E \exp(-\gamma(E)|\xi|). \tag{3.17}$$

The resolvent equation restricted to the boundary ∂Z^d_+ becomes

$$[D(\xi - \xi_0, E + i\epsilon) + \Delta_{d-1} + gv(\xi)]\, R_V((0, \xi_0), (0, \xi); E + i\epsilon) =$$

$$= (\delta(\xi - \xi_0) + E + i\epsilon) R_V((0, \xi_0), (0, \xi); E + i\epsilon).$$

On this way we have obtained $(d-1)$-dimensional eigenvalue problem which, however, depends non-linearly on the spectral parameter E (see also section 2). Nevertheless, the techniques developed in the spectral theory of random operators can be properly adapted to handle this non-linear spectral problem in the strong localization regime and we have:

Theorem 3.3 *Suppose that density $p(v)$ satisfies $\sup_{v \in \mathbf{R}} |p(v)| < \infty$. Then for $\forall \delta > 0$ there is $g(\delta) > 0$ so that the estimate*

$$\sup_{\epsilon>0} |R_V((0, \xi_0), (0, \xi); E + i\epsilon)| < C_{E,\xi_0,V} \exp(-\gamma(E)|\xi - \xi_0|), \tag{3.18}$$

holds for $|g| > g(\delta)$, each fixed $|E| > 2d + \delta$ and for a.e. V. The same estimate holds for each fixed g if $|E|$ is taken large enough, $|E| > E_0(g)$.

Remark. The result holds under more general condition on density $p(v)$.

Remark. Naturally, the constant $\gamma(E)$ in (3.17) may differ from one in Theorem 3.3. For notational simplicity, we will always use the letter $\gamma(E)$ for the E-dependent constant figuring in the exponential decay of the quantity in question.

From the estimate(3.18) and relation (3.16) it follows that under the conditions of the Theorem 3.3, the resolvent

$$\sup_{\epsilon>0} |R_V(X_0, X; E + i\epsilon)| \leq C_{V,X_0,E} \exp(-\gamma(E)|X_0 - X|_+),$$

in the large coupling/high energy regime described in the Theorem 3.3. The Simon-Wolff's theorem (see [9] and [10]) yields the following result:

Theorem 3.4 [11, 12] *Under the conditions of Theorem 3. 3 we have:*
a)For each $\delta > 0$ there is $g(\delta) > 0$ so that for $|g| > g(\delta)$,

$$\Sigma \cap \{E \mid |E| \geq 2d + \delta\} \subset \Sigma_{pp}.$$

The corresponding eigenfunctions decay exponentially.
b) For each $g \neq 0$ there is $E(g)$ so that

$$\Sigma \cap \{E \mid |E| \geq E(g)\} \subset \Sigma_{pp}.$$

The corresponding eigenfunctions decay exponentially.

Consider now the quasiperiodic potential

$$V(\xi) = g \tan \pi[(\alpha, \xi) + \omega] \tag{3.19}$$

where $\alpha = (\alpha_1, ..\alpha_{d-1})$ is a Diophantine vector, i.e.

$$|(\alpha, \xi) + m| \geq C|\xi|^\beta$$

for all $\xi \in Z^{d-1} \backslash \{0\}$, $m \in Z$ and some positive C and β; $\omega \in [0, 1]$ is the "randomness" parameter.

The potential (3.19) can model the quasiperiodically (strongly) corrugated surface. The case of the Schrödinger operator with the similar potential is analyzed in [10]. This operator has pure point spectrum coinciding with $\mathbf{R}$ for all $g \neq 0$ and almost all $\omega \in [0, 1]$ with respect to the Lebesgue measure.

Similarly, for the surface potential (3.19) we have :

Theorem 3.5 [13] *Let H_V be the operator defined by (2.7)-(2.9) and (3.19). Then the spectrum of H_V is $\mathbf{R}$ and its part lying outside of $\sigma(H_0) = [-2d, 2d]$ is pure point for almost all $\omega \in [0, 1]$ with respect to the Lebesgue measure. The eigenvalues are simple and dense on $\mathbf{R}\backslash[-2d, 2d]$ and the corresponding eigenfunctions decay exponentially in ξ.*

Combining Proposition 2.1 (which also holds for the model (2.7)-(2.9)) and Theorems 3.4 and 3.5 we conclude that in the strong localization regime the eigenfunctions are the surface solutions of respective equation and that they decay exponentially not only in the transverse coordinates x but also in the longitudinal coordinates ξ. In other words,

In the cases treated in Theorems 3.4 and 3.5 the surface waves are localized by strong fluctuations of the random potential.

On the other hand, according to Theorem 3.1, the absolutely continuous spectrum fills the interval $[-2d, 2d]$ for all strengths of the coupling. Thus we are naturally lead to the following questions:

(I) Is the spectrum of H_V purely absolutely continuous for g small?

(II) What is the nature of the spectrum on the interval $[-2d, 2d]$ for the intermidate and large values of g? Do we have embedded eigenvalues in a.c. spectrum?

(III) Do surface solutions exist on the interval $[2d, 2d]$? Are they propagating? What are the respective conditions?

Concerning (I), if $d = 3$ and the random potential is placed only along a line (the polymer case), then the conjecture can be established using a version of Kato's smooth perturbation theory [4]. Concerning (III) the answer is affirmative if $d = 3$ and i.i.d. random variables $v(\xi_1, \xi_2)$ do not depend on ξ_2. Indeed, in this case the dependence of solutions on ξ_2 is harmonic and the corresponding energies belong to the absolutely continuous spectrum of H_V. On the other hand, since the analogue of the respective non-linear spectral problem (3.15) is one-dimensional, one might hope to obtain some information modifying the existing techniques of 1-d random Schrödinger operator theory. In [4] we modify and extend the technique developed in [14], [15], to prove that if $d = 2$, then for all g $\Sigma \cap \{E \mid |E| > 4\} \subset \Sigma_{pp}$. This allows us to prove that $\sigma_S \cap [-6, 6] \neq \emptyset$. The corresponding solutions propagate along ξ_2-axis and exponentially decay in ξ_1 and x.

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