JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

M. HOFFMANN-OSTENHOF
T. HOFFMANN-OSTENHOF
N. NADIRASHVILI

Interior Hölder estimates for solutions of Schrödinger equations and the regularity of nodal sets

Journées Équations aux dérivées partielles (1994), p. 1-9
<http://www.numdam.org/item?id=JEDP_1994____A13_0>

INTERIOR HÖLDER ESTIMATES FOR SOLUTIONS OF SCHRÖDINGER EQUATIONS AND THE REGULARITY OF NODAL SETS

M. Hoffmann-Ostenhof[1]
T. Hoffmann-Ostenhof[2,3]
N. Nadirashvili[3]

Institut für Mathematik, Universität Wien[1]
Institut für Theoretische Chemie, Universität Wien[2]
International Erwin Schrödinger Institute for Mathematical Physics[3]

1. Introduction and Statement of the Results

Let Ω be an open bounded set in $\mathbb{R}^n, n \geq 2$ and let $V \in L^1(\Omega)$ be real valued. We consider real valued solutions $u \not\equiv 0$ which satisfy

$$\Delta u = Vu \tag{1.1}$$

in the distributional sense, that means

$$\int u(-\Delta + V)\chi dx = 0 \text{ for every } \chi \in C_0^\infty(\Omega).$$

In a recent paper two of us [HO2] investigated the local behaviour of such solutions under rather mild assumptions on the potential V, namely we assumed that $V \in K^{n,\delta}(\Omega)$ for some $\delta \in (0, 2)$, that is

$$\lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{R}^n} \int_{|x-y|<\varepsilon} \chi_\Omega \frac{|V(y)|}{|x-y|^{n-2+\delta}} dy = 0 \tag{1.2}$$

where χ_Ω denotes the characteristic function of Ω.

For $\delta = 0$ and $n \geq 3$ (1.2) defines the Kato class $K^n(\Omega)$ (for $K^2(\Omega)$, $|x-y|^{2-n}$ becomes $|\ln |x-y||$) first introduced by Kato [K] and studied by Aizenman and Simon [AS] in their seminal paper on the Harnack inequality. The class $K^{n,\delta}$ was investigated by Simon [S] who showed that solutions to (1.1) with $V \in K^{n,\delta}$, $\delta > 0$ are Hölder continuous. We note that unlike the more traditional L^p-conditions on V, (1.2) allows for the physically important case of many body interactions (see [AS], [S] for a discussion). But for $\delta \in (0, 2)$, $p > n/(2-\delta)$ it is easy to see that

$$L^p(\Omega) \subset K^{n,\delta}(\Omega). \tag{1.3}$$

One of the main results in [HO2] was the following representation result.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

Theorem 0.

Suppose $u \not\equiv 0$ is a real valued distributional solution of (1.1) and that V satisfies (1.2). Let $x_0 \in \Omega$, then either there is a homogeneous harmonic polynomial $P_M \not\equiv 0$ of degree $M \in \mathbb{N}_0$ such that

$$u(x) = P_M(x - x_0) + \mathcal{O}(|x - x_0|^{M+\min(1,\delta')}) \tag{1.4}$$

for all $\delta' < \delta$, or

$$u = \mathcal{O}(|x - x_0|^\alpha) \quad \text{for all } \alpha > 0. \tag{1.5}$$

Remarks 1.1.

(i) Originally this result was stated for $n \geq 3$. But $V \in K^{2,\delta}(\Omega)$, $\Omega \subset \mathbb{R}^2$ implies $V \otimes I \in K^{3,\delta}(\Omega \times J)$ where I denotes the identity on $\mathbb{R}^1$ and J an interval. (We thank I. Herbst for pointing this out to us).

(ii) Representation results like Theorem 0 have been previously obtained for more general types of equations, but under more restrictive assumptions on the potential [B, CF, A, HS, R].

(iii) For a large class of potentials (e.g. $V \in L^p$, $p \geq n/2$, $n \geq 3$) it is known that (1.5) implies $u \equiv 0$ (see e.g. [Ke]). This property is called a strong unique continuation property. Unfortunately it has not as yet been shown for $V \in K^{n,\delta}$ except for K^3 [Sa].

In this paper we shall improve upon this result in various ways. Namely we shall obtain interior estimates for solutions to (1.1) in the neighbourhood of zeros (Theorem 1). These estimates will enable us to show that, roughly speaking, zero sets are by one degree smoother than the corresponding solutions (Theorem 2). We illustrate this with an explicit example in $\mathbb{R}^2$: Set

$$u = \begin{cases} x - y & \text{for } x \leq 0 \\ \sinh x - y\cosh x & \text{for } x > 0 \end{cases}$$

then u satisfies in the distributional sense $\Delta u = Vu$ in $\mathbb{R}^2$ with $V = 0$ for $x \leq 0$ and $V = 1$ for $x > 0$. The level sets of u, $\{(x, y) \in \mathbb{R}^2\colon\ u(x, y) = c\}$, $c \in \mathbb{R}$, can be represented by the family of functions

$$y_c(x) = \begin{cases} x - c & \text{for } x \leq 0 \\ \tanh x - \frac{c}{\cosh x} & \text{for } x > 0. \end{cases}$$

y_c is smooth away from $x = 0$. If $c = 0$, y_0 has a jump in the third derivative while for $c \neq 0$, y_c has a jump already in the second derivative. So the zero set is by one degree smoother than the other level sets.

In the following we will consider not only $V \in K^{n,\delta}(\Omega)$ but also more regular potentials namely $V \in C^{k,\alpha}(\Omega)$, $k \in \mathbb{N}_0$ and $\alpha \in [0, 1]$ where $C^{k,\alpha}(\Omega)$ denote the usual Hölder spaces [GT]. We introduce the following norms:

$$||V||_{K^{n,\delta}(\Omega)} = \sup_{x\in\mathbb{R}^n} \int_{|x-y|<1} \chi_\Omega \frac{|V(y)|}{|y - x|^{n-2+\delta}} dy.$$

For the Hölder spaces $C^{k,\alpha}(\Omega)$ we define for $\alpha \in [0,1]$

$$|V|_{0,\alpha,\Omega} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|V(x)-V(y)|}{|x-y|^\alpha}$$

and correspondingly

$$|V|_{k,\alpha,\Omega} = \sum_{l=0}^{k-1} \sup_{|\beta|=l} \sup_{x\in\Omega} |D^\beta V| + \sup_{|\beta|=k} |D^\beta V|_{0,\alpha,\Omega}$$

where we used the usual multi-index notation [GT]. If $\partial\Omega$ is sufficiently smooth

$$C^{k',\alpha'}(\Omega) \subseteq C^{k,\alpha}(\Omega) \subset K^{n,\delta'}(\Omega) \subseteq K^{n,\delta}(\Omega) \tag{1.6}$$

if $0 < \delta \leq \delta' < 2$ and $k + \alpha \leq k' + \alpha'$ [GT]. Without loss we assume that

$$\overline{B} = \overline{B}_1 = \{x \in \mathbb{R}^n : |x| \leq 1\} \subset \Omega \tag{1.7}$$

and we set $||V||_{n,\delta} = ||V||_{K^{n,\delta}(B)}$ and $|V|_{k,\alpha} = |V|_{k,\alpha,B}$.

Consider now a solution of (1.1) with $V \in K^{n,\delta}(\Omega)$. According to Theorem 0 we can talk of the order of vanishing of u in a point x_0. We define

$$\mathcal{N}_u^{(M)} = \{x_0 \in \Omega : u \text{ has a zero of order } \geq M \quad \text{in } x_0\}.$$

Theorem 1. *(A priori estimates)*

Suppose that u is real valued and satisfies (1.1) in the distributional sense and that $\overline{B} \subset \Omega$.

(i) If $V \in K^{n,\delta}(\Omega)$ with $\delta \in (0,1]$ then there exists a constant C not depending on u,

$$C = C(n, M, \delta, ||V||_{n,\delta}),$$

such that for every $x_0 \in \mathcal{N}_u^{(M)} \cap \overline{B}_{1/2}$

$$|u(x) - P_M(x - x_0)| \leq C(\sup_B |u|)|x - x_0|^{M+\delta} \text{ for } x \in \overline{B}, \tag{1.8}$$

for some harmonic homogeneous polynomial P_M of degree M.

(ii) If $V \in K^{n,\delta}(\Omega)$ with $\delta \in (1,2)$ then there exists a constant C not depending on u,

$$C = C(n, M, \delta, ||V||_{n,\delta}),$$

such that for every $x_0 \in \mathcal{N}_u^{(M)} \cap \overline{B}_{1/2}$

$$|u(x) - P_M(x - x_0) - P_{M+1}(x - x_0)| \leq C(\sup_B |u|)|x - x_0|^{M+\delta} \text{ for } x \in \overline{B}, \tag{1.9}$$

for some harmonic homogeneous polynomials P_M, P_{M+1} of degree $M, M+1$ respectively.

(iii) If $V \in C^{k,\alpha}(\Omega)$ for some $k \in \mathbb{N}_0$, $\alpha \in (0,1)$, then there exists a constant C not depending on u,

$$C = C(n, M, \alpha, k, |V|_{k,\alpha}),$$

such that for every $x_0 \in \mathcal{N}_u^{(M)} \cap \overline{B}_{1/2}$

$$\begin{aligned} (1.10) \qquad & |u(x) - P_M(x - x_0) - P_{M+1}(x - x_0) - \sum_{i=M+2}^{M+k+2} p_i(x - x_0)| \\ & \leq C(\sup_B |u|)|x - x_0|^{M+k+2+\alpha} \qquad \text{for } x \in \overline{B}, \end{aligned}$$

for some harmonic homogeneous polynomials P_M, P_{M+1} with degree $M, M+1$ repectively and for some homogeneous polynomials p_i of degree i.

Remarks 1.2.

(i) If $M = 0$, (1.8) and (1.10) is well known, see [S] respectively [GT]. Interior estimates of this type (Schauder estimates) are ubiquitous in the theory of elliptic partial differential equations.

(ii) As an immediate consequence we have with the aid of (1.3) the corresponding Hölder estimates for $V \in L^p$ with $p > n/2$.

(iii) The δ, respectively α dependence is sharp in (1.8, 1.9, 1.10) as can be easily seen by working out radial examples.

(iv) The constants are in principle computable in the sense that our proofs contain no steps based on mere existence results.

(v) In [HO2] it was also shown that given any harmonic homogeneous polynomial $P_M \not\equiv 0$ then there exists a neighbourhood of x_0 such that u satisfies (1.1) and (1.4) for $x \to x_0$.

(vi) There are cases where a refined treatment is necessary for obtaining optimal results, in particular $V \in C^{k,\alpha}$ for $\alpha = 0$ or 1 and for the physically important Coulombic case for which $V \in K^{n,\delta}(\Omega)$ for every $\delta < 1$ [S]. For this case representation results like Theorem 0 have been recently obtained [HO2S1], [HO2S2].

The requirement $\alpha \in (0,1)$ is the same as for the usual interior Schauder estimates [GT]. For $\alpha = 0$, $\alpha = 1$ logarithms might turn up.

Theorem 1 will be the starting point for the proof of our results on the regularity of nodal sets of solutions to (1.1). We first explain what we mean by the regularity of a nodal set. Pick a solution u to (1.1) with $V \in K^{n,\delta}(\Omega)$ for $\delta > 0$, and let

$$\mathcal{N}_1(u) = \mathcal{N}_u^{(1)} \backslash \mathcal{N}_u^{(2)}$$

so that $\mathcal{N}_1(u) = \{x_0 \in \mathcal{N}_u^{(1)} : u(x) = 0,\ |\nabla u(x_0)| \neq 0\}$. We say that $\mathcal{N}_1(u)$ is locally a $C^{k,\alpha}$ hypersurface for some $k \in \mathbb{N}_0$ and $\alpha \in [0,1]$ if for each $x_0 \in \mathcal{N}_1(u)$ there is an $\varepsilon > 0$ such that $\mathcal{N}_1(u) \cap B_\varepsilon(x_0)$ can be represented as the graph of a $C^{k,\alpha}$ function. Here $B_\varepsilon(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\}$.

Theorem 2. *(Regularity of nodal sets)*
Let $\Omega \subset \mathbb{R}^n$ and $u \not\equiv 0$ be a real valued distributional solution to $\Delta u = Vu$ with $V \in K^{n,\delta}(\Omega)$.
(i) If $V \in K^{n,\delta}(\Omega)$, $\delta \in (0,1]$ then $\mathcal{N}_1(u)$ is locally a $C^{1,\delta}$ hypersurface.
(ii) If $V \in K^{n,\delta}(\Omega)$, $\delta \in (1,2)$ then $\mathcal{N}_1(u)$ is locally a $C^{2,\delta-1}$ hypersurface.
(iii) If $V \in C^{k,\alpha}(\Omega)$ for some $k \in \mathbb{N}_0$, $\alpha \in (0,1)$ then $\mathcal{N}_1(u)$ is locally a $C^{3+k,\alpha}$ hypersurface.

Remarks 1.3.

(i) So indeed for the cases we treat the zero set of a solution to (1.1) is by one degree smoother than the solution itself.

(ii) It should not be too difficult to extend the methods of Caffarelli and Friedman [CF] and Hardt and Simon [HS] to show that $\mathcal{N}_u^{(2)}$ has at most $(n-2)$-dimensional Hausdorff dimension.

(iii) Recently very interesting estimates on the $(n-1)$-dimensional Hausdorff measure of the nodal sets of eigenfunctions of Laplacians on compact manifolds or of Schrödinger operators on bounded domains have been obtained [DF,HS,CM,D]. Our present results imply that this question even makes sense for $V \in K^{n,\delta}(\Omega)$ for $\delta < 1$, where the eigenfunctions are just in $C^{0,\delta}$.

That nodal domains cannot have cusps has been shown for $V \in C^\infty$ in [HO2a]

(iv) Starting from Theorem 1 also the sets $\mathcal{N}_u^{(k)}$, $k \geq 2$, can be investigated. This poses many interesting questions.

(v) For a very general class of potentials Kröger and Sturm [KS] have some interesting results for quotients of positive solutions which might be related to our results.

(vi) Weaker forms of Theorem 1 and 2 have been announced in [HO2N].

2. Remarks on the proofs of Theorem 1 and Theorem 2

Some of the ideas used here are already in [HO2]. The new results however require some rather technically involved iterations and we refer to the full paper [HO2N1] for details.

A few remarks might be appropriate here and we mention here mainly the case $V \in K^{n,d}(\Omega)$ for $\delta \leq 1$.

First we assume that $\overline{B} \subset \Omega$ and that the origin $0 \in \mathcal{N}_u^{(M)}$. So we want to show that there is a homogeneous harmonic polynomial $P_M(x)$ so that

$$|u(x) - P_M(x)| \leq c(n, M, \delta, ||V||_{n,\delta})(\sup_{\overline{B}} |u|)|x|^{M+\delta} \tag{2.1}$$

The full result can then be obtained by translation and scaling and some simple estimates.

We use polar coordinates, $x = r\omega$, and we assume for simplicity $n \geq 3$. A homogeneous harmonic polynomial P_l of degree l can be written as

$$P_l = r^l \sum_{m=0}^{h(l)-1} c_{l,m} Y_{l,m} \quad \text{with} \quad h(l) = \frac{(2l+n-2)(l+n-3)!}{(n-2)!l!}$$

where $c_{l,m}$ are suitable constants and the orthonormal surface harmonics $\{Y_{l,m}\}$ $l \in \mathbb{N}_0, m = 0, 1, \ldots h(l) - 1$ span $L^2(S^{n-1})$. In polar coordinates $-\Delta = \frac{-\partial^2}{\partial r^2} - -\frac{n-1}{r}\frac{\partial}{\partial r} + \frac{L^2}{r^2}$ where $-L^2$ is the Laplace Beltrami operator on the unit sphere S^{n-1} and we have $L^2 Y_{l,m} = l(l+n-2)Y_{l,m}$.

We split u into two parts

$$u = \mathcal{P}_M u + \mathcal{Q}_M u$$

where $\mathcal{P}_M$ projects on the $Y_{l,m}$ with $l \leq M$ and $\mathcal{P}_M + \mathcal{Q}_M = I$ on $L^2(S^{n-1})$. So we want to show that in $\overline{B}$

$$\mathcal{P}_M u = P_M + \mu \quad \text{with} \quad |\mu| \leq const(\sup_{\overline{B}} |u|) r^{M+\delta}$$

and that

$$|\mathcal{Q}_M u| \leq const(\sup_{\overline{B}} |u|) r^{M+\delta}.$$

For $l \leq M$ we consider

$$f_{e,m} = r^{\frac{n-1}{2}} \int_{S^{n-1}} Y_{l,m}(\omega) V(r\omega) d\omega$$

and we obtain in $(0, R)$ for $R \leq 1$

$$-f''_{l,m} + \frac{\beta_l(\beta_l - 1)}{r^2} f_{l,m} = F_{l,m} \tag{2.2}$$

where $\beta_l = l + \frac{n-1}{2}$ and

$$F_{l,m}(r) = -r^{\frac{n-1}{2}} \int_{S^{n-1}} (Vu)(r\omega) Y_{l,m}(\omega) d\omega. \tag{2.3}$$

(2.2) is obtained by considering $\int_{S^{n-1}} Y_{l,m}(\omega)(-\Delta + V)u d\omega$. It is an inhomogeneous ODE so that $f_{l,m} = c_{l,m} r^{\beta_l} +$ inhomogeneous solution. If we assume (see Theorem 0) that $u \sim r^M$ near the origin, then for $l < M$, $c_{l,m} = 0$. To estimate the inhomogeneous part we use variation of constants and we can bound $|F_{l,m}|$ in terms of $W(r) = \int |V(r\omega)| d\omega$ and $\nu_\delta(r) = \int_0^r t^{1-\delta} W(t) dt$. Through these terms the $||V||_{n,\delta}$ comes into the estimates for $\mathcal{P}_M u$.

The second part, namely $\mathcal{Q}_M u$, is harder to estimate. We introduce

$$\varphi = r^{\frac{n-1}{2}} \sqrt{\int_{S^{n-1}} |\mathcal{Q}_M u|^2 d\omega}.$$

For this function we obtain using methods developed in [HO2] from

$$-\int_{S^{n-1}} (\mathcal{Q}_M u) \Delta (\mathcal{Q}_M u) d\omega = -\int_{S^{n-1}} (\mathcal{Q}_M u) \mathcal{Q}_M (Vu) d\omega$$

a non linear differential inequality for φ, namely

$$-\varphi''\varphi + \frac{(\beta_M + 1)\beta_M}{r^2}\varphi^2 \leq C(M,n)r^{n-1}W(r) \sup_{|x|=r} |u| \sup_{|x|=r} |Q_M u|. \tag{2.4}$$

(2.4) is to be understood in a suitably generalised sense. Since $|Q_M u| \leq |\mathcal{P}_M u| + |u|$ we can use differential inequality techniques to get a bound on φ, but since (2.4) is non linear we get only, roughly speaking that $|u| \sim r^M$ implies $\varphi \sim r^{M+\delta/2}$. We combine (2.4) with a subsolution estimate for $Q_M u$: For $x \in B_R(0), R \leq 1$, and $r < |R - |x||$

$$\begin{aligned}|(Q_M u)(x)| \leq{}& C_1 r^{-n} \int_{|x-y|<r} |(Q_M u)(y)|dy \\ &+ C_2 \int_{|x-y|<r} |x-y|^{2-n} |Q_M(Vu)(y)|dy\end{aligned}$$

¿From this inequality we obtain after some estimates

$$\begin{aligned}\sup_{|x|=r} |(Q_M u)(x)| \leq{}& C_3 r^{-\frac{n+1}{2}} \int_0^{2r} \varphi(t)dt \\ &+ C_4(M,n) \sup_{|y|\leq 2r} |u(y)|r^\delta ||V||_{K^{n,\delta}(B_{2r}(0))}\end{aligned} \tag{2.5}$$

The bounds obtained from (2.4) for φ and (2.5) permits us to set up an infinite iteration which finally enables us to show that $|Q_M u| \leq C(\sup_B |u|)r^{M+\delta}$. Of course the above considerations present only the rough ideas of the proof of Theorem 1 for $V \in K^{n,\delta}(\Omega), \delta \leq 1$. We note in particular that to obtain the explicit constant on Theorem 1 we first show that for $x \in \overline{B}$

$$|u(x)| \leq C(\sup_{\overline{B}} |u|)r^M \text{ for } C = C(M, n, \delta, ||V||_{n,\delta})$$

and proceed then further by the sketched bootstrap arguments keeping control of the constants to derive the refined estimates.

For $\delta \in (1,2)$ the proof is essentially the same. We use the result for $\delta \in (0,1)$ and then split u such that $u = \mathcal{P}_{M+1}u + Q_{M+1}u$ and investigate these terms separately. For $V \in C^{k,\alpha}(\Omega)$ there are various simplifications due to the fact that we can derive a differential inequality (analogous to (2.4)) for φ now given by

$$\varphi = \sqrt{\int_{S^{n-1}} |Q_{M+k+2}u|^2 d\omega}$$

which is linear. Here one uses also first the results for $V \in K^{n,\delta}$ for $\delta \in (1,2)$ and then proceeds from $C^{0,\alpha}$ to $C^{k,\alpha}$ inductively. On the other hand complications arise since one has to expand also V in surface harmonics to obtain the full result.

The proof of Theorem 2 is based on the a priori estimates of Theorem 1. Basic is the following Lemma

Lemma:. *Let $u \in C^0(\overline{B})$ be a real valued function and $\mathcal{N}$ a closed subset of $\overline{B}_{1/4}$. If for some $\delta \in (0,1]$ and $j \in \mathbb{N}$ there is a constant C_0 such that*

$$|u(x) - p_j^{(y)}(x-y)| \leq C_0(\sup_B |u|)|x-y|^{j+\delta} \quad \forall y \in \mathcal{N}$$

for $x \in \overline{B}$ where the $p_j^{(y)}$ are polynomials of degree j, then $u \in C^{j,\delta}(\mathcal{N})$.

The proof of this Lemma can be reduced to some classical one dimensional estimates for polynomials. The Lemma tells us that for our case the solutions are in $\mathcal{N}_u^{(M)}$ by M degrees smoother. Therefrom we derive an implicit funtion theorem: Assuming that $u(x_0) = 0, \nabla u(x_0) = (0, \dots 0, 1)$, then the level set $u(x) = 0$ is locally the graph of a continuous function φ. This together with the regularity of u on $\mathcal{N}_u^1$ leads to the desired regularity properties of φ.

REFERENCES

[A] G. Alessandrini, *Singular solutions and the determination of conductivity by boundary measurements*, J. Diff. Equ. **84** (1990), 252-272.

[AS] M. Aizenman, B. Simon, *Brownian motion and Harnack inequality for Schrödinger operators*, Commun. Pure Appl. Math. **35** (1982), 209-273.

[B] L. Bers, *Local behaviour of solutions of general linear elliptic equations*, Commun. Pure Appl. Math. **8** (1955), 473-496.

[CF] L. A. Caffarelli, A. Friedman, *Partial regularity of the zero-set of solutions of linear and superlinear elliptic equations*, J. Differential Equations **60** (1985), 420-433.

[CM] S. Chanillo, B. Muckenhoupt, *Nodal geometry on Riemannian manifolds*, J. Diff. geometry **34** (1991), 85-71.

[D] R.-T. Dong, *Nodal sets of eigenfunctions on Riemann surfaces*, J. Diff. Geometry **36** (1992), 493-506.

[DF] H. Donelly, Ch. Fefferman, *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. math. **93** (1988), 161-183.

[GT] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations 2nd ed.*, Springer, Berlin, 1983.

[HO2] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, *Local properties of solutions of Schrödinger equations*, Commun. PDE **17** (1992), 491-522.

[HO2N] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, N. Nadirashvili, *Regularity of the nodal sets of solutions to Schrödinger equations, Math. Results in Quantum Mechanics, Int. Conf. in Blossin, Germany, May 17-21 1993, Ed. by M. Demuth, P. Exner, H. Neidhardt, V. Zagrebnov, p.19-25, Operator Theory:Advances and Applications, Vol. 70,*, Birkhäuser, Basel, 1994.

[HO2N1] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, N. Nadirashvili, *Interior Hölder estimates for solutions of Schrödinger equations and the regularity of nodal sets*, to be submitted (1994).

[HO2S1] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, H. Stremnitzer, *electronic wavefunctions near coalescence points*, Phys. Rev. Letters **68** (1992), 3857-3860.

[HO2a] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, *On the local behaviour of nodes of solutions of Schrödinger equations in dimension* ≥ 3, Commun. PDE **15** (1990), 435-451.

[HS] R. Hardt, L. Simon, *Nodal sets for solutions of elliptic equations*, J. Diff. Geometry **30** (1989), 505-522.

[K] T. Kato, *Schrödinger operators with singular potentials*, Is. J. Math. **13** (1973), 135-148.

[Ke] C. Kenig, *Restriction theorems, Carleman estimates, uniform Sobolev inequalities and their application*, Lecture Notes in Mathematics **1384** (1989), 69-89.

[KS] P. Kröger, K.-Th. Sturm, *Hölder continuity of normalized solutions of the Schrödinger equation*, (to appear) Math. Ann. (1994).

[R] L. Robbiano, *Sur les zeros des solutions d'inequalités differentielles elliptiques*, Commun. PDE **12** (1987), 903-919.

[S] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. **7** (1982), 447-526.

[Sa] E. Sawyer, *Unique continuation for Schrödinger operators in dimension three or less*, Ann. Inst. Fourier (Grenobles) **33** (1984), 189-200.

M. Hoffmann-Ostenhof: Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

T. Hoffmann-Ostenhof: Institut für Theoretische Chemie, Universität Wien, Währinger Strasse 17, A-1090 Wien, Austria and International Erwin Schrödinger Institute for Mathematical Physics, Pasteurgasse 6/7, A-1090 Wien, Austria

E-mail address: hoho@itc.univie.ac.at

N. Nadirashvili: International Erwin Schrödinger Institute for Mathematical Physics, Pasteurgasse 6/7, A-1090 Wien, Austria and Institute of Earth Physics, Moscow, Russia