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INVERSE SCATTERING PROBLEMS IN SEVERAL DIMENSIONS

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§1. Refined inverse scattering problem for a real-valued potential

Consider the Schrödinger equation in $\mathbf{R}^n$:

$$(-\Delta + q(x) - k^2)u = 0, \tag{1}$$

where $q(x)$ is a real-valued potential,

$$u(x) = e^{ik\omega\cdot x} + v(x,\omega,k)$$

is the sum of the incident plane wave $e^{ik\omega\cdot x}$, $k > 0$, $|\omega| = 1$, and the reflected wave $v(x,\omega,k)$. We assume that $v(x,\omega,k)$ has the following asymptotics:

$$v(x,\omega,k) = \frac{e^{ik|x|}}{|x|^{\frac{n-1}{2}}} \left(a(\theta,\omega,k) + O\left(\frac{1}{|x|}\right)\right), \tag{2}$$

where $\theta = \frac{x}{|x|}$ and $|x| \to \infty$. Function $a(\theta,\omega,k)$ is called the scattering amplitude. It is convenient as in [ER1] to consider an integral equation

$$h(\xi,\zeta,k) + \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \frac{\tilde{q}(\xi-\eta)h(\eta,\zeta,k)}{|\eta|^2 - (k+i0)^2} d\xi = -\tilde{q}(\xi-\zeta), \tag{3}$$

where $\tilde{q}(\xi) = \int_{\mathbf{R}^n} q(x)e^{-ix\cdot\xi}dx$ is the Fourier transform of $q(x)$. Then

$$a(\theta,\omega,k) = \frac{1}{4\pi}\left(\left(\frac{k}{2\pi}\right)^{\frac{1}{2}} e^{-i\frac{\pi}{4}}\right)^{n-3} h(k\theta,k\omega,k). \tag{4}$$

Since a (θ,ω,k), $\theta \in S^{n-1}$, $\omega \in S^{n-1}$, $k \in \mathbf{R}_+$, is a function of $2n-1$ variables while $q(x)$ depends on n variables the inverse scattering problem of determining $q(x)$ from the scattering amplitude $a(\theta,\omega,k)$ is clearly an overdetermined problem (see [F], [N], [NK], [Me] where this problem was studied).

One can try to determine $q(x)$ using only a part of the scattering amplitude. The backscattering amplitude $a(\theta,-\theta,k)$ is a natural candidate since it depends on n variables ($\theta \in S^{n-1}, k \in \mathbf{R}_+$) as the potential $q(x)$. It was shown in [ER1] that the inverse backscattering is a well-posed problem when we don't require that $q(x)$ is a real-valued potential i.e. we allow complex-valued potentials.

However when $q(x)$ is real-valued the inverse backscattering problem is still overdetermined. Note that $\tilde{q}(\xi) = \overline{\tilde{q}(-\xi)}$ if $q(x)$ is real-valued but the backscattering amplitude $b(\xi) = a(\frac{\xi}{|\xi|}, \frac{-\xi}{|\xi|}, |\xi|)$, obtained from a real-valued potential $q(x)$ does not satisfy the relation $b(\xi) = \overline{b(-\xi)}$ i.e. $b(\xi)$ is not a Fourier transofrm of a real-valued function.

In order to find a natural inverse problem in the case of real-valued potentials consider the one-dimensional case. In this case $b(\xi)$ is a complex-valued function on the real line satisfying the following relations:

$$\begin{aligned} b(\xi) &= r^+(\xi), \quad \xi > 0, \\ b(\xi) &= r^-(\xi), \quad \xi < 0, \end{aligned} \tag{5}$$

where $r^+(\xi)$ and $r^-(\xi)$ are the right and the left reflection coefficients. Note that

$$r^{\pm}(\xi) = \overline{r^{\pm}(-\xi)}, \quad \xi \in \mathbf{R}^1 .$$

It is known (see [Ma]) that in the absence of bound states one can recover real-valued potential on the line from one reflection coeffient (either left or right). It will follow from our results that bound states play no role in the inverse scattering problems for $n \geq 2$. Hence in several dimensions the analog of a reflection coefficient would be $b(\xi)$ restricted to a half-space and the inverse problem will be the following: Choose arbitrarily a unit vector $\nu \in \mathbf{R}^n$. Denote by $\overline{\mathbf{R}^n_\nu}$ the closed half-space $\{\xi : \xi \cdot \nu \geq 0\}$. The inverse problem will consist of the recovering of a real-valued potential $q(x)$ from the restriction of the backscattering amplitude to $\overline{\mathbf{R}^n_\nu}$.

As we shall show even this inverse problem is still slightly overdetermined.

If $c(x)$ is real-valued function then its Fourier transform $\tilde{c}(\xi)$ is not an arbitrary function on $\overline{\mathbf{R}^n_\nu}$ since it must satisfied the relation $\tilde{c}(\xi) = \overline{\tilde{c}(-\xi)}$ for $\xi \in \overline{\mathbf{R}^n_\nu}$ such that $\xi \cdot \nu = 0$.

Let $\chi(t) \in C_0^\infty(\mathbf{R}^+)$ be a cutoff function such that $\chi(t) = 1$ for $|t| < \frac{1}{4}$, $\chi(t) = 0$ for $|t| > \frac{1}{2}$ and $\chi(t) = \chi(-t)$. For each $\xi \in \mathbf{R}^n$ we have the following decomposition

$$\xi = (\xi \cdot \nu)\nu + \xi_\nu , \tag{6}$$

where $\xi_\nu \cdot \nu = 0$. Denote by P_ν the following operator acting on functions defined on $\overline{\mathbf{R}^n_\nu}$:

$$\begin{aligned}(P_\nu f)(\xi) &= f(\xi) - \chi(\xi \cdot \nu) f(\xi_\nu) + \chi(\xi \cdot \nu)\frac{(f(\xi_\nu) + \overline{f(-\xi_\nu)})}{2} \\ &= f(\xi) + \frac{1}{2}\, \chi(\xi \cdot \nu)(\overline{f(-\xi_\nu)} - f(\xi_\nu)).\end{aligned} \tag{7}$$

Note that

$$(P_\nu f)(\xi_\nu) = \overline{(P_\nu f)(-\xi_\nu)} \tag{8}$$

and

$$(P_\nu f)(\xi) = f(\xi) \tag{9}$$

if $f(\xi) = \overline{f(-\xi)}$ i.e. $f(\xi)$ is a Fourier transform of a real-valued function.
As in [ER1] denote by $H_{\alpha,N}$ the closure of $C_0^\infty(\mathbf{R}^n)$ in the norm

$$(10) \qquad \|f\|_{\alpha,N} = \sup_{\xi,h\in\mathbf{R}^n} (1+|\xi|)^N \left(\frac{|f(\xi+h)-f(\xi)|}{|h|^\alpha} + |f(\xi)|\right)$$

and denote by $H^r_{\alpha,N}$ the subspace of $H_{\alpha,N}$ consisting of functions satisfying $f(\xi) = \overline{f(-\xi)}$.

Let $\hat{P}_\nu$ be the extension of $P_\nu f$ to $\mathbf{R}^n$ by defining

$$(11) \qquad (\hat{P}_\nu f)(\xi) = \overline{(P_\nu f)(-\xi)} \quad \text{when} \quad \xi \notin \overline{\mathbf{R}^n_\nu}.$$

Then $\hat{P}_\nu$ is a linear bounded operator from $H_{\alpha,N}$ onto $H^r_{\alpha,N}$.

Consider the case $n \geq 3$ dimensions. (The case $n = 2$ need a special consideration.) It was proven in [ER1] that there is an open dense set $O \subset H_{\alpha,N}$ such that $O_r = O \cap H^r_{\alpha,N}$ is also an open dense set in $H^r_{\alpha,N}$ and such that

$$(12) \qquad b(\xi) = S(\tilde{q})$$

is analytic map of O into $H_{\alpha,N}$. Therefore

$$(13) \qquad S_r(\tilde{q}) = \hat{P}_\nu S(\tilde{q})$$

is a real-analytic map of O_r to $H^r_{\alpha,N}$.

Moreover the following result holds:

THEOREM 1. *There is an open dense set $\mathcal{O}'_r \subset \mathcal{O}_r$ in $H^r_{\alpha,N}$ such that the map $\hat{P}_\nu S(\tilde{q})$ is a local homeomorphism in $H^r_{\alpha,N}$ in a neighborhood of any $\tilde{q} \in \mathcal{O}'_r$.*

Note the result above shows that the data $b(\xi)$ on $\overline{R^n_\nu}$ are still overdetermined since on the plane $\xi \cdot \nu = 0$ one only uses the combination $b(\xi_\nu) + \overline{b(-\xi_\nu)}$.

The proof of Theorem 1 follows the same steps as the proof of Theorem B in [ER1]. First we show that the Frechet derivative of $d(\hat{P}_\nu S(\tilde{q}))$ is a Fredholm operator of index zero for any $\tilde{q} \in \mathcal{O}_r$ and it is invertible when $\|\tilde{q}\|_{\alpha,N}$ is small. Since $\hat{P}_\nu S(\tilde{q})$ is a real analytic map we get that $\hat{P}_\nu S(\tilde{q})$ is a local homeomorphism for any $\tilde{q}$ belonging to an open dense set of the connected component of O_r containing zero potential. It was shown in [ER1] that $\mathcal{O}$ is a connected set in $H_{\alpha,N}$ but O_r is not a connected set in $H^r_{\alpha,N}$. To prove that $\hat{P}_\nu S(\tilde{q})$ is a local homeomorphism for any $\tilde{q}$ belonging to an open dense set $\mathcal{O}'_r \subset \mathcal{O}_r$ we shall follow an approach similar to [ER3]:

Consider equation (3) for $z = k + i\tau$, $k \in \mathbf{R}^1$, $\tau \in \overline{\mathbf{R}^1_+}$:

$$h(\xi, \zeta, k + i\tau) + \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{\tilde{q}(\xi - \eta) h(\eta, \zeta, k + i\tau)}{|\eta|^2 - (k + i\tau)^2} \, d\xi = -\tilde{q}(\xi - \zeta). \tag{14}$$

It follows from [ER1] that solution $h(\xi, \zeta, k)$ exists for any real k assuming that $\tilde{q} \in \mathcal{O}$. Therefore the scattering amplitude is defined for all $\theta \in S^{n-1}$, $\omega \in S^{n-1}$, $k \in \mathbf{R}^1$:

$$a(\theta, \omega, k) = C_{n,k} h(k\theta, k\omega, k) \tag{15}$$

where $\tilde{q} \in \mathcal{O}$ and

$$C_{n,k} = \frac{1}{4\pi} \left(\left(\frac{k}{2\pi} \right)^{\frac{1}{2}} e^{\frac{-i\pi}{4}} \right)^{n-3}. \tag{16}$$

In particular the backscattering amplitude

$$b(\theta, k) = a(\theta, -\theta, k) = C_{n,k} h(k\theta, k\theta, k) \tag{17}$$

is a function on $S^{n-1} \times \mathbf{R}^1$. Note that $b(\theta, 0) = C_{n,k}h(0,0,0)$ is independent of θ. Denote by $\overline{S_\nu^{n-1}}$ the closed half-sphere $\omega \cdot \nu \geq 0$. In the case of complex-valued potentials $q(x)$ we shall consider the inverse problem of recovering $\tilde{q}(\xi)$ from the restriction of $b(\theta, k)$ to $\overline{S_\nu^{n-1}} \times \mathbf{R}^1$. More precisely, denote by $P_\nu^{(1)}$ the following operator acting on functions $c(\theta, k)$, $\theta \in \overline{S_\nu^{n-1}}$, $k \in \mathbf{R}^1$, such that $c(\theta, 0)$ is independent of θ:

$$(P_\nu^{(1)}c)(\theta, k) = c(\theta, k) + \frac{1}{2}\chi(k\theta \cdot \nu)\left(c\left(-\frac{\theta_\nu}{|\theta_\nu|}, -k\theta_\nu\right) - c\left(\frac{\theta_\nu}{|\theta_\nu|}, k\theta_\nu\right)\right), \tag{18}$$

where

$$\theta = (\theta \cdot \nu)\nu + \theta_\nu .$$

For any $\tilde{q}(\xi) \in \mathcal{O} \subset H_{\alpha,N}$ define the following map:

$$\tilde{q}(\xi) \xrightarrow{S_\nu^{(1)}} P_\nu^{(1)}b(\theta, k), \quad \theta \in \overline{S_\nu^{n-1}}, \quad k \in \mathbf{R}^1 , \tag{19}$$

where $b(\theta, k)$ is the backscattering amplitude.

THEOREM 2. *Denote $b_1(\xi) = (P_\nu^{(1)}b)(\theta, k)$ where $\xi = k\theta$. If $\tilde{q}(\xi) \in \mathcal{O} \subset H_{\alpha,N}$ then $b_1(\xi) \in H_{\alpha,N}$. Moreover $S_\nu^{(1)}$ is an analytic map from $\mathcal{O}$ to $H_{\alpha,N}$ and the Frechet derivative $dS_\nu^{(1)}$ is a Fredholm operator of index zero for any $\tilde{q} \in \mathcal{O}$.*

Since $dS_\nu^{(1)}$ is invertible for small $\|\tilde{q}\|_{\alpha,N}$ and $\mathcal{O}$ is connected we obtain the following theorem:

Theorem 3. *There is an open dense set $\mathcal{O}_1 \subset \mathcal{O}$ such that $S_\nu^{(1)}(\tilde q)$ is a local homeomorphism in a neighborhood of any $\tilde q \in \mathcal{O}_1$.*

Consider now the case of real-valued potentials. If $\tilde q(\xi) \in \mathcal{O}_r = \mathcal{O} \cap H^r_{\alpha,N}$ than taking the complex conjugate of (14) we get

$$\overline{h(\xi,\zeta,k)} = h(-\xi,-\zeta,-k). \tag{20}$$

Therefore if $\tilde q \in \mathcal{O}_r$ we have

$$\overline{b(\theta,k)} = \overline{C_{n,k}}\ \overline{h(k\theta,-k\theta,k)} = C_{n,-k}h(-k\theta,k\theta,-k) = b(\theta,-k). \tag{21}$$

Hence

$$\begin{aligned}\overline{b_1(\xi)} &= \overline{b(\theta,k)} + \frac{1}{2}\chi(k\theta\cdot\nu)\left(\overline{b\left(-\frac{\theta_\nu}{|\theta_\nu|},-k|\theta_\nu|\right)} - \overline{b\left(\frac{\theta_\nu}{|\theta_\nu|},k|\theta_\nu|\right)}\right)\\ &= b(\theta,-k) + \frac{1}{2}\chi(k\theta\cdot\nu)\left(b\left(-\frac{\theta_\nu}{|\theta_\nu|},k|\theta_\nu|\right) - b\left(\frac{\theta_\nu}{|\theta_\nu|},-k|\theta_\nu|\right)\right) = b_1(-\xi).\end{aligned} \tag{22}$$

Therefore $b_1(\xi) \in H^r_{\alpha,N}$ and Theorem 3 implies the following result:

Theorem 4. *The restriction of the map $S_\nu^{(1)}$ to $O_r = \mathcal{O} \cap H^r_{\alpha,N}$ is a local real analytic homeomorphism in $H^r_{\alpha,N}$ in a neighborhood of any $\tilde q \in O_1 \cap H^r_{\alpha,N}$.*

Note that $\mathcal{O}_1 \cap H^r_{\alpha,N}$ is open and dense in $H^r_{\alpha,N}$. Now we shall show that for $\tilde q \in O_r$ the restriction of the map $b_1(\xi) = P_\nu^{(1)}S(\tilde q)$ to $\overline{\mathbf{R}^n_\nu}$ coincide with the map $P_\nu S(\tilde q)$ (see (7)). Since $b(\theta,k) = \overline{b(\theta,-k)}$ when $\tilde q(\xi) \in O_r$ we obtain from (18)

$$(P_\nu^{(1)}b)(\theta,k) = b(\theta,k) + \frac{1}{2}\chi(k\theta\cdot\nu)\left(\overline{b\left(-\frac{\theta_\nu}{|\theta_\nu|},\ k\theta_\nu\right)} - b\left(\frac{\theta_\nu}{|\theta_\nu|},k\theta_\nu\right)\right). \tag{23}$$

Denote $\xi = k\theta$. Then $\xi_\nu = k\theta_\nu$ and $\xi \in \overline{\mathbf{R}^n_\nu}$ when $k \geq 0$, $\theta \in \overline{S^{n-1}_\nu}$. Comparing (23) and (7) we obtain

$$(P_\nu b)(\xi) = (P^{(1)}_\nu b)(\theta, k) \quad \text{for} \quad \xi = k\theta, k \geq 0, \theta \in \overline{S^{n-1}_\nu} \tag{24}$$

Therefore

$$\hat{P}_\nu b(\xi) = P^{(1)}_\nu b(\theta, k), \quad \xi = k\theta \tag{25}$$

for any $\tilde{q} \in \mathcal{O}_r$ and Theorem 1 follows from Theorem 4 with $\mathcal{O}'_r = \mathcal{O}_1 \cap H^r_{\alpha,N}$.

§2 A new well-posed inverse scattering problem in two dimensions

The singularity of the backscattering amplitude at $k = 0$ makes the inverse backscattering problem in two dimensions more difficult than in $n \geq 3$ dimensions.

One way to overcome this difficulty was proposed in [ER2]. It was shown there that for a dense open set of $\tilde{q}(\xi) \in H_{\alpha,N}$ we have

$$b(\xi) = \frac{2\pi}{2\pi\beta + \ln|\xi|}\chi(|\xi|\delta_0^{-1}) + b_0(\xi), \tag{26}$$

where $\chi(t)$ is the same cutoff function as in (7), $\delta_0 > 0$ is small, $\beta \neq 0$ is a constant, $b_0(\xi) \in H_{\alpha,N}$ and $b_0(0) = 0$. It was proposed in [ER2] to consider β and $b_0(\xi)$ as "coordinates" of the backscattering amplitude $b(\xi)$ and it was proven that the map $\tilde{q}(\xi) \to \frac{1}{\beta}\chi(|\xi|\delta_0^{-1}) + b_0(\xi)$ is a local homeomorphism in $H_{\alpha,N}$ in a neighborhood of any $\tilde{q} \in \mathcal{O}_2$ where $\mathcal{O}_2$ is an open dense set in $H_{\alpha,N}$. In this section we consider another inverse scattering problem in two dimensions.

Fix $k_0 > 0$ and denote

$$\xi = \frac{1}{2}(k_0\theta - k_0\omega), \tag{27}$$

where θ and ω are unit vectors in $\mathbf{R}^2$. Denote by $\xi^{\perp}$ the rotation of ξ by the angle $\frac{\pi}{2}$ clockwise.

For any $\xi \neq 0$, $|\xi| \leq k$ we define

$$k_0\theta_+ = \xi + \sqrt{k_0^2 - |\xi|^2}\,\frac{\xi^{\perp}}{|\xi|}\,, k_0\omega_+ = -\xi + \sqrt{k_0^2 - |\xi|^2}\,\frac{\xi^{\perp}}{|\xi|} \tag{28}$$

$$k_0\omega_- = \xi - \sqrt{k_0^2 - |\xi|^2}\,\frac{\xi^{\perp}}{|\xi|}\,,\ k_0\omega_- = -\xi - \sqrt{k_0^2 - |\xi|^2}\,\frac{\xi^{\perp}}{|\xi|}\,. \tag{28'}$$

Then $|\theta_{\pm}| = 1$, $|\omega_{\pm}| = 1$. Vice versa any two vectors $k_0\theta$, $k_0\omega$ where $|\theta| = |\omega| = 1$, $\theta \neq \omega$, can be represented either in the form (28) or in the form (28′) where $\xi = \frac{1}{2}(k_0\theta - k_0\omega)$. Let $h(\xi, \zeta, k)$ be the solution of equation (3) for $k > 0$. It was proven in [ER1] that

$$h(\xi, \zeta, k) = h(-\zeta, -\xi, k), \forall (\xi, \zeta, k)\,. \tag{29}$$

It follows from (29) that

$$\begin{aligned} h(k_0\theta, k_0\omega, k_0) &= h(\xi + \sqrt{k_0^2 - |\xi|^2}\,\frac{\xi^{\perp}}{|\xi|}\,, -\xi + \sqrt{k_0^2 - |\xi|^2}\,\frac{\xi^{\perp}}{|\xi|}\,, k_0) \\ &= h\left(\xi - \sqrt{k_0^2 - |\xi|^2}\,\frac{\xi^{\perp}}{|\xi|}\,, -\xi - \sqrt{k_0^2 - |\xi|^2}\,\frac{\xi^{\perp}}{|\xi|}\,, k_0\right)\,, \end{aligned}$$

where $\theta \neq \omega$, $\xi = \frac{1}{2}(k_0\theta - k_0\omega)$ and either (28) on (28′) holds. Note that $h(k_0\theta, k_0\omega, k_0)$ is a single-valued function of $\xi = \frac{1}{2}(k_0\theta - k_0\omega)$ for $0 < |\xi| \leq k_0$. Let $\chi(t)$ be the same cutoff function as in (7) and θ_0 be a fixed unit vector, $|\theta_0| = 1$. Denote by $b_2(\xi)$ the following function:

$$b_2(\xi) = C_{2,k_0} h(\xi, -\xi, |\xi|) \quad \text{for} \quad |\xi| \geq k_0\,, \tag{31}$$

$$b_2(\xi) = C_{2,k_0}(h(k_0\theta, k_0\omega, k_0) - \chi\left(\frac{1}{2}|\theta-\omega|\right) h(k_0\theta, k_0\theta, k_0)$$

$$+\chi\left(\frac{1}{2}\ |\theta-\omega|\right) h(k_0\theta_0, k_0\theta_0, k_0) \quad \text{for} \quad |\xi| < k_0 \quad \text{where}$$

$$\xi = \frac{1}{2}(k_0\theta - k_0\omega) \quad \text{for} \quad \omega \neq \theta \quad \text{and} \quad \xi = 0, \ \frac{\xi_\perp}{|\xi|} = \theta \quad \text{for} \quad \omega = \theta. \tag{31'}$$

Here $C_{2,k_0} = \frac{1}{4\pi}\left(\frac{2\pi}{k_0}\right)^{\frac{1}{2}} e^{i\frac{\pi}{4}}$ is the same as in (16), $n = 2$. Note that $b_2(\xi)$ is equal to the backscattering amplitude for $k \geq k_0$ and it contains less information than the full scattering amplitude for $k = k_0$: we have subtracted the forward scattering amplitude $C_{2,k_0}h(k_0\theta, k_0\theta, k_0)$ in the neighborhood $|\omega - \theta| < \frac{1}{4}$. Although $h(k_0\theta, k_0\omega, k_0)$ is a discontinuous function of ξ at $\xi = 0$, $b_2(\xi)$ is continuous with respect to ξ at $\xi = 0$. Moreover $b_2(\xi) \in H_{\frac{\alpha}{2},N}$ assuming that $\tilde{q}(\xi) \in H_{\alpha,N}$, $0 < \alpha < 1$, $N > 0$.

THEOREM 5. *There is an open dense set U in $H_{\alpha,N}(\mathbf{R}^2) \times H_{\alpha,N}(\mathbf{R}^2)$ such that for any pair $(\tilde{q}, \tilde{q}') \in U$, $b_2(\xi) = b_2'(\xi)$ implies $\tilde{q} = \tilde{q}'$. Moreover, for $(\tilde{q}, \tilde{q}') \in U$ and $\tilde{q} - \tilde{q}'$ small in $H_{\alpha,N}(\mathbf{R}^2)$ we have*

$$\|\tilde{q} - \tilde{q}'\|_{\frac{\alpha}{4},N} \leq C\|b_2 - b_2'\|_{\frac{\alpha}{2},N},$$

i.e. the map $\tilde{q} \to b_2(\xi)$ is well-posed.

Note that one can modify $b_2(\xi)$ by taking $\chi(\frac{1}{2}|\theta-\omega|) \int_{|\theta|=1} h(k_0\theta, k_0\theta, k_0)p(\theta)d\theta$ instead of $\chi(\frac{1}{2}|\theta - \omega|)h(k_0\theta_0, k_0\theta_0, k_0)$. Here $p(\theta)$ is arbitrary such that $\int_{|\theta|=1} p(\theta)d\theta = 1$.

Another inverse scattering problem for potential without compact support in two dimensions was studied in [No].

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