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FOCUSING AND ABSORBTION OF NONLINEAR OSCILLATIONS

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The subject of this talk is the propagation of nonlinear oscillations *across focal points*. When rays converge near a focus, amplitudes grow. These large amplitudes can lead to more strongly nonlinear phenomena than in other regions.

We consider focusing waves in the regime of nonlinear geoemtric optics which is characterized by the fact that the prinicpal profile satisfies a *nonlinear* equation. For the semilinear wave equation

$$\Box u + F(Du) = 0, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^d$$

this means that the amplitude and wavelength are of order ε as $\varepsilon \to 0$. For piecewise smooth conormal solutions the analogous strength of waves occurs for solutions whose gradient is discontinuous. For smaller amplitudes or smoother conormal solutions, the principal profile (resp. prinicipal symbol) satisfies a *linear* equation.

The nonlinear effects can lead to catastrophic breakdown of solutions [JMR2]. We consider problems for which solutions are guaranteed to exist for all $t \geq 0$, in particular, long enough for oscillations to cross focal points.

Consider the dissipative wave equation

1) $$0 = \Box u + F(u_t), \quad F(s) := as|s|^{p-2}, \quad p \geq 2, \quad a > 0.$$

For Cauchy data $\partial_t^j u(0,\cdot) \in H^{1-j}(\mathbb{R}^d)$ $j=0,1$, there is a unique solution

$$u \in C([0,\infty[\,;H^1(\mathbb{R}^d)) \cap C^1([0,\infty[\,;L^2(\mathbb{R}^d))$$

and the evolution is contractive in the sense that for two solutions u and v

$$2)\quad \frac{d}{dt}\int (u_t-v_t)^2 + |\nabla_x(u-v)|^2\, dx = -\int (u_t-v_t)(F(u_t)-F(v_t))\, dx \le 0.$$

We study spherical wavefronts which focus at the origin. The goal is to describe what is observed after the focus. The analogous problem for general caustics is open, but formal calculations suggest that the phenomenon we describe extends to that case.

We first present computations which motivate the results and a part of the proof. They also suggest some, as yet unproved, results concerning the smoothing of focusing conormal wavefronts for strongly dissipative equations.

Linear spherical wavefronts.

For incoming *linear* geometric optics solutions of $\Box u=0$,

$$e^{i(t+r)/\varepsilon}\left\{f(t+r,x/|x|)r^{-(d-1)/2} + \cdots\right\},$$

the outgoing wave is given by

$$e^{i(t-r)/\varepsilon}\left\{(-1)^{(d-1)/2}f(t-r,-x/|x|)r^{-(d-1)/2} + \cdots\right\}.$$

The formula is interpreted as follows. Conservation of energy implies that $r^{(d-1)/2}$xamplitude is constant on rays. The rate of change per unit length of the phase is equal to one. Crossing the focus there is an additional phase factor $(-1)^{(d-1)/2}$, a Keller-Maslov index.

Transport equations for jumps.

Consider a radial solution $u(t,r)$ of the semilinear wave equation

$$3)\qquad 0 = \Box u + F(u_t) = u_{tt}-u_{rr}- (d-1)u_r/r + F(u_t)$$

supposed to be piecewise smooth with discontinuities in its first derivatives on the incoming light cone $t+r=0$, $t<0$.

Derivatives in $|x|>0$ of such piecewise smooth functions are given by

$$\partial_t f = 2[f]\delta(t+r) + [f_t]h(t+r) + \cdots$$

$$\partial_r f = 2[f]\delta(t+r) + [f_r]h(t+r) + \cdots$$

where $[\cdot]$ denotes the jump measured from $t+r>0$ to $t+r<0$, h is the Heaviside function and the terms $\cdots$ are smoother. Then,

$$\Box u + F(u_t) =$$

$$2[u_t-u_r]\delta(t+r) + [(2\partial_t-2\partial_r-(d-1)/r)u_t+F(u_t)]h(t+r) + \cdots .$$

Setting the coefficient of the most singular term equal to zero shows that u_t-u_r must be continuous. Setting the coefficient of the next term equal to zero yields

$$(2\partial_t-2\partial_r-(d-1)/r)[u_t] + [F(u_t)] = 0.$$

If u is constant in front of the wave, then $u_t=0$ in $t+r<0$, and one finds the nonlinear transport equation

4) $$(2\partial_t-2\partial_r-(d-1)/r)[u_t] + F([u_t]) = 0.$$

Transport equation for oscillations.

Consider oscillatory asymptotic solutions

5) $$u^\varepsilon \sim \varepsilon V_1(t,r,(t+r)/\varepsilon) + \varepsilon^2 V_2(t,r,(t+r)/\varepsilon) + \cdots$$

with smooth $V_j(t,r,\theta)$ 2π periodic in θ. Plugging into (3) yields

$$\Box u^\varepsilon + F(u^\varepsilon_t) = (2\partial_t-2\partial_r-(d-1)/r)\partial_\theta V_1+F(\partial_\theta V_1) + O(\varepsilon).$$

Setting the leading order term equal to zero yields a transport for $W:=\partial_\theta V_1$ which is identical to (4),

6) $$(2\partial_t - 2\partial_r - (d-1)/r)W + F(W) = 0.$$

Note that θ is a parameter and intergrating the transport equation for W with respect to θ shows that the identity $\int W d\theta = 0$ propagates.

Solution of the transport equation.

Parametrize the focussing ray as $(-r,r)$. Thus as r decreases to zero, $t=-r$ increases to zero and the ray approaches the origin. The transport equation yields an ordinary differential equation for $w(r) := [u_t(-r,r)]$ or $w(r) := W(-r,r,\theta)$,

7) $$2dw/dr + (d-1)w/r - aw|w|^{p-2} = 0.$$

Then $z := r^{(d-1)/2} w$ satisfies

8) $$2dz/dr - ar^{-\beta} z|z|^{p-2} = 0, \qquad \beta := (d-1)(p-2)/2.$$

It follows for $\beta \neq 1$ that

9) $$d/dr\left[-|z|^{-p}/p - ar^{-\beta+1}/(2(-\beta+1))\right] = 0.$$

Consider strong dissipation, that is $a>0$ and $\beta \geq 1$. Then the $\int r^{-\beta} dr$ term is not integrable at $r=0$. Equation (9) shows that $z(r)$ tends to zero as r decreases to zero.

On the other hand, for $0<\beta<1$, z converges to a nonzero constant as r tends to zero.

Thus, all of the energy is absorbed by the dissipative term when $\beta \geq 1$ and a positive fraction survives if $0<\beta<1$.

For $d=3$, the condition $\beta \geq 1$ is equivalent to $p \geq 3$ which corresponds to a friction term $u_t|u_t|^{p-1}$ with at least quadratic growth.

Explosive solutions.

If $a<0$, the $F(u_t)$ term supplies energy. If $\beta \geq 1$, then solutions of the equation for z which are nonzero for some $r_0>0$ explode *before* the ray $(-r,r)$ reaches the origin.

Applying known results about piecewise smooth solutions (in dimension 1+1 for radial solutions) one proves blowup theorems *before the wave reaches the focus.*

Formal analysis after a strongly dissipative focus.

Suppose that a piecewise smooth solution survives focusing and emerges as piecewise smooth solution in $|x|>0$ with discontinuities on the outgoing light cone. Along outgoing rays one has the transport equation

$$10) \qquad (2\partial_t+2\partial_r-(d-1)/r)[u_t] + [F(u_t)] = 0.$$

The strong monotonicity of the dissipative p^{th} power nonlinearity implies that,

$$11) \qquad [u_t][F(u_t)] \geq c|[u_t]|^p, \qquad c=a/2^p.$$

Multiply the transport equation by $r^{d-1}[u_t]$ and use

$$r^{d-1}[u_t](2\partial_t+2\partial_r-(d-1)/r))[u_t] = (\partial_t+\partial_r)(r^{d-1}[u_t]^2)$$

to find the transport inequality for $\zeta := r^{(d-1)}[u_t]^2$

$$12) \qquad d\zeta/dr + ar^{-\beta}|\zeta|^{p/2} \leq 0, \qquad \beta := (d-1)(p-2)/2.$$

The same analysis as for the explosive solutions shows that if $\beta \geq 1$ and ζ is nonzero at a point on the outgoing ray, then tracing *backward* along the ray, ζ must explode *before arriving at the focus.* Thus if the solution is piecewise smooth away from the origin, one must have $\zeta=0$ on the outgoing ray. This predicts that *the discontinuity in* $\nabla_{t,x}u$ *is smoothed after the focus.*

The same type of formal argument on an outgoing ray shows that *if* an oscillation survives focusing and emerges as a wave with an asymptotic expansion

$$u^{\varepsilon} \sim \varepsilon \underline{V}_1(t,r,(t-r)/\varepsilon) + \varepsilon^2 \underline{V}_2(t,r,(t-r)/\varepsilon) + \cdots$$

then for $\beta \geq 1$, the outgoing profile $\underline{V}_1$ must be identically zero.

The two formal arguments suggest that incoming conormal singularities or oscillations which are of critical size (discontinuities in gradient, or as in (5)), the natural principal symbol of the outgoing wave vanishes. In both cases, this is a smoothing effect.

Main Result.

We prove that the smoothing effect for oscillatory solutions does occur. More generally, we prove such a smoothing for arbitrary families which are uniformly smooth in the angular variables. The smoothing is expressed by the fact that families with bounded energy emerge on the other side of the focus as families compact in the energy norm.

An additional analysis is required to identify the strong limits of the family, and, to show that solutions with profiles as in (4) are described with profiles up to the focus (see [JMR4]).

Consider a family of solutions u^{ε} of (1) with Cauchy data $u^{\varepsilon}(0,x)$, $u^{\varepsilon}_t(0,x)$ which are of unformly bounded energy together with their angular derivatives. Precisely, for $1 \leq k, l \leq d$ let

13) $$\Gamma_{k,l} := x_l \partial/\partial x_k - x_k \partial/\partial x_l \,,$$

and, consider initial data such that the families

3 $$u^{\varepsilon}(0,x), \quad \nabla_{t,x} u^{\varepsilon}(0,x), \text{ and } \Gamma_{k,l} \nabla_{t,x} u^{\varepsilon}(0,x), \quad 0 < \varepsilon \leq 1$$

are uniformly bounded in $L^2(\mathbb{R}^d)$. Then the families

$$u^{\varepsilon}(t,x), \quad \nabla_{t,x} u^{\varepsilon}(t,x), \text{ and } \Gamma_{k,l} \nabla_{t,x} u^{\varepsilon}(t,x), \quad t \geq 0, \ 0 < \varepsilon \leq 1$$

are uniformly bounded in $L^2(\mathbb{R}^d)$, and

$$\{u_t^\varepsilon\} \text{ bounded in } L^p([0,\infty[\times\mathbb{R}^d).$$

The fact that one has extra regularity in the angular directions forces the principal directions of propagation to be radial. Introduce

14) $$v_\pm^\varepsilon := (\partial_t \mp \partial_r)u^\varepsilon/2, \qquad \partial_r := |x|^{-1}x\cdot\nabla .$$

In $x\neq 0$, the differential equation (1) takes the form

15) $$2(\partial_t \pm \partial_r)v_\pm^\varepsilon - (d-1)(v_+^\varepsilon + v_-^\varepsilon)/r - \sum_{j<k} \Gamma_{k,l}^2 u^\varepsilon/r^2 + F(v_+^\varepsilon + v_-^\varepsilon) = 0.$$

Definition. A bounded family z^ε in $L^q(\mathcal{O})$ is compact at $\underline{y}\in\mathcal{O}$ if there is a neighborhood $\omega\subset\mathcal{O}$ of $\underline{y}$ such that the restrictions of z^ε to ω lie in a compact subset of $L^q(\mathcal{O})$.

Example. The angular regularity of u^ε implies that if $q\le 2$ and $\underline{x}\neq 0$, then $\nabla_{t,x}u^\varepsilon$ is compact in L^q at $(\underline{t},\underline{x})$ if and only if both v_+^ε and v_-^ε are compact in L^q at $(\underline{t},\underline{x})$.

For angularly smooth solutions, compactness of $v_\pm^\varepsilon$ propagates along integral curves of $\partial_t \pm \partial_r$ until they reach the origin.

THEOREM. (Propagation of compactness). Suppose that u^ε is angularly smooth family as above. If the Cauchy data $v_-^\varepsilon(0,\cdot)$ is compact in L^2 at $\underline{x}\neq 0$ then v_-^ε is compact in L^2 at the points $\{(t,\underline{x}-t\underline{x}/|\underline{x}|):0\le t<|\underline{x}|\}$ on the focu~~s~~sing ray through $\underline{x}$. Similarly if $v_+^\varepsilon(0,\cdot)$ is compact in L^2 at $\underline{x}\neq 0$ then v_+^ε is compact in L^2 at all points $\{(t,\underline{x}+t\underline{x}/|\underline{x}|):0\le t\}$ of the outgoing ray through $\underline{x}$.

The rays which have passed through a focus are the $\partial_t+\partial_r$ rays in the forward light cone $\{t>|x|\}$. Our main result expresses the fact that oscillations which may be present in the initial data do not survive a passage through the focus.

THEOREM. (**Absorbtion of oscillations**). If $a>0$ and $1\leq|\beta|=(d-1)(p-2)/2$, then v_+^ε is compact in L^2 at all points of the punctured forward light cone $\{t>|x|>0\}$.

Outline of the proofs.

Step 1. Extract limits using uniform bounds.

Let $\Omega:=]0,T[\times\mathbb{R}^d$. Passing to a subsequence, one can suppose that

16) $u^\varepsilon \rightarrow \underline{u}$ and $\Gamma_{k,l}u^\varepsilon \rightarrow \Gamma_{k,l}\underline{u}$ weakly $H^1(\Omega)$ and strongly in $L^2_{loc}(\Omega)$.

17) Similar convergence holds for the Cauchy data.

18) $F(u_t^\varepsilon) \longrightarrow \underline{F}$ weakly in $L^{p/(p-1)}(\Omega)$.

It follows that the limit $\underline{u}$ is the unique solution of

$$\Box\underline{u} + \underline{F} = 0, \quad \underline{u}(0,\cdot) = \underline{u}_0, \quad \underline{u}_t(0,\cdot) = \underline{u}_1.$$

A key ingredient is to determine $\underline{F}$ which is often not equal $F(\lim u_t^\varepsilon)$.

Step 2. Independence of the Young measures.

Any lack of compactness in $\nabla_{t,x}u^\varepsilon$ must come from u_t^ε and u_r^ε. Introduce

$$w_\pm^\varepsilon := (\partial_t \mp \partial_r)(u^\varepsilon - \underline{u})/2.$$

Then the $w_\pm^\varepsilon$ and $\Gamma_{k,l}w_\pm^\varepsilon$ tend weakly to zero and are bounded in $L^2(\Omega)$. Passing to a subsequence we can suppose that there exist Young measures $\mu(y,d\lambda_+ d\lambda_-)$ which are probability measures on $\mathbb{R}^2$ (with running point $\lambda=(\lambda_+,\lambda_-)$) depending measurably on $y\in\Omega$ and such that for all $f\in C(\mathbb{R}^2_\lambda:\mathbb{R})$ such that $f=o(|\lambda|^2)$ as $\lambda\longrightarrow\infty$, and

$\forall\varphi\in C_0^{\infty}(\Omega)$

$$\int\varphi(y)f(w_+^\varepsilon(y),w_-^\varepsilon(y))dy\longrightarrow \int\varphi(y)f(\lambda_+,\lambda_-)\mu(y,d\lambda_+d\lambda_-)dy.$$

Then the Young measures $\mu_+(y,d\lambda_+)$ of the sequence w_+^ε is given by

$$\int\psi(\lambda_+)\mu_+(y,d\lambda_+) = \int \psi(\lambda_+)\mu(y,d\lambda_+d\lambda_-)$$

with a similar formula for μ_-.

A crucial step is to show that the Young measure μ is given by

19) $$\mu(y,d\lambda_+d\lambda_-) = \mu_+(y,d\lambda_+)\otimes\mu_-(y,d\lambda_-).$$

This identity expresses the indpendence of the waves $w_\pm^\varepsilon$ which is linked to the fact that there are no nontrivial resonance relations involving the phases $t\pm r$.

To prove (19) it suffices to show that for $f_\pm\in C_0^\infty(\mathbb{R})$, $z_\pm^\varepsilon:=f_\pm(w_\pm^\varepsilon)$ satisfy

$$\text{weak}\lim(z_+^\varepsilon z_-^\varepsilon) = (\text{weak}\lim z_+^\varepsilon)(\text{weak}\lim z_-^\varepsilon).$$

This follows from a variant of the div-curl lemma based on the angular regularity of the $z_\pm^\varepsilon$ together with the wave equation which implies that

$$(\partial_t\pm\partial_r)z_\pm^\varepsilon \quad \text{is bounded in } L^q_{loc}(\Omega\setminus\{x=0\}),\ q>1.$$

Step 3. Transport equations for the Young measures.
Write the equation (2) in terms of the w's, multiply by $\varphi(y)f(w_\pm^\varepsilon)$, and, pass to the limit using the crucial independence relation (19). This yields a coupled system of transport equations for $\mu_\pm$,

20) $$2(\partial_t\pm\partial_r)\mu_\pm \mp (d-1)\partial_{\lambda_\pm}(\lambda_\pm\mu_\pm/r) - \partial_{\lambda_\pm}(F_\pm(y,\lambda_\pm)\mu_\pm = 0,$$

21) $$F_\pm(y,\lambda_\pm) := \int \tilde{F}(y,\lambda_\pm+\tau)\mu_\mp(y,d\tau) - \underline{F}(y)$$

22) $$\tilde{F}(y,\tau) := F(\partial_t\underline{u}(y) + \tau).$$

This contrasts sharply with resonant problems for which the Young measures at time t=0 do not determine their values in t>0.

Step 4. Transport inequality for the variance of $\mu_\pm$.

The strategy for proving compactness (in L^q for q<2) is to prove that the variances $\sigma_\pm$ of $\mu_\pm$ vanish. To do that, one derives a transport equation for $\sigma_+(y) := \int \lambda_+^2 \mu_+(y, d\lambda_+)$ (similarly σ_-),

23) $$(\partial_t + \partial_r)\sigma_+ + (d-1)\sigma_+/r + h_+ = 0 \text{ on } x \neq 0$$

where

$$h_+(y) := \int \lambda_+ F_+(y, \lambda_+)\mu_+(y, d\lambda_+).$$

Strict monotonicity of F yields

24) $$h_+(y) \geq c(\sigma_+(y))^{p/2}.$$

The propagation of compactness in L^q for q<2 follows immediately from the transport inequality (23)-(24).

For absorbtion, one reasons as in the formal arguments. If σ_+ is nonzero at a point of an outgoing ray, then tracing *backward*, the transport inequality (23)-(24) is explosive and σ_+ must approach infinity at $r_0>0$. This contradicts the energy bounds on $\nabla_{t,x} u^\varepsilon$.

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