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Global Solutions of Nonlinear Parabolic Equations*

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Consider the Navier-Stokes equations in two dimensions, describing the motion of viscous incompressible fluid in all of $\mathbb{R}^2$. Using the well-known vorticity formulation [4] the equations can be written as follows.

$$
\begin{aligned}
&\xi_t + (\mathbf{u}\cdot\nabla)\xi = \nu\Delta\xi, \text{ in } (\mathbf{x},t) \in \mathbb{R}^2 \times \bar{\mathbb{R}}_+, \\
&\mathbf{u}(\mathbf{x},t) = (\mathbf{K} * \xi)(\mathbf{x},t),
\end{aligned} \tag{1}
$$

where the kernel $\mathbf{K}(\mathbf{x}) = \frac{1}{2\pi}|\mathbf{x}|^{-2}(-x^2, x^1), \mathbf{x} = (x^1, x^2)$, and ξ is the scalar vorticity.

Equation (1) is supplemented by the initial condition,

$$\xi(\mathbf{x},0) = \xi_0(\mathbf{x}). \tag{2}$$

Suppose that we know that $|\mathbf{u}(\mathbf{x},t)| \le C$ for $(\mathbf{x},t) \in \mathbb{R}^2 \times \bar{\mathbb{R}}_+$. It then follows form (1) that,

$$\xi_t - \nu\Delta\xi \le C|\nabla\xi|. \tag{3}$$

Inequality (3) motivates our study of the following scalar equation, in any space dimension,

$$u_t = \Delta u + \mu|\nabla u| \text{ in } \mathbb{R}^n \times \bar{\mathbb{R}}_+, \tag{4}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n. \tag{5}$$

In (4) we are assuming that $\mu \neq 0$ is a real constant.

The precise results concerning existence of global solutions of (1) will be stated elsewhere. However, the method of proof is similar to that used in the proof of the following theorem.

THEOREM 1. *Let $u_0(\mathbf{x}) \in C_0^3(\mathbb{R}^n)$(= three times continuously differentiable functions with compact support). Then there exists a unique classical solution to (4)-(5), in all of $\mathbb{R}^n \times \bar{\mathbb{R}}_+$. Furthermore, this solution satisfies the maximum - minimum principle,*

$$\sup_{\mathbb{R}^n\times[0,\infty)} u(\mathbf{x},t) = \sup_{\mathbb{R}^n} u_0(\mathbf{x}), \quad \inf_{\mathbb{R}^n\times[0,\infty)} u(\mathbf{x},t) = \inf_{\mathbb{R}^n} u_0(\mathbf{x}). \tag{6}$$

PROOF: A detailed proof may be found in [2]. We outline here the main ingredients of the proof.

Let $G(\mathbf{x},t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ be the heat kernel. We set $u^{(-1)} \equiv 0$ and solve iteratively,

$$u_t^{(k)} - \Delta u^{(k)} = \mu |\nabla u^{(k-1)}|, \quad k = 0, 1, 2, \dots,$$

where $u^{(k)}(\mathbf{x}, 0) = u_0(\mathbf{x})$. Thus,

$$\begin{aligned} u^{(k)}(\mathbf{x},t) = &\int_{\mathbb{R}^n} G(\mathbf{x}-\mathbf{y},t) u_0(\mathbf{y}) d\mathbf{y} \\ &+ \mu \int_{\mathbb{R}^n} \int_0^t G(\mathbf{x}-\mathbf{y}, t-s) |\nabla u^{(k-1)}(\mathbf{y},s)| d\mathbf{y} ds. \end{aligned} \tag{7}$$

Differentiating (7) with respect to $\mathbf{x}$ yields,

$$\begin{aligned} \nabla u^{(k)}(\mathbf{x},t) = &\int_{\mathbb{R}^n} G(\mathbf{x}-\mathbf{y},t) \nabla u_0(\mathbf{y}) d\mathbf{y} \\ &+ \mu \int_{\mathbb{R}^n} \int_0^t \nabla_{\mathbf{x}} G(\mathbf{x}-\mathbf{y}, t-s) |\nabla u^{(k-1)}(\mathbf{y},s)| d\mathbf{y} ds. \end{aligned} \tag{8}$$

Set,

$$A_k(t) = \sup\{|\nabla u^{(k)}(\mathbf{x},s)| / \mathbf{x} \in \mathbb{R}^n, 0 \le s \le t\},$$

then it follows from (8) and the properties of G that, with some constant C,

$$A_k(t) \le A_0(t) + C \int_0^t (t-s)^{-\frac{1}{2}} A_{k-1}(s) ds,$$

which implies that, fixing $T > 0$, there exists a constant L, depending only on C, T, such that, for $k = 1, 2, \dots,$

$$A_k(t) \le 2A_0(T) e^{Lt}, \quad 0 \le t \le T. \tag{9}$$

Thus, the sequence $\{u^{(k)}(\mathbf{x},t)\}$, which consists of twice continuously differentiable functions which decay as $|\mathbf{x}| \to \infty$, is seen to possess uniformly bounded gradients (with respect to $\mathbf{x}$) in every strip of the form $\mathbb{R}^n \times [0,T]$. It follows from (7) that the sequence itself is uniformly bounded in such strips.

Differentiating (8) once more it follows similarly that the sequence $\{\nabla^2_{\mathbf{x}} u^{(k)}(\mathbf{x},t)\}$ is also uniformly bounded in strips, and it can be further shown that it is uniformly Hölder continuous.

Finally, using ideas similar to the above, one shows that for every $T > 0$,

$$\lim_{k\to\infty} \sup\{|u^{(k)}(\mathbf{x},t) - u^{(k-1)}(\mathbf{x},t)|/(\mathbf{x},t) \in \mathbb{R}^n \times [0,T]\} = 0.$$

Using the Ascoli-Arzela theorem it follows that one can extract uniformly convergent (in every strip) subsequences $\{\Delta u^{(k_j)}(\mathbf{x},t)\}$ and $\{\nabla u^{(k_j-1)}(\mathbf{x},t)\}$, hence also $\{u_t^{(k_j)}(\mathbf{x},t)\}$. This implies that $\{u^{(k_j)}(\mathbf{x},t)\}$ converges uniformly to $u(\mathbf{x},t)$, which is a classical solution of (4)-(5).

Q.E.D.

Observe that the initial function $u_0(\mathbf{x})$ was assumed to be in $C_0^3(\mathbb{R}^n)$. However, the following theorem shows that the (nonlinear) solution operator can be extended to initial values in $L^2(\mathbb{R}^n)$.

THEOREM 2. *For $u_0(\mathbf{x}) \in C_0^3(\mathbb{R}^n)$ let $S_t u_0 = u(\mathbf{x},t), 0 \le t < \infty$, be the solution given by Theorem 1. Then S_t can be extended continuously to $L^2(\mathbb{R}^n)$.*

PROOF: Let $u(\mathbf{x},t), v(\mathbf{x},t)$ be two solutions of (4) such that $u(\mathbf{x},0) = u_0(\mathbf{x}), v(\mathbf{x},0) = v_0(\mathbf{x})$. Then,

$$(u-v)_t = \Delta(u-v) + \mu(|\nabla u| - |\nabla v|).$$

Multiplication by $u - v$ and integration over $\mathbb{R}^n$ yields, for all $\varepsilon > 0$,

$$\frac{d}{dt}\int_{\mathbb{R}^n} (u-v)^2 d\mathbf{x} \le -\int_{\mathbb{R}^n} |\nabla(u-v)|^2 d\mathbf{x} + \varepsilon|\mu| \int_{\mathbb{R}^n} |\nabla(u-v)|^2 d\mathbf{x} + \frac{|\mu|}{\varepsilon}\int_{\mathbb{R}^n} (u-v)^2 d\mathbf{x},$$

from which we infer, in view of Grønwall's inequality,

$$\|S_t u_0 - S_t v_0\|_{L^2(\mathbb{R}^n)} \le e^{2\mu^2 t}\|u_0 - v_0\|_{L^2(\mathbb{R}^n)}.$$

Q.E.D.

It is not clear to us whether or not S_t extends continuously also to $L^1(\mathbb{R}^n)$. For various reasons, the space $L^1(\mathbb{R}^n)$ is more natural, and indeed more interesting, than $L^2(\mathbb{R}^n)$ in our case. In particular, the case where $\mu > 0$ is of special interest, and constitutes a continuous analogue of a discrete "averaging and penalization" model [5]. In this case ($\mu < 0$) the effect of Δu in the right-hand side of (4) is to "diffuse" the profile of u, however without changing its L^1 - norm. On the other hand, the negative term $\mu|\nabla|$ serves to diminish the size of u pointwise, especially in places with large $|\nabla u|$, i.e., where the diffusive effect of the Laplacian has not yet taken over completely. In conclusion, Eq. (4) in the case of $\mu < 0$ represents a "competition" between a diffusive operator, preserving the L^1-norm and a damping operation, effective especially in large-gradient locations. For the case of smooth nonnegative initial data we have the following result.

THEOREM 3. *Assume* $0 \leq \mathbf{u}_0(\mathbf{x}) \in C_0^4(\mathbb{R}^n)$ *and* $\mu < 0$. *Then* $u(\mathbf{x}, \mathbf{t})$, *the solution to (4)-(5), decays as* $t \to +\infty$ *in the following sense.*

There exists a constant A_n, *depending only on* n, *such that*

$$\sup_{0 \leq t < \infty} t^{\frac{|\mu|}{1+|\mu|} A_n} \int_{\mathbb{R}^n} u(\mathbf{x}, t) d\mathbf{x} < \infty.$$

PROOF: The detailed proof can be found in [2]. We give here a very brief outline.

(a) Let $u_0(\mathbf{x}) = 0$ for $|\mathbf{x}| > R$, and set, $M_j = \sup_{\mathbb{R}^n} |\frac{\partial u_0}{\partial x_j}|, j = 1, \ldots, n$. Using the comparison principle for parabolic equations one shows that, with some constant C depending only on n (and not on μ, R etc.),

$$|u(\mathbf{x}, t)| \leq \min\{\bar{M}, \bar{M} n \exp((C + |\mu| n^{1/2} + 1)t + R - |\mathbf{x}|)\}, \tag{9}$$

where $\bar{M} = \max\{M_1, \ldots, M_n, \|u_0\|_{L^\infty(\mathbb{R}^n)}\}$.

(b) Denoting by A_n a generic constant depending only on n, the estimate (a) yields,

$$\int_{|\bar{x}| \geq 4 a_n t} |u(\mathbf{x}, t)| d\mathbf{x} \leq \bar{M} A_n e^{-t}, \tag{10}$$

where $a_n = C + |\mu| n^{1/2} + 1$.

(c) Using the Sobolev imbedding theorem and the Hölder inequality one gets,

$$\int\limits_{|\bar{x}|\leq 4a_n t} |u(\mathbf{x},t)|\,d\mathbf{x} \leq tA_n(1+|\mu|)\int\limits_{\mathbb{R}^n} |\nabla u(\mathbf{x},t)|d\mathbf{x}, \tag{11}$$

so that in conjunction with (10),

$$\int\limits_{\mathbb{R}^n} |\nabla u(\mathbf{x},t)|d\mathbf{x} \geq \frac{A_n}{(1+|\mu|)t}[\int\limits_{\mathbb{R}^n} u(\mathbf{x},t)d\mathbf{x} - \bar{M}e^{-t}]. \tag{12}$$

(d) Setting $Q(t) = \int\limits_{\mathbb{R}^n} u(\mathbf{x},t)d\mathbf{x}$ and using Eq. (4) and (12) we finally get

$$Q'(t) \leq \frac{\mu}{1+|\mu|}\cdot\frac{A_n}{t}[Q(t) - \bar{M}e^{-t}],$$

from which the assertion follows.

Q.E.D.

Note that in the proof of Theorem 3 we have made substantial use of the positivity of the solution $u(\mathbf{x},\mathbf{t})$. It is not clear to us what are the decay properties of the solution (if any) for nonsmooth or nonpositive initial data.

Finally, we mention the papers [1,3] for blow-up results concerning closely related nonlinear parabolic equations.

References

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