Journées

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Évian, 4 juin–8 juin 2007

Armen Shirikyan

Controllability of nonlinear PDE's: Agrachev–Sarychev approach

J. É. D. P. (2007), Exposé n° IV, 11 p.

<http://jedp.cedram.org/item?id=JEDP_2007____A4_0>

Journées *Équations aux dérivées partielles*
Évian, 4 juin–8 juin 2007
GDR 2434 (CNRS)

Controllability of nonlinear PDE's: Agrachev–Sarychev approach

Armen Shirikyan

Abstract

This short note is devoted to a discussion of a general approach to controllability of PDE's introduced by Agrachev and Sarychev in 2005. We use the example of a 1D Burgers equation to illustrate the main ideas. It is proved that the problem in question is controllable in an appropriate sense by a two-dimensional external force. This result is not new and was proved earlier in the papers [AS05, AS07] in a more complicated situation of 2D Navier–Stokes equations.

Contents

1. Introduction

In the paper [AS05], Agrachev and Sarychev introduced a new approach for investigating the controllability of nonlinear PDE's. They studied the 2D Navier–Stokes

MSC 2000: 35Q35, 93B05, 93C20.
Keywords: Burgers equation, approximate controllability, exact controllability in projection, Agrachev–Sarychev method.

equations on a torus controlled by a finite-dimensional external force and proved the properties of approximate controllability and exact controllability in observed projections. These results were later extended to the Euler and Navier–Stokes systems on various 2D and 3D manifolds; see [AS06, Rod06, Shi06, Rod07, AS07, Shi07].

The aim of this paper is to illustrate the Agrachev–Sarychev approach on the simple example of the 1D viscous Burgers equation. We thus consider the problem

$$\partial_t u - \nu\partial_x^2 u + u\partial_x u = h(t,x) + \eta(t,x), \tag{1.1}$$

$$u(0,t) = u(\pi,t) = 0, \tag{1.2}$$

$$u(0,x) = u_0(x), \tag{1.3}$$

where $x \in (0,\pi)$, $t > 0$, $\nu > 0$ is a parameter, h and u_0 are given functions, and η is a control with range in a finite-dimensional space. We wish to study controllability properties of problem (1.1), (1.2).

To introduce the necessary concepts and formulate the main result, let us fix a constant $T > 0$, a function $h \in L^2(Q_T)$, where $Q_T = (0,T)\times(0,\pi)$, and a finite-dimensional space $E \subset L^2(0,\pi)$. To simplify the notation, we shall write

$$H = L^2(0,\pi), \quad H_0^1 = H_0^1(0,\pi), \quad \mathcal{X}_T = C(0,T;H)\cap L^2(0,T;H_0^1);$$

see Notation for more details. Let us denote by $\mathcal{R} : H \times L^2(0,T;E) \to \mathcal{X}_T$ the operator that takes a pair (u_0,η) to the solution $u \in \mathcal{X}_T$ of (1.1) – (1.3) and by $\mathcal{R}_t : H \times L^2(0,T;E) \to H$ its restriction at time $t \in [0,T]$. It is well known that the operators $\mathcal{R}$ and $\mathcal{R}_t$ are uniformly Lipschitz continuous on bounded subsets of their domain of definition; see [Lio69, Tay97].

Definition 1.1. We shall say that problem (1.1), (1.2) is *controllable at time T by an E-valued control* if for any constant $\varepsilon > 0$, any functions $u_0, \hat{u} \in H$, and any finite-dimensional subspace $F \subset H$ there is a control $\eta \in C^\infty(0,T;E)$ such that

$$\|\mathcal{R}_T(u_0,\eta) - \hat{u}\| < \varepsilon, \tag{1.4}$$

$$\mathsf{P}_F\mathcal{R}_T(u_0,\eta) = \mathsf{P}_F\hat{u}, \tag{1.5}$$

where $\|\cdot\|$ denotes the L^2 norm, and $\mathsf{P}_F : H \to H$ stands for the orthogonal projection in H onto F.

We shall prove the following result:

Main Theorem. *Let E be the vector space spanned by the function $\sin x$ and $\sin(2x)$. Then for any $\nu > 0$ and $T > 0$ problem (1.1), (1.2) is controllable at time T by an E-valued control.*

The rest of the paper is organised as follows. In Section 2, we show that the controllability in the sense of Definition 1.1 is a consequence of the so-called *uniform approximate controllability*. We then outline the proof of the latter property. In Section 3, we give the details of the proof.

In conclusion, let us emphasise once again that this paper contains no new results, and the Main Theorem stated above can be regarded as a simple particular case of more general results established in [AS05, AS07].

Notation

Let $J \subset \mathbb{R}$ be an open finite interval and let X be a Banach space. We use the following functional spaces.

$C(J;X)$ denotes the space of continuous functions $f : \bar{J} \to X$, where $\bar{J}$ is the closure of J. This space is endowed with the norm $\sup_{t\in J} \|f(t)\|_X$.

$L^2(J;X)$ stands for the space of Bochner-measurable Â functions $f : J \to X$ such that

$$\|f\|_{L^2(J;X)} = \left(\int_J \|f(t)\|_X^2 dt\right)^{1/2} < \infty.$$

In the case $X = \mathbb{R}$, we write simply $L^2(J)$ and $\|f\|$.

$H^k = H^k(J)$ is the Sobolev space of order k on the interval J.

$H_0^1 = H_0^1(J)$ denotes the space of scalar functions that belong to the Sobolev class H^1 and vanish at the endpoints of J.

2. Proof of the Main Theorem

2.1. Reduction to uniform approximate controllability

Let us fix a constant $T > 0$, a function $h \in L^2(Q_T)$, and a finite-dimensional subspace $E \subset H = L^2(0,\pi)$. Â Recall that we denote by $\mathcal{R}_t : H \times L^2(0,T;E) \to H$ the resolving operator for problem (1.1) – (1.3).

Definition 2.1. Let us fix a constant $\varepsilon > 0$, a function $u_0 \in H$, and a compact set $\mathcal{K} \subset H$. Problem (1.1), (1.2) is said to be $(\varepsilon, u_0, \mathcal{K})$*-controllable at time T by an E-valued control* if there is a continuous mapping $\Psi : \mathcal{K} \to L^2(0,T;E)$ such that

$$\sup_{u\in\mathcal{K}} \|\mathcal{R}_T(u_0, \Psi(u)) - u\| < \varepsilon. \tag{2.1}$$

In what follows, the time T and the control space E are fixed, and we shall simply say that problem (1.1), (1.2) is $(\varepsilon, u_0, \mathcal{K})$*-controllable.*

Definition 2.2. Problem (1.1), (1.2) is said to be *uniformly approximately controllable* if it is $(\varepsilon, u_0, \mathcal{K})$-controllable for any $\varepsilon > 0$, $u_0 \in H$, and $\mathcal{K} \Subset H$.

The Main Theorem stated in the Introduction will be deduced from the following result.

Theorem 2.3. *Let E be the vector span of the functions $\sin x$ and $\sin(2x)$. Then for any $\nu > 0$ and $h \in L^2(Q_T)$ problem* (1.1), (1.2) *is uniformly approximately controllable by an E-valued control.*

The proof of this result is sketched in Subsection 2.2, and the details are given in Section 3. We now prove the Main Theorem.

Proof of the Main Theorem. Let us fix a constant $\varepsilon > 0$, functions $u_0, \hat{u} \in H$, and a finite-dimensional space $F \subset H$. Without loss of generality, we can assume that $\hat{u} \in F$; otherwise, we can replace F by the larger space spanned by F and $\hat{u}$.

Let us denote by $B_F(R)$ the ball in F of radius R centred at origin and define $\mathcal{K} = B_F(\|\hat{u}\| + \varepsilon)$. Since $\mathcal{K}$ is a compact subset of H, in view of Theorem 2.3, we can construct a continuous mapping $\Psi : \mathcal{K} \to L^2(0,T;E)$ satisfying inequality (2.1).

Furthermore, since $\mathcal{K} \subset H$ is compact and $C^\infty(0,T;E)$ is dense in $L^2(0,T;E)$, we can assume that the range of Ψ is contained in $C^\infty(0,T;E)$; otherwise, we can replace the function Ψ by its convolution with a mollifying kernel. Let us consider the mapping

$$\Phi : \mathcal{K} \to F, \quad \Phi(u) = \mathsf{P}_F \mathcal{R}_T(u_0, \Psi(u)).$$

It follows from (2.1) that Φ is a continuous mapping satisfying the inequality

$$\sup_{u \in \mathcal{K}} \|\Phi(u) - u\| < \varepsilon.$$

The Brouwer theorem (e.g., see [Tay97]) implies that the image of Φ contains the ball $B_F(\|\hat{u}\|)$. In particular, there is $\bar{u} \in \mathcal{K}$ such that $\Phi(\bar{u}) = \hat{u}$. Setting $\eta = \Psi(\bar{u})$, we see that

$$\mathsf{P}_F \mathcal{R}_T(u_0, \eta) = \hat{u}. \tag{2.2}$$

Furthermore, it follows from (2.1) and (2.2) that

$$\|\mathcal{R}_T(u_0,\eta) - \hat{u}\| = \|\mathcal{R}_T(u_0,\eta) - \mathsf{P}_F \mathcal{R}_T(u_0,\eta)\| \le \|\mathcal{R}_T(u_0,\Psi(\bar{u})) - \bar{u}\| < \varepsilon,$$

where we used the facts that $\bar{u} \in F$ and that P_F is an orthogonal projection. This completes the proof of the Main Theorem. □

2.2. Scheme of the proof of Theorem 2.3

Let us fix a constant $\varepsilon > 0$, a function $u_0 \in H$, and a compact set $\mathcal{K} \Subset H$. We need to show that problem (1.1), (1.2) is $(\varepsilon, u_0, \mathcal{K})$-controllable by an E-valued control.

Step 1: Extension principle. Let $G \subset H^2 \cap H_0^1$ be an arbitrary finite-dimensional subspace. Along with (1.1), consider the equation

$$\partial_t u - \nu \partial_x^2 (u + \zeta(t,x)) + (u + \zeta(t,x)) \partial_x (u + \zeta(t,x)) = h(t,x) + \eta(t,x), \tag{2.3}$$

where η and ζ are G-valued control functions. We shall say that problem (2.3), (1.2) is $(\varepsilon, u_0, \mathcal{K})$-controllable by G-valued controls (η, ζ) if there is a continuous mapping $\widehat{\Psi} : \mathcal{K} \to L^2(0,T; G \times G)$ such that

$$\sup_{u \in \mathcal{K}} \|\widehat{\mathcal{R}}_T(u_0, \widehat{\Psi}(u)) - u\| < \varepsilon, \tag{2.4}$$

where $\widehat{\mathcal{R}}_t : H \times L^2(0,T; G \times G) \to H$ stands for the operator that takes the triple (u_0, η, ζ) to the solution $u(t,\cdot)$ of problem (2.3), (1.2), (1.3).

Even though Eq. (2.3) is "more controlled" than Eq. (1.1), it turns out that the property of uniform approximate controllability is equivalent for them. Namely, we have the following result.

Proposition 2.4. *Problem* (1.1), (1.2) *is* $\hat{A}$ $(\varepsilon, u_0, \mathcal{K})$*-controllable if and only if so is problem* (2.3), (1.2).

Step 2: Convexification principle. Now let $N \subset H^2 \cap H_0^1$ be another finite-dimensional subspace such that

$$N \subset G, \quad B(N) \subset G, \tag{2.5}$$

where $B(u) = u \partial_x u$. Denote by $\mathcal{F}(N,G) \subset H^2 \cap H_0^1$ the largest vector space spanned by the functions of the form[1]

$$\eta + \xi \partial_x \tilde{\xi} + \tilde{\xi} \partial_x \xi, \tag{2.6}$$

where $\eta, \xi \in G$ and $\tilde{\xi} \in N$. It is easy to see that $\mathcal{F}(N,G)$ is a well-defined finite-dimensional space. The following proposition, which is an infinite-dimensional analogue of the well-known convexification principle for controlled ODE's (e.g., see [AS04, Theorem 8.7]), is a key point of the proof of Theorem 2.3.

Proposition 2.5. *Let $N, G \subset H^2 \cap H_0^1$ be finite-dimensional subspaces satisfying* (2.5)*. Then* (2.3), (1.2) *is $(\varepsilon, u_0, \mathcal{K})$-controllable by a $G \times G$-valued control if and only if* (1.1), (1.2) *is $(\varepsilon, u_0, \mathcal{K})$-controllable by an $\mathcal{F}(N,G)$-valued control.*

Step 3: Saturating property. Propositions 2.4 and 2.5 imply the following result, which is a kind of "relaxation property" for Â the controlled Navier–Stokes system.

Proposition 2.6. *Let $N, G \subset H^2 \cap H_0^1$ be finite-dimensional subspaces satisfying* (2.5)*. Then problem* (1.1), (1.2) *is $(\varepsilon, u_0, \mathcal{K})$-controllable by a G-valued control if and only if it is $(\varepsilon, u_0, \mathcal{K})$-controllable by an $\mathcal{F}(N,G)$-valued control.*

We now introduce the subspaces $E_k = \{\sin(jx), 1 \leq j \leq k\}$, so that the space E defined in the Main Theorem coincides with E_2. We wish to apply Proposition 2.6 to the subspaces $N = E_1$ and $G = E_k$.

Lemma 2.7. *For any integer $k \geq 2$, we have $\mathcal{F}(E_1, E_k) = E_{k+1}$.*

Proposition 2.6 and Lemma 2.7 imply that, for any integer $k \geq 2$, problem (1.1), (1.2) is $(\varepsilon, u_0, \mathcal{K})$-controllable by an E_k-valued control if and only if it is $(\varepsilon, u_0, \mathcal{K})$-controllable by an E_{k+1}-valued control. Thus, Theorem 2.3 will be established if we find an integer $N \geq 2$ such that problem (1.1), (1.2) is $(\varepsilon, u_0, \mathcal{K})$-controllable by an E_N-valued control. We shall be able to do that due to the *saturating property*

$$\bigcup_{k=2}^{\infty} E_k \text{ is dense in } H, \tag{2.7}$$

which is a straightforward consequence of the definition of E_k.

Step 4: Case of a large control space. It is easy to construct a continuous mapping $\Psi_0 : \mathcal{K} \to L^2(0,T;H)$ such that

$$\sup_{u \in \mathcal{K}} \|\mathcal{R}_T(u_0, \Psi_0(u)) - u\| < \varepsilon. \tag{2.8}$$

Since $\mathcal{K} \subset H$ is a compact set, the image $\Psi_0(\mathcal{K})$ is compact in $L^2(0,T;H)$. Using (2.7), it is not difficult to approximate Ψ_0, within any accuracy $\delta > 0$, by a continuous function $\Psi : \mathcal{K} \to L^2(0,T;H)$ with range in $L^2(0,T;E_N)$:

$$\sup_{u \in \mathcal{K}} \|\Psi_0(u) - \Psi(u)\| < \delta. \tag{2.9}$$

Since $\mathcal{R}_t(u_0, \eta)$ is Lipschitz continuous on bounded subsets, inequalities (2.8) and (2.9) with $\delta \ll 1$ imply (2.1). This completes the proof of Theorem 2.3.

[1]Note that a function of the form (2.6) does not necessarily belong to $H^2 \cap H_0^1$, and therefore the space $\mathcal{F}(N,G)$ may be not larger than G.

3. Approximate controllability

In this section, we prove Theorem 2.3. To simplify the presentation, we shall assume that $\mathcal{K}$ consists of a single point $\hat{u} \in H$. The proof in the general case can be carried out by similar arguments, following carefully the dependence of all the objects on the final point $\hat{u}$; cf. [Shi07]. In what follows, the constant ε, the functions u_0, and the subset $\mathcal{K}$ are fixed, and we shall say simply *ε-controllable* rather than *$(\varepsilon, u_0, \mathcal{K})$-controllable*.

3.1. Extension principle

In this subsection, we prove Proposition 2.4. It is clear that if problem (1.1), (1.2) is ε-controllable, then so is problem (2.3), (1.2), because it suffices to take $\zeta \equiv 0$. Let us establish the converse assertion.

Let $(\hat{\eta}, \hat{\zeta}) \in L^2(0, T; G)$ be an arbitrary control such that

$$\|\widehat{\mathcal{R}}_T(u_0, \hat{\eta}, \hat{\zeta}) - \hat{u}\| < \varepsilon. \tag{3.1}$$

In view of continuity of $\widehat{\mathcal{R}}_T(u_0, \eta, \zeta)$ with respect to $\zeta \in L^2(0, T; H)$, there is no loss of generality in assuming that

$$\hat{\zeta} \in C^\infty(0, T; G), \quad \hat{\zeta}(0) = \hat{\zeta}(T) = 0. \tag{3.2}$$

Consider the function $u(t, x) = \widehat{\mathcal{R}}_t(u_0, \hat{\eta}, \hat{\zeta}) + \hat{\zeta}(t, x)$. It is straightforward to see that it belongs to $\mathcal{X}_T$ and satisfies Eqs. (1.1), (1.2) with $\eta = \hat{\eta} + \partial_t \hat{\zeta} \in L^2(0, T; G)$. Moreover, it follows from (3.1) and (3.2) that

$$u(0) = u_0, \quad \|u(T) - \hat{u}\| = \|\widehat{\mathcal{R}}_T(u_0, \hat{\eta}, \hat{\zeta}) - \hat{u}\| < \varepsilon.\hat{\text{A}}$$

Thus, problem (1.1), (1.2) is ε-controllable.

3.2. Convexification principle

Let us prove Proposition 2.5. It follows from the extension principle that if problem (2.3), (1.2) is ε-controllable by a $G \times G$-valued control, then (1.1), (1.2) is ε-controllable by a G-valued control and all the more by an $\mathcal{F}(N, G)$-valued control. The proof of the converse assertion is divided into several steps. We need to show that if $\eta_1 \in L^2(0, T; H)$ is an $\mathcal{F}(N, G)$-valued control such that

$$\|\mathcal{R}_T(u_0, \eta_1) - \hat{u}\| < \varepsilon, \tag{3.3}$$

then there are $\eta, \zeta \in L^2(0, T; G)$ such that

$$\|\widehat{\mathcal{R}}_T(u_0, \eta, \zeta) - \hat{u}\| < \varepsilon. \tag{3.4}$$

Step 1. We first show that it suffices to consider the case in which η_1 is a piecewise constant function. Indeed, suppose Proposition 2.5 is proved in that case and denote $G_1 = \mathcal{F}(N, G)$. For a given $\eta_1 \in L^2(0, T; G_1)$, we can find a sequence $\{\eta^m\}$ of piecewise constant G_1-valued functions such that

$$\|\eta_1 - \eta^m\|_{L^2(0,T;G_1)} \to 0 \quad \text{as } m \to \infty.$$

By continuity of $\mathcal{R}_t$, there is an integer $n \geq 1$ such that

$$\|\mathcal{R}_T(u_0, \eta^n) - \hat{u}\| < \varepsilon. \tag{3.5}$$

Since the result is true for piecewise constant controls, for any $\delta > 0$ there are $\eta, \zeta \in L^2(0,T;G)$ such that

$$\|\mathcal{R}_T(u_0,\eta^n) - \widehat{\mathcal{R}}_T(u_0,\eta,\zeta)\| < \delta. \tag{3.6}$$

Comparing (3.5) and (3.6), for a sufficiently small $\delta > 0$ we arrive at (3.4).

Step 2. We now consider the case of piecewise constant G_1-valued controls. A simple iteration argument combined with the continuity of $\mathcal{R}_t$ and $\widehat{\mathcal{R}}_t$ shows that it suffices to consider the case of one interval of constancy. Thus, we shall assume that $\eta_1(t) \equiv \eta_1 \in G_1$.

We shall need the lemma below, whose proof is given at the end of this subsection. Recall that $B(u) = u\partial_x u$.

Lemma 3.1. *For any $\eta_1 \in \mathcal{F}(N,G)$ and any $\delta > 0$ there is an integer $k \geq 1$, constants $\alpha_j > 0$, and vectors $\eta, \zeta^j \in G$, $j = 1, \dots, k$, such that*

$$\sum_{j=1}^k \alpha_j = 1, \tag{3.7}$$

$$\Big\|\eta_1 - B(u) - \Big(\eta - \sum_{j=1}^k \alpha_j \Big(B(u+\zeta^j) - \nu\partial_x^2\zeta^j\Big)\Big)\Big\| \leq \delta \quad \textit{for any } u \in H^1. \tag{3.8}$$

We fix a small $\delta > 0$ and choose constants $\alpha_j > 0$ and vectors $\eta, \zeta^j \in G$ satisfying (3.7), (3.8). Let us consider the equation

$$\partial_t u - \nu\partial_x^2 u + \sum_{j=1}^k \alpha_j \Big(B(u+\zeta^j(x)) - \nu\partial_x^2\zeta^j(x)\Big) = h(t,x) + \eta(x). \tag{3.9}$$

This is a Burgers-type equation, and using the same arguments as in the case of the Burgers equation, it can be proved that problem (3.9), (1.2), (1.3) has a unique solution $\tilde{u} \in \mathcal{X}_T$. On the other hand, we can rewrite (3.9) in the form

$$\partial_t u - \nu\partial_x^2 u + u\partial_x u = h(t,x) + \eta_1(x) - r_\delta(t,x), \tag{3.10}$$

where $r_\delta(t,x)$ stands for the function under sign of norm on the left-hand side of (3.8) in which $u = \tilde{u}(t,x)$. Since $\mathcal{R}_t$ is Lipschitz continuous on bounded subsets, there is a constant $C > 0$ depending only on the L^2 norm of η_1 such that

$$\begin{aligned}\|\mathcal{R}_T(u_0,\eta_1) - \tilde{u}(T)\| &= \|\mathcal{R}_T(u_0,\eta_1) - \mathcal{R}_T(u_0,\eta_1 - r_\delta)\| \\ &\leq C\|r_\delta\|_{L^2(0,T;H)} \leq C\sqrt{T}\delta,\end{aligned}$$

where we used inequality (3.8). Combining this with (3.3), we see that if $\delta > 0$ is sufficiently small, then

$$\|\tilde{u}(T) - \hat{u}\| < \varepsilon. \tag{3.11}$$

We shall show that there is a sequence $\zeta_m \in L^2(0,T;G)$ such that

$$\|\widehat{\mathcal{R}}_T(u_0,\eta,\zeta_m) - \tilde{u}(T)\| \to 0 \quad \text{as } m \to \infty. \tag{3.12}$$

In this case, inequalities (3.11) and (3.12) with Â $m \gg 1$ will imply the required estimate (3.4) in which $\zeta = \zeta_m$.

Step 3. Following a classical idea in the control theory, we define a sequence $\zeta_m \in L^2(0,T;G)$ by the relation $\zeta_m(t) = \zeta(mt/T)$, where $\zeta(t)$ is a 1-periodic G-valued function such that

$$\zeta(t) = \zeta^j \quad \text{for } 0 \le t - (\alpha_1 + \cdots + \alpha_{j-1}) \le \alpha_j,\ j = 1, \dots, k.$$

Let us rewrite (3.9) in the form

$$\partial_t u - \nu \partial_x^2 (u + \zeta_m(t,x)) + B(u + \zeta_m(t,x)) = h(t,x) + \eta(x) + f_m(t,x),$$

where we set $f_m = f_{m1} + f_{m2}$,

$$f_{m1} = -\nu \partial_x^2 \zeta_m + \nu \sum_{j=1}^{k} \alpha_j \partial_x^2 \zeta^j, \tag{3.13}$$

$$f_{m2} = B(\tilde u + \zeta_m) - \sum_{j=1}^{k} \alpha_j B(\tilde u + \zeta^j). \tag{3.14}$$

We now define an operator $K : L^2(0,T;H) \to \mathcal{X}_T$ that takes a function f to the solution $u(t,x)$ of the equation

$$\partial_t u - \nu \partial_x^2 u = f(t,x),$$

supplemented with initial and boundary conditions (1.2), (1.3) with $u_0 = 0$. In other words,

$$(Kf)(t) = \int_0^t e^{\nu(t-s)A} f(s)\, ds,$$

where A stands for the operator $\frac{d^2}{dx^2}$ with the domain $\mathcal{D}(A) = H^2 \cap H_0^1$. Setting $v_m = \tilde u - K f_m$, we see that $v_m \in \mathcal{X}_T$ satisfies the equation

$$\partial_t v - \nu \partial_x^2 (v + \zeta_m) + B(v + \zeta_m + K f_m) = h + \eta. \tag{3.15}$$

Suppose we have shown that

$$\|K f_m\|_{\mathcal{X}_T} \to 0 \quad \text{as } m \to \infty. \tag{3.16}$$

Then, by the Lipschitz continuity of the resolving operator for (3.15) on bounded subsets, we have

$$\|\widehat{\mathcal{R}}_T(u_0, \eta, \zeta_m) - \tilde u(T)\| \le \|\widehat{\mathcal{R}}_T(u_0, \eta, \zeta_m) - v_m(T)\| + \|K f_m(T)\| \to 0$$

as $m \to \infty$. Thus, it remains to prove (3.16).

Step 4. We first note that $\{f_m\}$ is a bounded sequence in $L^2(0,T;H)$. It follows that

$$\|K f_m\|_{C(0,T;H^1)} + \|K f_m\|_{L^2(0,T;H^2)} \le C_1, \tag{3.17}$$

where we denote by C_i, $i = 1, 2, \dots$, positive constants not depending on m. Furthermore, we have the interpolation inequalities

$$\|v\| \le C_2 \|v\|_1^{1/2} \|v\|_{-1}^{1/2}, \quad \|v\|_1 \le C_3 \|v\|_2^{2/3} \|v\|_{-1}^{1/3} \quad \text{for } v \in H^2 \cap H_0^1.$$

Combining this with (3.17), we obtain

$$\begin{aligned} \|K f_m\|_{\mathcal{X}_T} &\le \|K f_m\|_{C(0,T;H)} + \|K f_m\|_{L^2(0,T;H^1)} \\ &\le C_4 \Big(\|K f_m\|_{C(0,T;H^{-1})}^{1/2} + \|K f_m\|_{L^2(0,T;H^{-1})}^{1/3} \Big). \end{aligned}$$

Thus, convergence (3.16) will be established if we show that

$$\|K f_m\|_{C(0,T;H^{-1})} \to 0 \quad \text{as } m \to \infty. \tag{3.18}$$

Step 5. To prove (3.18), we write

$$(Kf_m)(t) = F_m(t) + G_m(t), \tag{3.19}$$

where

$$F_m(t) = \int_0^t f_m(s)\,ds, \quad G_m(t) = \nu \int_0^t Ae^{\nu(t-s)A} F_m(s)\,ds.$$

Since $\|Ae^{\tau A}\|_{\mathcal{L}(H,H^{-1})} \le C_5 \tau^{-1/2}$ for $\tau > 0$, where $\|\cdot\|_{\mathcal{L}(H,H^{-1})}$ stands for the usual norm of operators from H to H^{-1}, we have

$$\begin{aligned}\|G_m\|_{C(0,T;H^{-1})} &\le \nu \sup_{t\in[0,T]} \int_0^t \|Ae^{\nu(t-s)A}\|_{\mathcal{L}(H,H^{-1})} \|F_m(s)\|\,ds \\ &\le C_6\,\|F_m\|_{C(0,T;H)}.\end{aligned}$$

Comparing this with (3.19), we see that (3.18) will be established if we show that

$$\|F_m\|_{C(0,T;H)} \to 0 \quad \text{as } m \to \infty. \tag{3.20}$$

This convergence is a straightforward consequence of relations (3.13) and (3.14); cf. [Shi06, Section 3.3]. The proof of Proposition 2.5 is complete.

Proof of Lemma 3.1. It suffices to find functions $\eta, \tilde\zeta^j \in G$, $j = 1, \dots, m$, such that

$$\Big\|\eta_1 - \eta + \sum_{j=1}^k B(\tilde\zeta^j)\Big\| \le \delta. \tag{3.21}$$

If such vectors are constructed, then we can set $k = 2m$,

$$\alpha_j = \alpha_{j+m} = \frac{1}{2}, \quad \zeta^j = -\zeta^{j+m} = \tilde\zeta^j \quad \text{for } j = 1, \dots, m.$$

To construct $\eta, \tilde\zeta^j \in G$ satisfying (3.21), note that if $\eta_1 \in \mathcal{F}(N,G)$, then there are functions $\tilde\eta_j, \xi_j \in G$ and $\tilde\xi_j \in N$ such that

$$\eta_1 = \sum_{j=1}^k \Big(\tilde\eta_j - \xi_j \partial_x \tilde\xi_j - \tilde\xi_j \partial_x \xi_j\Big). \tag{3.22}$$

Now note that, for any $\varepsilon > 0$,

$$\xi_j \partial_x \tilde\xi_j + \tilde\xi_j \partial_x \xi_j = B(\varepsilon\xi_j + \varepsilon^{-1}\tilde\xi_j) - \varepsilon^2 B(\xi_j) - \varepsilon^{-2} B(\tilde\xi_j).$$

Combining this with (3.22), we obtain

$$\eta_1 - \sum_{j=1}^k \Big(\tilde\eta_j + \varepsilon^{-2} B(\tilde\xi_j)\Big) + \sum_{j=1}^k B(\varepsilon\xi_j + \varepsilon^{-1}\tilde\xi_j) = \varepsilon^2 \sum_{j=1}^k B(\xi_j).$$

Choosing $\varepsilon > 0$ sufficiently small and setting

$$\eta = \sum_{j=1}^k \Big(\tilde\eta_j + \varepsilon^{-2} B(\tilde\xi_j)\Big), \quad \tilde\zeta^j = \varepsilon\xi_j + \varepsilon^{-1}\tilde\xi_j,$$

we arrive at (3.21). □

3.3. Saturating property

Let us prove Lemma 2.7 and the inclusion $B(E_1) \subset E_2$. For $\xi = \sin(jx)$ and $\tilde{\xi} = \sin x$, we have

$$\begin{aligned}\xi \partial_x \tilde{\xi} + \tilde{\xi} \partial_x \xi &= \sin(jx) \cos x + j \sin x \cos(jx) \\ &= \frac{1}{2}\Big((j+1)\sin(j+1)x - (j-1)\sin(j-1)x\Big). \qquad (3.23)\end{aligned}$$

It follows that $B(E_1) \subset E_2$ and $\mathcal{F}(E_1, E_k) \subset E_{k+1}$. Furthermore, taking $j = k$ in (3.23), we write

$$\sin(k+1)x = \frac{k-1}{k+1}\sin(k-1)x + \frac{2}{k+1}\Big(\sin(kx)\,\partial_x \sin x + \sin x\,\partial_x \sin(kx)\Big).$$

This relation implies that the function $\sin(k+1)x$ belongs to $\mathcal{F}(E_1, E_k)$ and therefore $E_{k+1} \subset \mathcal{F}(E_1, E_k)$.

3.4. Case of a large control space

We wish to construct a control $\eta \in L^2(0, T; E_N)$ with a large integer $N \geq 2$ such that

$$\|\mathcal{R}_T(u_0, \eta) - \hat{u}\| < \varepsilon. \qquad (3.24)$$

To this end, consider the function $u_\mu(t, x)$ defined as

$$u_\mu(t, x) = T^{-1}\Big(te^{\mu A}\hat{u} + (T - t)e^{\nu A}u_0\Big),$$

where A denotes the operator $\frac{d^2}{dx^2}$ with the domain $\mathcal{D}(A) = H^2 \cap H_0^1$, and $\mu > 0$ is a small constant that will be chosen later. The function u_μ belongs to the space $\mathcal{X}_T$ and satisfies Eqs. (1.1) – (1.3), in which

$$\eta = \eta_\mu := \partial_t u_\mu - \nu \partial_x^2 u_\mu + u_\mu \partial_x u_\mu - h.$$

This function belongs to $L^2(0, T; H)$. Furthermore,

$$\|u_\mu(T) - \hat{u}\| = \|e^{\mu A}\hat{u} - \hat{u}\| \to 0 \quad \text{as } \mu \to 0. \qquad (3.25)$$

Choosing $\mu > 0$ sufficiently small in (3.25) and approaching $\eta_\mu \in L^2(0, T; H)$ by continuous H-valued functions, we can find $\tilde{\eta} \in C(0, T; H)$ such that

$$\|\mathcal{R}_T(u_0, \tilde{\eta}) - \hat{u}\| < \varepsilon. \qquad (3.26)$$

Let us denote by $\mathsf{P}_k : H \to H$ the orthogonal projection in H onto the subspace E_k. In view of the saturating property (2.7), we have

$$\sup_{t \in [0,T]} \|\mathsf{P}_k \tilde{\eta}(t) - \tilde{\eta}(t)\| \to 0 \quad \text{as } k \to \infty.$$

By continuity of $\mathcal{R}_t$, we obtain

$$\|\mathcal{R}_T(u_0, \mathsf{P}_k\tilde{\eta}) - \mathcal{R}_T(u_0, \tilde{\eta})\| \to 0 \quad \text{as } k \to \infty.$$

Combining this with (3.26), we see that for a sufficiently large $N \geq 1$ the function $\eta = \mathsf{P}_N \tilde{\eta}$ satisfies (3.24). This completes the proof of Theorem 2.3 in the case $\mathcal{K} = \{\hat{u}\}$.

References

[AS04] A. A. Agrachev and Yu. L. Sachkov, *Control Theory from Geometric Viewpoint*, Springer-Verlag, Berlin, 2004.

[AS05] A. A. Agrachev and A. V. Sarychev, *Navier–Stokes equations: controllability by means of low modes forcing*, J. Math. Fluid Mech. **7** (2005), 108–152.

[AS06] ______, *Controllability of 2D Euler and Navier–Stokes equations by degenerate forcing*, Commun. Math. Phys. **265** (2006), no. 3, 673–697.

[AS07] ______, *Solid controllability in fluid dynamics*, Instabilities in Models Connected with Fluid Flow. I (C. Bardos and A. Fursikov, eds.), Springer, 2007, pp. 1–35.

[Lio69] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[Rod06] S. S. Rodrigues, *Navier-Stokes equation on the rectangle: controllability by means of low mode forcing*, J. Dyn. Control Syst. **12** (2006), no. 4, 517–562.

[Rod07] ______, *Controllability of nonlinear PDE's on compact Riemannian manifolds*, Workshop on Mathematical Control Theory and Finance, vol. Lisbon, 10–14 April, 2007, pp. 462–493.

[Shi06] A. Shirikyan, *Approximate controllability of three-dimensional Navier–Stokes equations*, Commun. Math. Phys. **266** (2006), no. 1, 123–151.

[Shi07] ______, *Exact controllability in projections for three-dimensional Navier-Stokes equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **24** (2007), no. 4, 521–537.

[Tay97] M. E. Taylor, *Partial Differential Equations. I–III*, Springer-Verlag, New York, 1996-1997.

CNRS (UMR 8088), Département de Mathématiques Université de Cergy–Pontoise, Site de Saint-Martin, 2 avenue Adolphe Chauvin 95302 Cergy–Pontoise Cedex, France

Armen.Shirikyan@u-cergy.fr