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Global solution of the wave equation

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Let M be a C^∞ Riemannian manifold, dim M = n. In the present paper we consider the Cauchy problem for the wave equation

$$\begin{cases} (\frac{1}{i}\partial_t + \sqrt{-\Delta})\, u(t,x,y) = 0 \\ u(0,x,y) = \delta(x-y), \qquad x,y \in M, t \in [0,\infty[\end{cases}$$

where Δ denotes the Laplace-Beltrami operator on M. Let $\{g^{ij}\}$ be components of the metric tensor g on M, $h(x,\xi) = (\Sigma g^{ij}\xi_i\xi_j)^{1/2}$ be a smooth function on $T^*M\setminus 0$.

Let us introduce a smooth homogeneous canonical transformation G^t defined by the Hamiltonian flow generated by the Hamiltonian system of equations

(1) $\frac{d}{dt}\, x = h_\xi$, $\frac{d}{dt}\,\xi = h_x$,

with initial conditions $x(0) = y$, $\xi(0) = \eta$, $(y,\eta) \in T^*M$. $S_0\, G^t : T^*M \to T^*M$ and

$G^t(y,\eta) = (x^*,\xi^*) = (x^*(t,y,\eta), \xi^*(t,y,\eta))$,

where (x^*, ξ^*) is the solution of the system (1)

We consider a family of Lagrangian manifolds

$\Lambda^t = \{(x,\xi),(y,\eta) : (x,\xi) = G^t(y,-\eta)\} \subset T^*M \times T^*M$,

smoothly depending on the parameter t.

Definition . We say that function $\varphi \in C^\infty(\mathbb{R}^1 \times M \times \{T^*M\setminus 0\})$ parametrizes $\Lambda = U_{t\geq 0} \Lambda^t f$

(2) $\Lambda^t = \{(x,\varphi_x), (y,\varphi_y) ; \varphi_\eta = 0\}$

and φ is non degenerate, homogeneous phase function of positive type, such that differentials $d\varphi_{\eta_j}$, $j = 1,\dots,n$ are linearly independant (we use the terminology from [1, v. IV])

In this paper we construct a global (for all t) phase function parametrizing Λ and as a result we can modulo smooth functions represent the Schwartz Kernel of

the operator $\exp(-it\sqrt{-\Delta})$ as a simple oscillatory integral. It was thought for a long time that Lagrangian manifolds in general do not allow a global parametrization of the type (2). This is apparently the case if the phase function φ is real. We introduce here function φ which is a complex-valued function and $\operatorname{Im}\varphi \geq 0$. Computation of the principal symbol of the amplitude allows us to introduce 1-form Ω and to interpret the Maslov class as an element in the de Rham cohomology group. The value of this class on a closed curve gives the value of the Maslov index of the curve. (an analogous definition has been given by V. Arnold [2] only for the Lagrangian Grassmannian).

Complex phase functions have been applied to different problems earlier, see [3], [4], [5]. General method concerning homogeneous Lagrangian manifold was given in [6]. Some analogues one can also find in [7], [8]

Construction of the global phase function

Let Γ be a linear symmetric Levi-Civita connection corresponding to Riemannian metric g, $\Gamma^i_{jk} = \Gamma^i_{jk}$ be the Christoffel symbols of the connection Γ.

For sufficiently close points $x \in M$, $y \in M$ let us denote by $\gamma_{x,y}(\tau)$ the "shortest" geodesic connecting x and y. We shall choose the parameter τ such that $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(1) = y$. Put $v(x,y) = \gamma_{x,y}(0) \in T_x M$. Denote by $\phi_{x,y}$ the parallel transport from $T^*_x M$ into $T^*_y M$ along the geodesic $\gamma_{x,y}$. In the normal system of coordinates with centre x this operator can be represented by $n \times n$ matrix. We denote this matrix by $\phi(x,y)$.

Lemma 1 : If $x \to y$ then

$$\phi(x;y) = I + O(|x-y|^2).$$

Proof. In the normal system of coordinates $y = (y^1,\ldots,y^n)$ the components $\phi^i_j(x,y)$

satisly the following equations

$$\sum_k (y^k - x^k)\, \partial_{y^k} \phi^i_j (x,y) = \sum_K \Gamma^{\ell}_{k,j} (y)\, (y^k - x^k)\, \phi^i_{\ell} (x;y)$$

$$\phi^i_j(x;y)|_{y=x} = \delta^i_j.$$

Since $\Gamma^{\ell}_{kj}(x) = 0$ is the normal system of coordinates, it implies that $\partial_{yk}\phi^i_j(x,y)|_{y=x} = 0$. Lemma is proved.

Lemma 2 In an arbitrary system of coordinates

$$\xi^*(t,y,\eta).\, x^*_{\eta_k}(t,y,\eta) = 0\,, \quad k = 1,\ldots,n,$$

$$\xi^*(t,y,\eta).\, x^*_{y^k}(t,y,\eta) = \eta_k,\ k = 1,\ldots,n,$$

where $(x^*, \xi^*) = G^t(y,\eta)$.

Proof. If we consider ∂_{η_k} and ∂_{y^k} as vector fields on T^*M, then

$$\xi^*.\, x^*_{\eta_k} = \langle \xi\, dx,\, dG^t(\partial_{\eta_k}) \rangle,$$

$$\xi^*.\, x^*_{y^k} = \langle \xi\, dx,\, d\, G^t(\partial_{y^k}) \rangle$$

Since the transformation G^t preserves the canonical 1-form ξdx, it follows that

$$\xi^*.\, x^*_{\eta_k} = \langle \eta\, d\eta,\, \partial_{\eta_k} \rangle = 0$$

$$\xi^*.\, x^*_{y^k} = \langle \eta\, dy,\, \partial_{y^k} \rangle = \eta_k.$$

Thus the proof is complete

Lemma 3 . In an arbitrary system of coordinates

(3) $x^*_{\eta_k}.\, \xi^*_{y^{\ell}} - \xi^*_{\eta_k}.\, \xi^*_{y^{\ell}} = \delta_{k\ell}\,, \quad k,\ell = 1,\ldots,n$

(4) $x^*_{\eta_k}.\, \xi^*_{\eta_{\ell}} - \xi^*_{\eta_k}.\, x^*_{\eta_{\ell}} = 0\,, \quad k,\ell = 1,\ldots,n$

Proof

$$x^*_{\eta_k} \cdot \xi^*_{y^\ell} - \xi^*_{\eta_k} \cdot x^*_{\eta_\ell} = \langle dx \wedge d\xi,\ dG^t(\partial_{\eta_\ell}) \wedge dG^t(\partial_{y^\ell}) \rangle =$$

$$= \langle dy \wedge d\eta,\ \partial_{\eta_k} \wedge \partial_{y^\ell} \rangle = \delta_{k\ell}.$$

By analogy

$$x^*_{\eta_k} \cdot \xi^*_{\eta_\ell} - \xi^*_{\eta_k} \cdot x^*_{\eta_\ell} = \langle dx \wedge d\xi,\ dG^t(\partial_{\eta_\ell}) \wedge dG^t(\partial_{\eta_\ell}) \rangle$$

$$= \langle dy \wedge d\eta,\ \partial_{\eta_k} \wedge \partial_{\eta_\ell} \rangle = 0.$$

Lemma is proved.

Let $|\,.\,|_y$ be the length of the covector (vector) from $T^*M(T_yM)$ in the metric g. We introduce the following functions, which are positively homogeneous of degree 1 in η.

$$\varphi_1(t,x,y,\eta) = \xi^*(t,y,\eta).\ \nu\,(x^*(t,y,\eta),x)$$

$$\varphi_2(t,x,y,\eta) = \tfrac{1}{2}|\nu\,(x^*(t,y,\eta),x)|^2_{x^*(t,y,\eta)} \cdot |\eta|_y$$

(7) $$\varphi(t,x,y,\eta) = \varphi_1(t,x,y,\eta) + i\varphi_2(t,x,y,\eta)$$

Here it is enough for us to define the function ν for points x^*, $x \in M$ sufficiently close to each other. However we do not lose anything if we continue ν by a smooth non-zero function outside a neighborhood of

diag $(M \times M)$

Let us denote

(8) $$Z_k = \xi^*_{\eta_k} - i|\eta|_y\, x^*_{\eta_k}$$

and

(9) $$Z = \{Z_k\}_{k=1}^n$$

Lemme 4

(10) $$\partial_{\eta_k} \varphi(t,x,y,\eta) = \nu(x^*,x)\, Z_k + O\,(|x-x^*|^2),$$

(11) $$\partial_{y^k} \varphi(t,x^*,y,\eta) = -\eta_k,$$

(12) $\partial_{x^k} \varphi(t,x^*,y,\eta) = \xi^*_k$,

(13) $(\nabla_x \varphi)_{\eta_k} (t,x^*,y,\eta) = Z_k$.

Proof. In the normal system of coordinates with centre x

$$v(x^*,x) = -\phi_{x,x^*} v(x,x^*) = \phi(x;x^*)\,(x-x^*).$$

In view of Lemmas 1 and 2 we obtain that

(14) $\partial_{\eta_k} \varphi_1 = \phi(x;x^*)(x-x^*)\xi^*_{\eta_k} - \phi(x;x^*)\, x^*_{\eta_k} \xi^* + O(|x-x^*|^2) = v(x^*,x)\xi^*_{\eta_k}$

$+O(|x-x^*|^2)$

Now, in an arbitrary system of coordinates

(15) $\partial_{\eta_k} \varphi_2 = -i(\, x^*_{\eta_k}\,, v(x^*,x))_{x^*}.|\eta|_y + O(|x-x^*|^2)$,

where $(.,.)_x$ is the scalar product of vectors (covectors) from T_xM (T^*_xM) in the metric g. Combining (14) and (15) we have (10).

By analogy, in view of Lemma 2

$$\partial_{y^k} \varphi(t,x^*,y,\eta) = -\xi^* . x^*_{y_k} = -\eta_k$$

In the normal system of coordinate with centre x^*

$\partial_{x^k} \varphi(t,x^*,y,\eta) = \partial_{x^k} (\xi^* (x-x^*) + \frac{i}{2}|x-x^*|^2_{x^*} = \xi^*_k$.

By differentiation (14) and (15) with respect to x we obtain (13).

Theorem 1. The function φ is a non-degenerate phase function parametrizing Λ.

Proof. Assume first that the matrix Z introduced in (9) in non-degenerate. Then from (10) it follows that the condition $\varphi_\eta = 0$ is equivalent to $x = x^*$ for x close to x^*. In view of (11) and (12) we obtain that

$$\Lambda^t = \{(x,\varphi_x), (y,\varphi_y) : \varphi_\eta = 0\} \text{ for } t \in [0,\infty])$$

Now it is clear that (13) implies

$$(\det \varphi_{x,\eta})|_{\varphi_\eta = 0} = \det Z \neq 0,$$

and therefore the function φ is a non-degenerate phase function.

To prove that the matrix Z is non-degenerate let us suppose first that $\lambda = (\lambda_1,...,\lambda_n) \in \mathbb{R}^n$ and

$$\sum_{k=1}^{n} \lambda_k Z_k = 0,$$

where Z_k are columns of Z, see (8). Then both real and imaginary parts are equal to zero

$$(16)\quad \sum \lambda_k \xi^*_{\eta_k} = 0, \sum \lambda_x x^*_{\eta_k} = 0.$$

Multipling (3) by $\lambda_k.\lambda_\ell$ after summation using (16) we have

$$0 = \sum_{k,\ell=1}^{n} (\lambda_k x^*_{\eta_k} . \lambda_\ell \xi_{\ell\, y} - \lambda_k \xi^*_{\eta_k} . \lambda_\ell x_{\ell\, y}) = \sum_{k=1}^{n} \lambda_k^2.$$

This implies $\lambda_1 = ... = \lambda_n = 0$. Now if $\lambda_k = a_k + ib_k$, $k = 1,...,n$, then from the equation $\sum_{k=1}^{n} \lambda_k Z_k = 0$ and (4) it follows that

$$\sum_{k=1}^{n} \lambda_k Z_k . \overline{Z}_\ell = \sum_{k=1}^{n} \lambda_k \operatorname{Re}(Z_k . \overline{Z}_\ell) = 0 \text{ for } \ell = 1,...,n.$$

This means that

$$\sum_{k=1}^{n} a_k Z_k \overline{Z}_\ell = \sum_{k=1}^{n} b_k Z_\ell \overline{Z}_\ell = 0 \text{ for } \ell = 1,...,n. \text{ So } \sum_{k,\ell=1}^{n} a_k Z_k . a_\ell \overline{Z}_\ell = 0$$

$$\sum_{k,\ell=1}^{n} b_k . Z_k . b_k \overline{Z}_\ell = 0$$

Since Z_k are linearly independent over the set of real numbers we obtain that $a_k = b_k = 0$ for $k = 1,..,n$.

Theorem 2. Modulo the smooth functions the Schwartz kernel of the operator $u(t) = \exp(-it\sqrt{-\Delta})$ can be written in the form

$$(17)\quad (2\pi)^{-n} \int_{T^*_y M} e^{i\varphi(t,x,y,\eta)} a(t,y,\eta)\, d\eta,\ x,y \in M,\ t \in [0,\infty[\text{ where } \varphi \text{ is defined}$$

in (7) and $a \in C^{\infty}(\mathbb{R}^1 \times T^*M)$ is a classical amplitude with the principal symbol

(18) $a_0 = (\det Z)^{1/2}$

Proof To prove this theorem we need only to find the principal symbol of the amplitude. The problem is reduced to solution of the transport equations. Let us assume that a is a classical amplitude function. Then

$$(\tfrac{1}{i}\partial_t + \sqrt{-\Delta})\, e^{i\varphi} a =$$

$$= \tfrac{1}{i}\, a_t + [\varphi_t + h(x, \nabla_x \varphi) + \tfrac{1}{2i}\, h_{\xi\xi}\, \varphi_{xx} + O(|\eta|^{-1})]a =$$

$$= \tfrac{1}{i}\, a_t + [\xi_t^* \nu - \xi^* x_t^* - i x_t^* \nu\, |\eta| + h(x, \xi^* + i\nu\, |\eta| + O(|\nu|^2 |\eta|)$$

$$+ \tfrac{1}{2i}\, h_{\xi\xi}\, \varphi_{xx} + O(|\nu|^2 |\eta|)]a =$$

$$\tfrac{1}{i}\, a_t + \tfrac{1}{2}[h_{xx}\nu.\nu + 2i h_{x\xi}\, \nu.\nu\, |\eta| - h_{\xi\xi}\nu\nu|\eta|^2 + \tfrac{1}{i}\, h_{\xi\xi}\, \varphi_{xx} + R_1]a,$$

where $R_1 = O(|\nu|^3 |\eta|^2 + |\nu|^2 |\eta| + |\nu| + |\nu||\eta|^{-1})$, when $|\eta| \to \infty$, $|\nu| \to 0$.

Since Z is non-degenerate we obtain that

$$\nu e^{i\varphi} = \tfrac{1}{i}\, Z^{-1} \nabla_\eta\, e^{i\varphi}.$$

Formaly integrating by part we have

(19) $$(\tfrac{1}{i}\partial_t + \sqrt{-\Delta}\,)\, u =$$

$$(2\pi)^{-n} \int_{T^*M_y} [\tfrac{1}{i} a_t + \tfrac{1}{2i}\, \mathrm{Tr}\,(Z^{-1} h_{xx}\, x_\eta^* + 2i\, Z^{-1} h_{x\xi} x_\eta^* - |\eta|^2 Z^{-1} h_{\xi\xi} x_\eta^*) a +$$

$$h_{xx}\varphi_{xx} a + R_2]\, e^{i\varphi}\, d\eta\,,$$

where $R_2 = O(|\nu|^2|\eta| + |\eta|^{-1})$. From the definitions (8), (9) of Z we have

(20) $$|\eta|^2\, Z^{-1}\, h_{\xi\xi}\, x_\eta^* = i|\eta| Z^{-1}\, h_{\xi\xi}\, (Z - \xi_\eta^*)$$

Hamiltonian system of equations (1) implies

(21)
$$\frac{d}{dt}\, x_\eta^* = h_{\xi\xi}\, \xi_\eta^* + h_{\xi x}\, x_\eta^*$$

$$\frac{d}{dt}\, \xi_\eta^* = -h_{x\xi}\, \xi_\eta^* - h_{xx}\, x_\eta^*.$$

If we substitude (20) and (21) into (19) and put it equal to zero we obtain

the following equation for the principal symbol a_0 of the amplitude a

$$\frac{d}{dt}\, a_0 - \frac{1}{2} \mathrm{Tr}\, (Z_t\, Z^{-1})\, a_0 = 0.$$

Therefore

$$a_0 = (\det Z)^{1/2}$$

To finish the proof of the theorem 2 we should notice that further terms of the amplitude we obtain by solving recurrent system of transport equations.

Using the explicite formula for the principal symbol in oscillatory integral (17) we can now define the phase shift determined by the Maslov index.

Let us introduce 1-form Ω on Λ

$$\Omega = \frac{1}{2\pi}\, d(\mathrm{agr}\, \det^2 Z)$$

Theorem 3 The 1-form represents an element of the Maslov class in the de Rham cohomology group. The Maslov index of a closed curve on Λ is equal to the integral of the form Ω over this curve.

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