Journées

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Forges-les-Eaux, 7 juin–11 juin 2004

François Castella

The radiation condition at infinity for the high-frequency Helmholtz equation with source term: a wave packet approach

The radiation condition at infinity for the high-frequency Helmholtz equation with source term: a wave packet approach

François Castella

Abstract

We consider the high-frequency Helmholtz equation with a given source term, and a small absorption parameter $\alpha > 0$. The high-frequency (or: semi-classical) parameter is $\varepsilon > 0$. We let ε and α go to zero simultaneously. We assume that the zero energy is non-trapping for the underlying classical flow. We also assume that the classical trajectories starting from the origin satisfy a transversality condition, a generic assumption.

Under these assumptions, we prove that the solution u^ε radiates in the outgoing direction, **uniformly** in ε. In particular, the function u^ε, when conveniently rescaled at the scale ε close to the origin, is shown to converge towards the **outgoing** solution of the Helmholtz equation, with coefficients frozen at the origin. This provides a uniform (in ε) version of the limiting absorption principle.

Writing the resolvent of the Helmholtz equation as the integral in time of the associated semi-classical Schrödinger propagator, our analysis relies on the following tools: (i) For very large times, we prove and use a uniform version of the Egorov Theorem to estimate the time integral; (ii) for moderate times, we prove a uniform dispersive estimate that relies on a wave-packet approach, together with the above mentioned transversality condition; (iii) for small times, we prove that the semi-classical Schrödinger operator with variable coefficients has the same dispersive properties as in the constant coefficients case, uniformly in ε.

1. Introduction

We study the asymptotics $\varepsilon \to 0^+$ in the following scaled Helmholtz equation, with unknown w^ε,

$$i\varepsilon\, \alpha_\varepsilon\, w^\varepsilon(x) + \frac{1}{2}\Delta_x w^\varepsilon(x) + n^2(\varepsilon x) w^\varepsilon(x) = S(x). \qquad (1)$$

In this scaling, both the absorption parameter $\alpha_\varepsilon > 0$ is small, i.e.

$$\alpha_\varepsilon \to 0^+ \quad \text{as} \quad \varepsilon \to 0,$$

and the index of refraction $n^2(\varepsilon x)$ is almost constant,

$$n^2(\varepsilon x) \approx n^2(0).$$

The competition between these two effects is the key difficulty of the present work. Note that the limiting case $\alpha_\varepsilon = 0^+$ is actually allowed in our analysis.

In all our analysis, the variable x belongs to $\mathbf{R}^d$, for some $d \geq 3$. The index of refraction $n^2(x)$ is assumed to be given, smooth and non-negative[1]

$$\forall x \in \mathbf{R}^d, \quad n^2(x) \geq 0, \quad \text{and} \quad n^2(x) \in C^\infty(\mathbf{R}^d). \tag{2}$$

It is also supposed that $n^2(x)$ goes to a constant at infinity,

$$n^2(x) = n_\infty^2 + O\left(\langle x \rangle^{-\rho}\right) \quad \text{as} \quad x \to \infty, \tag{3}$$

for some, possibly small, exponant $\rho > 0$[2]. In the language of Schrödinger operators, this means that the potential $n_\infty^2 - n^2(x)$ is assumed to be either short-range or long range. Finally, the source term in (1) uses a function $S(x)$ that is taken sufficiently smooth and decays fast enough at infinity. We refer to the sequel for the very assumptions we need on the refraction index $n^2(x)$, together with the source S (see the statement of the main Theorem below).

Upon the L^2-unitary rescaling

$$w^\varepsilon(x) = \varepsilon^{d/2} u^\varepsilon(\varepsilon x),$$

the study of (1) is naturally linked to the analysis of the high-frequency Helmholtz equation,

$$i\varepsilon\alpha_\varepsilon u^\varepsilon(x) + \frac{\varepsilon^2}{2}\Delta_x u^\varepsilon(x) + n^2(x)u^\varepsilon(x) = \frac{1}{\varepsilon^{d/2}}S\left(\frac{x}{\varepsilon}\right), \tag{4}$$

where the source term $S(x/\varepsilon)$ now plays the role of a concentration profile at the scale ε. In this picture, the difficulty now comes from the interaction between the oscillations induced by the source $S(x/\varepsilon)$, and the ones due to the semiclassical operator $\varepsilon^2\Delta/2 + n^2(x)$. We give below more complete motivations for looking at the asymptotics in (1) or (4).

The goal of this talk is to prove that the solution w^ε to (1) converges (in the distributional sense) to the **outgoing solution** of the natural constant coefficient Helmholtz equation, i.e.

$$\lim_{\varepsilon \to 0} w^\varepsilon = w^{\text{out}}, \quad \text{where} \quad w^{\text{out}} \quad \text{is defined as the solution to}$$

$$i0^+ w^{\text{out}}(x) + \frac{1}{2}\Delta_x w^{\text{out}}(x) + n^2(0)w^{\text{out}}(x) = S(x). \tag{5}$$

---

[1] Our analysis could easily extended to the case where the refraction index is a function that changes sign. The only really important assumption on the sign of n is $n_\infty^2 > 0$. We do not give further details on this point.

[2] Here and below we use the standard notation $\langle x \rangle := (1 + x^2)^{1/2}$.

In other words,

$$
\begin{aligned}
w^{\text{out}} &= \lim_{\delta \to 0^+} \left(i\delta + \frac{1}{2}\Delta_x + n^2(0) \right)^{-1} S \\
&= i \int_0^{+\infty} \exp\left(it \left(\frac{1}{2}\Delta_x + n^2(0) \right) \right) S \, dt.
\end{aligned}
\tag{6}
$$

It is well-known that w^{out} can also be defined as the unique solution to $(\Delta_x/2 + n^2(0))w^{\text{out}} = S$ that satisfies the Sommerfeld radiation condition at infinity (stated here in $d = 3$ dimensions of space)

$$
\frac{x}{\sqrt{2}|x|} \cdot \nabla_x w^{\text{out}}(x) + in(0)w^{\text{out}}(x) = O\left(\frac{1}{|x|^2}\right) \quad \text{as} \ \ |x| \to \infty.
\tag{7}
$$

The main geometric assumptions we need on the refraction index to ensure the validity of (5) are twofolds. First, we need that the trajectories of the Hamiltonian $\xi^2/2 - n^2(x)$ at the zero energy are **not trapped**. This is a standard assumption in this context. It somehow prevents accumulation of energy in bounded regions of space. Second, it turns out that the trajectories that really matter in our analysis, are those that *start from the origin* $x = 0$, with zero energy $\xi^2/2 = n^2(0)$. In this perspective, we need that these trajectories satisfy a **transversality condition**: in essence, each such ray can self-intersect, but the self-intersection then has to be "tranverse" (see assumption (14)). This second assumption prevents accumulation of energy at the origin.

We wish to emphasize that the statement (5) is not obvious. In particular, if the transversality assumption (14) is not fullfilled, our analysis shows that (5) becomes false in general.

The central difficulty is the following. On the one hand, the vanishing absorption parameter α_ε in (1) leads to thinking that w^ε should satisfy the Sommerfeld radiation condition at infinity **with the variable refraction index** $n^2(\varepsilon x)$ (see (7)). Knowing that $\lim_{|x|\to\infty} n^2(\varepsilon x) = n_\infty^2$, this roughly means that w^ε should behave like $\exp(i2^{-1/2}n_\infty|x|)/|x|$ at infinity in x. On the other hand, the almost constant refraction index $n^2(\varepsilon x)$ in (1) leads to observe that w^ε naturally goes to a solution of the Helmholtz equation **with constant refraction index** $n^2(0)$. Hoping that we may follow the absorption coefficient α_ε continuously along the limit $\varepsilon \to 0$ in $n^2(\varepsilon x)$, the statement (5) becomes natural, and w^ε should behave like $\exp(i2^{-1/2}n(0)|x|)/|x|$ asymptotically. As we see, the strong non-local effects induced by the Helmholtz equation make the key difficulty in following the continuous dependence of w^ε upon both the absorption parameter $\alpha_\varepsilon \to 0^+$ and on the index $n^2(\varepsilon x) \to n^2(0)$.

2. Motivation

Let us now give some more detailed account on our motivations for looking at the asymptotics $\varepsilon \to 0$ in (1).

In [BCKP], the high-frequency analysis of the Helmholtz equation with source term is performed. More precisely, the asymptotic behaviour as $\varepsilon \to 0$ of the

following equation is studied[3]

$$i\varepsilon\alpha_\varepsilon u^\varepsilon(x) + \frac{\varepsilon^2}{2}\Delta_x u^\varepsilon(x) + n^2(x)u^\varepsilon(x) = \frac{1}{\varepsilon^{d/2}}S\left(\frac{x}{\varepsilon}\right),\qquad(8)$$

where the variable x belongs to $\mathbf{R}^d$, for some $d \geq 3$, and the index of refraction $n^2(x)$ together with the concentration profile $S(x)$ are as before (see [BCKP]). Later, the analysis of [BCKP] was extended in [CPR] to more general oscillating/concentrating source terms. The paper [CPR] studies indeed the high-frequency analysis $\varepsilon \to 0$ in

$$i\varepsilon\alpha_\varepsilon u^\varepsilon(x) + \frac{\varepsilon^2}{2}\Delta_x u^\varepsilon(x) + n^2(x)u^\varepsilon(x) =$$
$$\frac{1}{\varepsilon^q}\int_\Gamma S\left(\frac{x-y}{\varepsilon}\right)\, A(y)\,\exp\left(i\,\frac{\phi(x)}{\varepsilon}\right)\,d\sigma(y).\qquad(9)$$

(See also [CRu] for extensions - see [Fou] for the case where n^2 has discontinuities). In (9), the function S again plays the role of a concentration profile like in (8), but the concentration occurs this time around a smooth submanifold $\Gamma \subset \mathbf{R}^d$ of dimension p instead of a point. On the more, the source term here includes additional oscillations through the (smooth) amplitude A and phase ϕ. In these notations $d\sigma$ denotes the induced euclidean surface measure on the manifold Γ, and the rescaling exponent q depends on the dimension of Γ together with geometric considerations, see [CPR].

Both Helmholtz equations (8) and (9) modelize the propagation of a high-frequency source wave in a medium with scaled, variable, refraction index $n^2(x)/\varepsilon^2$. The scaling of the index imposes that the waves propagating in the medium naturally have wavelength ε. On the other hand, the source in (8) as well as (9) is concentrating at the scale ε, close to the origin, or close to the surface Γ. It thus carries oscillations at the typical wavelength ε. One may think of an antenna concentrated close to a point or to a surface, and emmitting waves in the whole space. The important phenomenon that these linear equations include precisely lies in the **resonant interaction** between the high-frequency oscillations of the source, and the propagative modes of the medium dictated by the index n^2/ε^2. This makes one of the key difficulties of the analysis performed in [BCKP] and [CPR].

A Wigner approach is used in [BCKP] and [CPR] to treat the high-frequency asymptotics $\varepsilon \to 0$. Up to a harmless rescaling, these papers establish that the Wigner transform $f^\varepsilon(x,\xi)$ of $u^\varepsilon(x)$ satisfies, in the limit $\varepsilon \to 0$, the stationnary transport equation

$$0^+ f(x,\xi) + \xi\cdot\nabla_x f(x,\xi) + \nabla_x n^2(x)\cdot\nabla_\xi f(x,\xi) = Q(x,\xi),\qquad(10)$$

where $f(x,\xi) = \lim f^\varepsilon(x,\xi)$ measures the energy carried by rays located at the point x in space, with frequency $\xi \in \mathbf{R}^d$. The limiting source term Q in (10) describes quantitatively the resonant interactions mentioned above. In the easier case of (8), one has $Q(x,\xi) = \delta\left(\xi^2/2 - n^2(0)\right)\,\delta(x)\,|\widehat{S}(\xi)|^2$, meaning that the asymptotic source of energy is concentrated at the origin in x (this is the factor $\delta(x)$), and it only carries

---

[3]note that we use here a slightly different scaling than the one used in [BCKP]. This a harmless modification that is due to mere convenience.

resonant frequencies ξ above this point (due to $\delta\left(\xi^2/2 - n^2(0)\right)$). A similar but more complicated value of Q is obtained in the case of (9). In any circumstance, equation (10) tells us that the energy brought by the source Q is propagated in the whole space through the transport operator $\xi \cdot \nabla_x + \nabla_x n^2(x) \cdot \nabla_\xi$ naturally associated with the semi-classical operator $-\varepsilon^2 \Delta_x/2 - n^2(x)$. The term $0^+ f$ in (10) specifies a radiation condition at infinity for f, that is the trace, as $\varepsilon \to 0$ of the absorption coefficient $\alpha_\varepsilon > 0$ in (8) and (9). It gives f as the outgoing solution

$$f(x, \xi) = \int_0^{+\infty} Q\left(X(s, x, \xi), \Xi(s, x, \xi)\right)\ ds.$$

Here $(X(s, x, \xi), \Xi(s, x, \xi))$ is the value at time s of the characteristic curve of $\xi \cdot \nabla_x + \nabla_x n^2(x) \cdot \nabla_\xi$ starting at point (x, ξ) of phase-space (see (13) below). Obtaining the radiation condition for f as the limiting effect of the absorption coefficient α_ε in (8) is actually the second main difficulty of the analysis performed in [BCKP] and [CPR].

It turns out that the analysis performed in [BCKP] relies at some point on the asymptotic behaviour of the scaled wave function $w^\varepsilon(x) = \varepsilon^{d/2} u^\varepsilon(\varepsilon x)$ that measures the oscillation/concentration behaviour of u^ε close to the origin. Similarly, in ([CPR]) one needs to rescale u^ε around any point $y \in \Gamma$, setting $w_y^\varepsilon(x) := \varepsilon^{d/2} u^\varepsilon(y + \varepsilon x)$ for any such y. We naturally have

$$i\varepsilon\alpha_\varepsilon w^\varepsilon(x) + \frac{1}{2}\Delta_x w^\varepsilon(x) + n^2(\varepsilon x) w^\varepsilon(x) = S(x),$$

in the case of (8), and a similar observation holds true in the case of (9). Hence the natural rescaling leads to the analysis of the prototype equation (1). Under appropriate assumptions on $n^2(x)$ and $S(x)$, it may be proved that w^ε, solution to (1), is bounded in the weighted L^2 space $L^2(\langle x\rangle^{1+\delta}\ dx)$, for any $\delta > 0$, uniformly in ε. For a fixed value of ε, such weighted estimates are consequences of the work by Agmon, Hörmander, [Ag], [AH]. The fact that these bounds are uniform in ε is a consequence of the recent (and optimal) estimates established by B. Perthame and L. Vega in [PV1], [PV2] (where the weighted L^2 space are replaced by a more precise homogeneous Besov-like space). The results in [PV1] and [PV2] actually need a virial condition of the type $2n^2(x) + x \cdot \nabla_x n^2(x) \geq c > 0$, a condition that *implies* our transversality assumption (14). We also refer to the work by N. Burq [Bu], Gérard and Martinez [GM], T. Jecko [J], as well as Wang and Zhang [WZ], for (not optimal) bounds in a similar spirit. Under the (weaker) assumptions we make in the present paper, a (weaker) bound may also be obtained as a consequence of our analysis. In any case, once w^ε is seen to be bounded, it naturally possesses a weak limit $w = \lim w^\varepsilon$ in the appropriate space. The limit w clearly satisfies in a weak sense the equation

$$\left(\frac{1}{2}\Delta_x + n^2(0)\right) w(x) = S(x). \tag{11}$$

Unfortunately, equation (11) does not specify $w = \lim w^\varepsilon$ in a unique way, and it has to be supplemented with a radiation condition at infinity. In view of the equation

(1) satisfied by w^ε, it has been **conjectured** in [BCKP] and [CPR] that $\lim w^\varepsilon$ actually satisfies

$$\lim w^\varepsilon = w^{\text{out}},$$

where w^{out} is the outgoing solution defined before. The present talk answers the conjecture formulated in these works. It also gives geometric conditions for the convergence $\lim w^\varepsilon = w^{\text{out}}$ to hold.

As a final remark, let us mention that our analysis is purely time-dependent. We wish to indicate that similar results than those in the present talk were recently and independently obtained by Wang and Zhang [WZ] using a stationary approach.

3. Main result

Our main theorem is the following

Main Theorem

Let w^ε satisfy $\quad i\varepsilon\alpha_\varepsilon w^\varepsilon(x) + \frac{1}{2}\Delta_x w^\varepsilon(x) + n^2(\varepsilon x)w^\varepsilon(x) = S(x), \quad$ *for some sequence $\alpha_\varepsilon > 0$ such that $\alpha_\varepsilon \to 0^+$ as $\varepsilon \to 0$. Assume that the source term S belongs to the Schwartz class $\mathcal{S}(\mathbf{R}^d)$. Suppose also that the index of refraction satisfies the following set of assumptions*

- *(smoothness, decay).   There exists an exponent $\rho > 0$, and a positive constant $n_\infty^2 > 0$ such that for any multi-index $\alpha \in \mathbf{N}^d$, there exists a constant $C_\alpha > 0$ with*

$$\left| \partial_x^\alpha \left(n^2(x) - n_\infty^2 \right) \right| \leq C_\alpha \, \langle x \rangle^{-\rho - |\alpha|}. \tag{12}$$

- *(non-trapping condition).   The trajectories associated with the Hamiltonian $\xi^2/2 - n^2(x)$ are not trapped at the zero energy. In other words, any trajectory $(X(t,x,\xi), \Xi(t,x,\xi))$ solution to*

$$\frac{\partial}{\partial t}X(t,x,\xi) = \Xi(t,x,\xi), \qquad\qquad X(0,x,\xi) = x,$$

$$\frac{\partial}{\partial t}\Xi(t,x,\xi) = \left(\nabla_x n^2\right)(X(t,x,\xi)), \qquad\qquad \Xi(0,x,\xi) = \xi, \tag{13}$$

with initial datum (x,ξ) such that $\xi^2/2 - n^2(x) = 0$, is assumed to satisfy

$$|X(t,x,\xi)| \to \infty, \quad \text{as } |t| \to \infty.$$

- *(tranversality condition).   The tranversality condition (14) on the trajectories starting from the origin $x = 0$, with zero energy $\xi^2/2 = n^2(0)$, is satisfied.*

Then, we do have the following convergence, weakly, when tested against any function $\phi \in \mathcal{S}(\mathbf{R}^d)$,

$$w^\varepsilon \to w^{\text{out}}.$$

Remark

The transversality assumption (14) requires that

> the set $S := \{(\eta, \xi, t) \in \mathbf{R}^{2d} \times]0, \infty[\text{ s.t.}$
> $X(t, 0, \xi) = 0,\ \Xi(t, 0, \xi) = \eta,\ \xi^2/2 = n^2(0)\}$ (14)
> is a smooth submanifold of $\mathbf{R}^{2d+1}$, **having a codimension** $> d + 2$.

In other words, zero energy trajectories issued from the origin and passing several times through the origin $x = 0$ should be "rare".

It is possible to prove that $\operatorname{codim} S \geq d + 2$ in any case. Our assumption thus means that the extreme case $\operatorname{codim} S = d + 2$ should be avoided.

To give a caricatural example, let us simply say that the flow of the harmonic oscillator (which is, strictly speaking, not included in our analysis), i.e. the case of a Hamiltonian $H(x, \xi) = \xi^2/2 + x^2/2$, gives $\operatorname{codim} S = d + 2$. In the case of a harmonic oscillator with rationally independent frequencies, i.e. $H(x, \xi) = \xi^2/2 + \omega_1 x_1^2/2 + \cdots + \omega_d x_d^2/2$ with $(\omega_1, \ldots, \omega_d)$ being $\mathbf{Q}$-independent, gives $\operatorname{codim} S = 0$. $\square$

The above theorem is not only a local convergence result, valid for test functions $\phi \in \mathcal{S}$. Indeed, by density of smooth functions in weighted L^2 spaces, it readily implies the following immediate corollary. It states that, provided w^ε is bounded in the natural weighted L^2 space, the convergence also holds weakly in this space. In other words, the convergence also holds globally.

Immediate corollary
With the notations of the main Theorem, assume that the source term S above satisfies the weaker decay property

$$\|S\|_B := \sum_{j \in \mathbf{Z}} 2^{j/2} \|S\|_{L^2(C_j)} < \infty,$$ (15)

where C_j denotes the annulus $\{2^j \leq |x| \leq 2^{j+1}\}$ in $\mathbf{R}^d$. Suppose also that the index of refraction satisfies the smoothness condition of the main Theorem, with the non-trapping and transversality assumptions replaced by the stronger

$$\bullet \ (virial\ condition) \qquad 2\sum_{j \in \mathbf{Z}} \sup_{x \in C_j} \frac{(x \cdot \nabla n^2(x))_-}{n^2(x)} < 1.$$ (16)

Then, we do have the convergence $w^\varepsilon \to w^{\mathrm{out}}$, weakly, when tested against any function ϕ such that $\|\phi\|_B < \infty$,

Remark
Here, the decay (15) assumed on the source S is the natural (and optimal) one. On the more, the above weak convergence holds in the optimal space, as we now explain.

It is well known that the resolvent of the Helmholtz operator maps the weighted L^2 space $L^2\left(\langle x \rangle^{1+\delta} dx\right)$ to $L^2\left(\langle x \rangle^{-1-\delta} dx\right)$ for any $\delta > 0$ ([Ag], [J], [GM]). It has been established (in the constant coefficients case) by Agmon and Hörmander [AH] that this may be improved into the following optimal result: the resolvent of the

Helmholtz operator sends the weighted L^2 space B defined in (15) to the dual weighted space B^* defined by

$$\|u\|_{B^*} := \sup_{j \in \mathbf{Z}} 2^{-j/2} \|u\|_{L^2(C_j)}. \tag{17}$$

This has been generalized to the non-constant coefficients case (that are non-compact perturbations of the constant coefficients case) by Perthame and Vega in [PV1] and [PV2]. Their work uses the assumption (16). In our perspective, the assumption (16) is of technical nature, and it may be replaced by **any** assumption ensuring that the solution w^ε to (1) satisfies the uniform bound

$$\|w^\varepsilon\|_{B^*} \le C_{d,n^2} \, \|S\|_B, \tag{18}$$

for some universal constant C_{d,n^2} that only depends on the dimension $d \ge 3$ and the index n^2. $\qquad\square$

Proof of the immediate Corollary
Under the virial assumption, it has been established in [PV1] that estimate (18) holds true. Hence, by density of the Schwartz class in the space B, one readily reduces the problem to the case when the source S and the test function ϕ belong to $\mathcal{S}(\mathbf{R}^d)$. The main Theorem now allows to conclude. $\qquad\square$

Needless to say, the central assumptions needed for the theorem are the non-trapping condition together with the transversality condition. To state the result very briefly, the heart of our proof lies in proving that under the above assumptions, the propagator $\exp\left(i\varepsilon^{-1}t\left(-\varepsilon^2\Delta_x/2 - n^2(x)\right)\right)$, or its rescaled value $\exp\left(it\left(-\Delta_x/2 - n^2(\varepsilon x)\right)\right)$, satisfy "similar" dispersive properties as the free Schrödinger operator $\exp\left(it\left(-\Delta_x/2 - n^2(0)\right)\right)$, *uniformly in ε*. This in turn is proved upon distinguishing between small times, moderate times, and very large times, each case leading to the use of different arguments and techniques.

4. Outline of the proof

Let w^ε be the solution to $i\varepsilon\alpha_\varepsilon w^\varepsilon + \frac{1}{2}\Delta w^\varepsilon + n^2(\varepsilon x)w^\varepsilon = S(x)$, with $S \in \mathcal{S}(\mathbf{R}^d)$. According to the statement of our main Theorem, we wish to study the asymptotic behaviour of w^ε as $\varepsilon \to 0$, in a weak sense. Taking a test function $\phi(x) \in \mathcal{S}(\mathbf{R}^d)$, and defining the duality product

$$\langle w^\varepsilon, \phi \rangle := \int_{\mathbf{R}^d} w^\varepsilon(x)\phi(x)\, dx,$$

we want to prove the convergence

$$\langle w^\varepsilon, \phi \rangle \to \langle w^{\mathrm{out}}, \phi \rangle \ \text{ as } \ \varepsilon \to 0.$$

where the outgoing solution of the (constant coefficient) Helmholtz equation w^{out} is defined in (5), (6) before.

4.1. First step: preliminary reduction - the time dependent approach

In order to prove the weak convergence $\langle w^\varepsilon, \phi \rangle \to \langle w^{\mathrm{out}}, \phi \rangle$, we define the rescaled function

$$u^\varepsilon(x) = \frac{1}{\varepsilon^{d/2}} w^\varepsilon \left(\frac{x}{\varepsilon} \right). \tag{1}$$

It satisfies $i\varepsilon\alpha_\varepsilon u^\varepsilon + \varepsilon^2/2\, \Delta u^\varepsilon + n^2(x)u^\varepsilon = 1/\varepsilon^{d/2}S\left(x/\varepsilon\right) =: S_\varepsilon(x)$, where for any function $f(x)$ we use the short-hand notation

$$f_\varepsilon(x) = \frac{1}{\varepsilon^{d/2}} f \left(\frac{x}{\varepsilon} \right).$$

Using now the function u^ε instead of w^ε, we observe the equality

$$\langle w^\varepsilon, \phi \rangle = \langle u^\varepsilon, \phi_\varepsilon \rangle. \tag{2}$$

This transforms the original problem into the question of computing the semiclassical limit $\varepsilon \to 0$ in the equation satisfied by u^ε. One sees in (2) that this limit needs to be computed *at the semiclassical scale* (i.e. when tested upon a smooth, concentrated function ϕ_ε).

In order to do so, we compute u^ε in terms of the semiclassical resolvent $(i\varepsilon\alpha_\varepsilon + (\varepsilon^2/2)\Delta + n^2(x))^{-1}$. It is the integral over the whole time interval $[0, +\infty[$ of the propagator of the Schrödinger operator associated with $\varepsilon^2\Delta/2 + n^2(x)$. In other words we write

$$
\begin{aligned}
u^\varepsilon &= \left(i\varepsilon\alpha_\varepsilon + \frac{\varepsilon^2}{2}\Delta + n^2(x) \right)^{-1} S_\varepsilon \\
&= i \int_0^{+\infty} \exp\left(it \left(i\varepsilon\alpha_\varepsilon + \frac{\varepsilon^2}{2}\Delta + n^2(x) \right) \right) S_\varepsilon \, dt.
\end{aligned} \tag{3}
$$

Now, defining the semi-classical propagator

$$U_\varepsilon(t) := \exp\left(i\frac{t}{\varepsilon} \left(\frac{\varepsilon^2}{2}\Delta + n^2(x) \right) \right) = \exp\left(-i\frac{t}{\varepsilon} H_\varepsilon \right), \tag{4}$$

associated with the semi-classical Schrödinger operator

$$H_\varepsilon := -\frac{\varepsilon^2}{2}\Delta - n^2(x), \tag{5}$$

we arrive at the final formula

$$\langle w^\varepsilon, \phi \rangle = \langle u^\varepsilon, \phi_\varepsilon \rangle = \frac{i}{\varepsilon} \int_0^{+\infty} e^{-\alpha_\varepsilon t} \langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon \rangle \, dt. \tag{6}$$
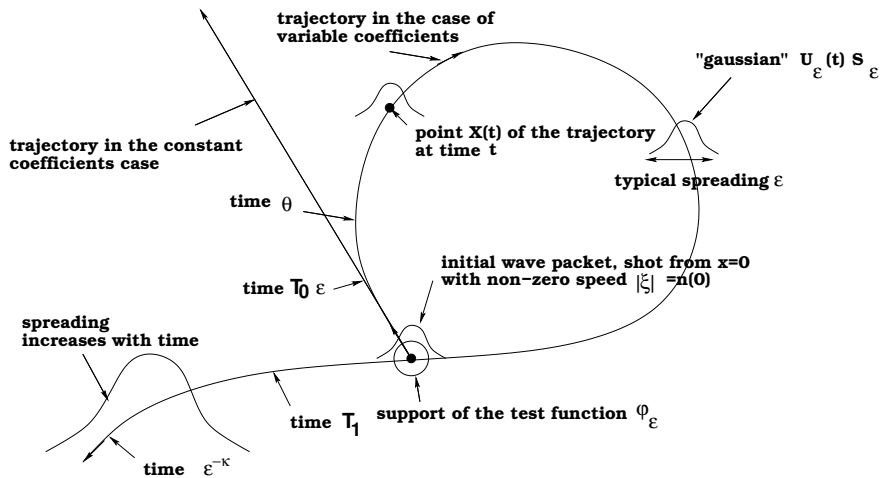
Our strategy is to pass to the limit in this very integral.

More precisely, we wish to prove that the quantity associated with the **non-constant coefficients** propagator (corresponding to the curved trajectory in the picture below), namely

$$\langle w^\varepsilon, \phi \rangle = \frac{i}{\varepsilon} \int_0^{+\infty} e^{-\alpha_\varepsilon t} \left\langle \exp\left(i\frac{t}{\varepsilon}\left(\frac{\varepsilon^2}{2}\Delta + n^2(x)\right)\right) S_\varepsilon, \phi_\varepsilon \right\rangle dt, \tag{7}$$

is asymptotic to the analogous quantity with **coefficients frozen at the origin** (corresponding to the straight line in the picture below), namely

$$\langle w^{\text{out}}, \phi \rangle = \frac{i}{\varepsilon} \int_0^{+\infty} \left\langle \exp\left(i\frac{t}{\varepsilon}\left(\frac{\varepsilon^2}{2}\Delta + n^2(0)\right)\right) S_\varepsilon, \phi_\varepsilon \right\rangle dt. \tag{8}$$

4.2. Second step: passing to the limit from (7) to (8)

In order to pass to the limit $\varepsilon \to 0$ in (7), we need to analyze the contributions of various time scales in the corresponding time integral. More precisely, we choose for the whole subsequent analysis two (large) cutoff parameters in time, denoted by T_0 and T_1, and one (small) cutoff parameter θ. We analyze the contributions to the time integral (7) that are due to the four regions

$$0 \leq t \leq T_0\varepsilon, \ T_0\varepsilon \leq t \leq \theta, \ T_0\varepsilon \leq t \leq T_1, \ \text{and} \ t \geq T_1.$$

We also choose a (small) exponent $\kappa > 0$, and we occasionally treat separately the contributions of very large times
$$t \geq \varepsilon^{-\kappa}.$$

Associated with these truncations, we take once and for all a smooth cutoff function χ defined on **R**, such that

$$\chi(z) \equiv 1 \ \text{when} \ |z| \leq 1/2, \ \chi(z) \equiv 0 \ \text{when} \ |z| \geq 1,$$
$$\chi(z) \geq 0 \ \text{for any} \ z. \tag{9}$$

To be complete, there remains to finally choose a (small) cutoff parameter in energy $\delta > 0$. Accordingly we distinguish in the L^2 scalar product $\langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon \rangle$ between

energies close to (or far from) the zero energy, which is critical for our problem. In other words, we set the self-adjoint operator

$$\chi_\delta\left(H_\varepsilon\right) := \chi\left(\frac{H_\varepsilon}{\delta}\right).$$

This object is perfectly well defined using standard functional calculus for self-adjoint operators. We decompose

$$\langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\rangle = \left\langle U_\varepsilon(t)\,\chi_\delta(H_\varepsilon S_\varepsilon, \phi_\varepsilon\right\rangle + \left\langle U_\varepsilon(t)\,(1-\chi_\delta)\,(H_\varepsilon)\,S_\varepsilon, \phi_\varepsilon\right\rangle.$$

Following the above described decomposition of times and energies, we study each of the subsequent terms:

- **The contribution of small times** is

$$A_\varepsilon := \frac{1}{\varepsilon}\int_0^{2T_0\varepsilon}\chi\left(\frac{t}{T_0\varepsilon}\right)\,\mathrm{e}^{-\alpha_\varepsilon t}\,\langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\rangle\,dt.$$

We prove that this term actually gives the dominant contribution in (6), provided the cutoff parameter T_0 is taken large enough. This (easy) analysis essentially boils down to manipulations on the time dependent Schrödinger operator $i\partial_t + \Delta_x/2 + n^2(\varepsilon x)$, for *finite times* t of the order $t \sim T_0$ at most. Indeed, it is readily seen, going back to the microscopic scale $x \to \varepsilon x$ and $t \to \varepsilon t$, that

$$\begin{aligned}
A_\varepsilon &= \int_0^{2T_0}\chi\left(\frac{t}{T_0}\right)\,\mathrm{e}^{-\varepsilon\alpha_\varepsilon t}\,\left\langle\exp\left(it\left(\frac{1}{2}\Delta_x + n^2(\varepsilon x)\right)\right)S, \phi\right\rangle\,dt\\
&\sim \int_0^{2T_0}\chi\left(\frac{t}{T_0}\right)\,\left\langle\exp\left(it\left(\frac{1}{2}\Delta_x + n^2(0)\right)\right)S, \phi\right\rangle\,dt \quad\text{for any } \textit{finite } T_0\\
&\sim \int_0^{+\infty}\left\langle\exp\left(it\left(\frac{1}{2}\Delta_x + n^2(0)\right)\right)S, \phi\right\rangle\,dt \quad\text{for } T_0 \text{ large enough}\\
&= \langle w^{\mathrm{out}}, \phi\rangle.
\end{aligned}$$

In view of this result, the main Theorem is proved once it is established that all other (subsequent) contributions are small. This is the task we now perform.

- **The contribution of moderate and large times, away from the zero energy**, is

$$B_\varepsilon := \frac{1}{\varepsilon}\int_{T_0\varepsilon}^{+\infty}(1-\chi)\left(\frac{t}{T_0\varepsilon}\right)\,\mathrm{e}^{-\alpha_\varepsilon t}\,\left\langle U_\varepsilon(t)\,(1-\chi_\delta)\,(H_\varepsilon)\,S_\varepsilon, \phi_\varepsilon\right\rangle\,dt.$$

We prove that this term has a vanishing contribution, provided T_0 is large enough. This easy result relies on a non-stationnary phase argument in time, recalling that $U_\varepsilon(t) = \exp(-itH_\varepsilon/\varepsilon)$ and the energy H_ε is larger than $\delta > 0$.

- **The contribution of very large times, close to the zero energy** is

$$C_\varepsilon := \frac{1}{\varepsilon} \int_{\varepsilon^{-\kappa}}^{+\infty} \mathrm{e}^{-\alpha_\varepsilon t} \left\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon, \phi_\varepsilon \right\rangle dt.$$

We prove that this term has a vanishing contribution as $\varepsilon \to 0$. To do so, we use results proved by X.P. Wang [Wa]: these essentially assert that the operator $\langle x \rangle^{-s} \, U_\varepsilon(t)\chi_\delta(H_\varepsilon) \, \langle x \rangle^{-s}$ has the natural size $\langle t \rangle^{-s}$ as time goes to infinity, provided the critical zero energy is non-trapping. Roughly, the semiclassical operator $U_\varepsilon(t)\chi_\delta(H_\varepsilon)$ sends rays initially close to the origin, at a distance of the order t from the origin, when the energy is non trapping. Quantitatively, this information allows us to estimate

$$\left| \left\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon, \phi_\varepsilon \right\rangle \right| \leq C_s \, \langle t \rangle^{-s}, \quad \forall s \geq 0,$$

and the contribution of this scalar product to the above integral vanishes (provided s is large, and κ is small):

$$C_\varepsilon = O(\varepsilon^{-1+s\kappa}), \quad \text{for any } s \geq 0.$$

Note that the need for considering polynomially large times here ($t \geq \varepsilon^{-\kappa}$), stems from the ε^{-1} in front of the integral in time that defines C_ε.

The most difficult terms are the last two that we describe now.

- **The contribution of large times, close to the zero energy** is

$$D_\varepsilon := \frac{1}{\varepsilon} \int_{T_1}^{\varepsilon^{-\kappa}} \mathrm{e}^{-\alpha_\varepsilon t} \left\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon, \phi_\varepsilon \right\rangle dt.$$

The treatment of this term is similar in spirit, though much harder, to the analysis performed in the previous term. Using only information on the localization properties of $U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon$ and ϕ_ε, we prove that this term has a vanishing contribution, provided T_1 is large enough. To do so, we use ideas of Bouzouina and Robert [BR], to establish a version of the Egorov theorem that holds true for *polynomially large times* in ε.

Roughly, the statement is the following. On the one hand, ϕ_ε is localised close to $x = 0$. On the other hand, the term $\chi_\delta\left(H_\varepsilon\right)S_\varepsilon$ is microlocalised close to $x = 0$ *and* $\xi^2/2 = n^2(x)$. The Egorov Theorem, in the version of [BR] then asserts that, **up to a remainder term $R_\varepsilon(t,x)$** (that is quite explicitly estimated), the propagated quantity $U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon$ is microlocalised close to the trajectories, at time t, issued from $x = 0$ and $\xi^2/2 = n^2(0)$. Now, the non-trapping assumption implies that, for large enough times, such trajectories are away from the origin. As a consequence, up to the remainder $R_\varepsilon(t,x)$ again, the scalar product $\left\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon, \phi_\varepsilon \right\rangle$ vanishes for large times, due to orthogonality of the supports. In other words

$$D_\varepsilon \sim \frac{1}{\varepsilon} \int_{T_1}^{\varepsilon^{-\kappa}} \left\langle R_\varepsilon(t,x), \phi_\varepsilon \right\rangle dt$$

IV–12

provided T_1 is large enough. Hence, there only remains to estimate the error term in Egorov's Theorem. The article [BR] gives the typical estimate

$$\|R_\varepsilon(t,x)\|_{L^2(\mathbf{R}^d)} \le C_{N,\delta}\ \varepsilon^N \sup_{\substack{1\le|\alpha|\le N \\ |x|\le\delta \\ |\xi^2/2-n^2(x)|\le\delta}} \left| \frac{\partial^\alpha}{\partial(x,\xi)} \left(X(t,x,\xi), \Xi(t,x,\xi)\right) \right|,$$

where the trajectory $(X(t),\Xi(t))$ has been defined in (13), and the initial data (x,ξ) run over a compact neighbourhood, of size δ, of $\{x=0, \xi^2/2 = n^2(x)\}$. In other words, the growth in time of $R_\varepsilon(t,x)$ is controlled by the growth of the *linearized flow*. In general, this term grows exponentially with time, wich is too strong a growth for our purpose. In our very case however, using the simplecticness of the flow $(X(t),\Xi(t))$, together with the fact that $n^2(x)$ goes to a constant at infinity, it turns out that the linearized flow has *polynomial growth* in time, i.e.

$$\sup_{\substack{1\le|\alpha|\le N \\ |x|\le\delta \\ |\xi^2/2-n^2(x)|\le\delta}} \left| \frac{\partial^\alpha}{\partial(x,\xi)} \left(X(t,x,\xi), \Xi(t,x,\xi)\right) \right| \le C_{N,\delta}\ t^{N^2}.$$

(The exponant N^2 here is very probably not optimal). As a consequence, we deduce the polynomial bound

$$\|R_\varepsilon(t,x)\|_{L^2(\mathbf{R}^d)} \le C_N\ \varepsilon^N\ t^{N^2},$$

from which it follows that

$$D_\varepsilon \sim \frac{1}{\varepsilon} \int_{T_1}^{\varepsilon^{-\kappa}} \varepsilon^N\ t^{N^2}\ dt \le \varepsilon^{N-N^2\kappa} \to 0,$$

provided κ is small.

• **The contribution of moderate times close to the zero energy** is

$$E_\varepsilon := \frac{1}{\varepsilon} \int_{T_0\varepsilon}^{T_1} (1-\chi)\left(\frac{t}{T_0\varepsilon}\right)\ \mathrm{e}^{-\alpha_\varepsilon t} \left\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right) S_\varepsilon, \phi_\varepsilon \right\rangle dt.$$

This is the most difficult term: contrary to all preceding terms, it **cannot** be analyzed using only geometric informations on the microlocal support of the relevant functions. Indeed, keeping in mind that the function $U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right) S_\varepsilon$ is localized on a trajectory initially shot from the origin, whereas ϕ_ε stays at the origin, it is clear that for times $T_0\varepsilon \le t \le T_1$, the support of $U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right) S_\varepsilon$ and ϕ_ε *may intersect*, due to trajectories passing *several times* at the origin. This might create a dangerous accumulation of energy at this point. For that reason, we need a precise evaluation of the semi-classical propagator $U_\varepsilon(t)$, for times up to the order $t \sim T_1$. This is done using the elegant wave-packet approach of M. Combescure and D. Robert [CRo] (see also [Ro], and the nice lecture [Ro2]), as we describe now.

Let us take a Gaussian wave packet centered at the point (q,p) in phase space:

$$\varphi_{q,p}^\varepsilon(x) := (\pi\varepsilon)^{-d/4}\ \exp\left(\frac{i}{\varepsilon}p\cdot\left(x-\frac{q}{2}\right)\right)\ \exp\left(-\frac{(x-q)^2}{2\varepsilon}\right).$$

It has been proved in [CRo] that, at least for bounded values of time, the propagator $U_\varepsilon(t)$ has a quite explicit action on $\varphi^\varepsilon_{q,p}(x,\xi)$, namely,

$$U_\varepsilon(t)\varphi^\varepsilon_{q,p}(x) = (\pi\varepsilon)^{-d/4}\ \exp\left(\frac{i}{\varepsilon}p_t \cdot \left(x - \frac{q_t}{2}\right)\right)\ \exp\left(-\Gamma(t,q,p)\frac{(x-q_t)^2}{2\varepsilon}\right)$$

$$\times \exp\left(\frac{i}{\varepsilon}S(t,q,p)\right)\ P_N(t,\varepsilon,q,p;(x-q_t)/\sqrt{\varepsilon})$$

$$+ \text{remainder.} \tag{10}$$

This formula states in essence that an intial wave packet centered at (q,p) in phase space becomes, after propagation through $U_\varepsilon(t)$, a gaussian wave packet centered at $(q_t, p_t) = (X(t,q,p), \Xi(t,q,p))$, with a new (complex) "variance" $\Gamma(t,q,p)$ (a $d \times d$ symmetric matrix, that is explicitely computable in terms of the classical flow), and an additional phase factor $S(t,q,p)$ (an "action", which is again explicitely computable in terms of the classical flow). In formula (10), the corrective factor $P_N(t,\varepsilon,q,p;(x-q_t)/\sqrt{\varepsilon})$ is a polynomial of degree $2N$ in its last variables, that depends smoothly upon t,ε,q,p, and the remainder term is of size ε^N, N being some large integer. The important point in (10) is that the (complex) phase

$$\frac{i}{\varepsilon}p_t \cdot \left(x - \frac{q_t}{2}\right) - \Gamma(t,q,p)\frac{(x-q_t)^2}{2\varepsilon} + \frac{i}{\varepsilon}S(t,q,p),$$

as well as the amplitude P_N, are "explicitely" known in terms of classical quantities.

Hence, projecting S_ε over the gaussian wave packets, we may write

$$E_\varepsilon \approx \frac{1}{\varepsilon}\int_{T_0\varepsilon}^{T_1}(1-\chi)\left(\frac{t}{T_0\varepsilon}\right)\left\langle U_\varepsilon(t)\chi_\delta\left(H_\varepsilon\right)S_\varepsilon, \phi_\varepsilon\right\rangle dt$$

$$= \frac{1}{\varepsilon}(2\pi\varepsilon)^{-d}\int_{\mathbf{R}^{2d}}dqdp\int_{T_0\varepsilon}^{T_1}\left\langle\chi_\delta\left(H_\varepsilon\right)S_\varepsilon, U_\varepsilon(-t)\varphi^\varepsilon_{q,p}\right\rangle\left\langle\varphi^\varepsilon_{q,p}\phi_\varepsilon\right\rangle dt$$

and, using (10), we arrive after some computations at a formula of the form (very roughly)

$$E_\varepsilon \approx \varepsilon^{-(d+2)/2}\int_{T_0\varepsilon}^{T_1}dtd\xi d\eta\ (1-\chi)\left(\frac{t}{T_0\varepsilon}\right)\ A(t,\xi,\eta)\ \exp\left(i\ \frac{\Phi(t,\xi,\eta)}{\varepsilon}\right). \tag{11}$$

This formula involves a rather explicit (complex) phase Φ and amplitude A. Our goal is to prove with the help of (11) that E_ε is negligible.

To do so, we wish to apply the stationary phase formula in (11). Since integration by parts *in time* will be needed, this step requires some care. Indeed, close to the lower bound $T_0\varepsilon$, integration by parts in time creates diverging factors, due to the term $(1-\chi)(t/T_0\varepsilon)$ in (11). This is why we now need to further distinguish in (11) between times $T_0\varepsilon \leq t \leq \theta$ (for which one cannot use a pure stationary phase approach), and later times $\theta \leq t \leq T_1$.
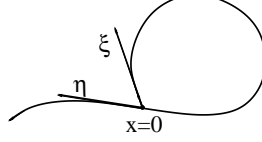
Times $\theta \leq t \leq T_1$

IV–14

For those times, one may use a stationary phase approach in t, ξ, η, to analyse the asymptotic behaviour of

$$E_\varepsilon^1 := \varepsilon^{-(d+2)/2} \int_\theta^{T_1} dt \int_{\mathbf{R}^{2d}} d\xi d\eta \ (1-\chi)\left(\frac{t}{\theta}\right) A(t,\xi,\eta) \ \exp\left(i \ \frac{\Phi(t,\xi,\eta)}{\varepsilon}\right).$$

It turns out that the stationary set $S := \{\mathrm{Im}\,\Phi = 0, \ \nabla_{t,\xi,\eta}\Phi = 0\}$ is exactly

$$S = \{(t,\xi,\eta) \in]\theta, +\infty[\times \mathbf{R}^{2d} \ \text{such that}$$
$$\xi^2/2 = n^2(0), \ X(t,0,\xi) = 0, \ \Xi(t,0,\xi) = \eta\}.$$

Assuming S is a smooth submanifold, we arrive at

$$E_\varepsilon^1 \sim \varepsilon^{[\mathrm{codim}\,S - d - 2]/2} \int_S dt d\xi d\eta \ (1-\chi)\left(\frac{t}{\theta}\right) \ A(t,\xi,\eta) \ \exp\left(i \ \frac{\Phi(t,\xi,\eta)}{\varepsilon}\right).$$

Thus, E_ε^1 vanishes asymptotically provided

$$\mathrm{codim}\,S > d+2, \quad \text{i.e.} \quad \dim S < d-1.$$

This is the geometric assumption (14) mentioned previously. Note that, in the case $\mathrm{codim}\,S = d+2$, it is in principle possible to compute the $O(1)$ quantity

$$\lim_{\varepsilon \to 0} E_\varepsilon^1 = \int_S dt d\xi d\eta \ (1-\chi)\left(\frac{t}{\theta}\right) \ A(t,\xi,\eta) \ \exp\left(i \ \frac{\Phi(t,\xi,\eta)}{\varepsilon}\right),$$

In the case $\lim_{\varepsilon \to 0} E_\varepsilon^1 \neq 0$, this observation gives a counterexample to the convergence $w^\varepsilon \to w^{\mathrm{out}}$.

For times $T_0\varepsilon \leq t \leq \theta$

For those times, the above argument fails, because one cannot use a stationary phase argument *in time*. In this case, one exploits at variance the fact that the classical trajectory associated with constant coefficients Hamiltonian $\xi^2/2 - n^2(0)$, is tangent with the classical trajectory associated with non-constant coefficients Hamiltonian $\xi^2/2 - n^2(x)$. In other words, one starts doing Taylor expansions *in the phase*, in the spirit of [Dsf], as we now explain.

Quantitatively, we write, after some computations

$$E_\varepsilon^2 := \varepsilon^{-(d+2)/2} \int_{T_0\varepsilon}^\theta dt \int_{\mathbf{R}^{2d}} d\xi d\eta \ A(t,\xi,\eta) \ \exp\left(i \ \frac{\Phi(t,\xi,\eta)}{\varepsilon}\right)$$
$$= \varepsilon^{-1} \int_{T_0\varepsilon}^\theta dt \int_{\mathbf{R}^d} d\xi \ \widetilde{A}(t,\xi) \ \exp\left(i \ \frac{\widetilde{\Phi}(t,\xi)}{\varepsilon}\right),$$

for some new amplitude and phase $\widetilde{A}(t,\xi)$ and $\widetilde{\Phi}$, that are computable in terms of A and Φ. In essence, we have here absorbed $\varepsilon^{-d/2}$ upon making the stationary phase

argument of the previous step *in the variable η only*. There remains to absorb the factor ε^{-1}, that corresponds to the stationary phase argument in time used in the previous step.

Here, we write, upon rescaling time by $t \to \varepsilon t$,

$$
\begin{aligned}
E_\varepsilon^2 &= \varepsilon^{-1} \int_{T_0\varepsilon}^\theta dt \int_{\mathbf{R}^d} d\xi \; \widetilde{A}(t,\xi) \, \exp\left(i \, \frac{\widetilde{\Phi}(t,\xi)}{\varepsilon} \right) \\
&= \int_{T_0}^{\theta/\varepsilon} dt \int_{\mathbf{R}^d} d\xi \; \widetilde{A}(\varepsilon t,\xi) \, \exp\left(i \, t \, \frac{\widetilde{\Phi}(\varepsilon t,\xi)}{\varepsilon t} \right).
\end{aligned}
$$

The difficulty now is to get integrability in the new time variable t, close to infinity. This is obtained upon exploiting the fact that $\varepsilon t \le \theta$ is a small parameter, and writing the Taylor expansion

$$
\frac{\widetilde{\Phi}(\varepsilon t,\xi)}{\varepsilon t} = \left(\partial_t \widetilde{\Phi} \right)(0,\xi) + O(\theta) = \xi^2/2 + O(\theta),
$$

where the second equatlity stems from an explicit computation. As a consequence, as time t becomes large, while εt remains $O(\theta)$, we have the standard dispersive estimate

$$
\int_{\mathbf{R}^d} d\xi \; \widetilde{A}(\varepsilon t,\xi) \, \exp\left(i \, t \, \frac{\widetilde{\Phi}(\varepsilon t,\xi)}{\varepsilon t} \right) = O(t^{-d/2}),
$$

from which it follows that

$$
E_\varepsilon^2 = \int_{T_0}^{\theta/\varepsilon} dt \int_{\mathbf{R}^d} d\xi \; \widetilde{A}(\varepsilon t,\xi) \, \exp\left(i \, t \, \frac{\widetilde{\Phi}(\varepsilon t,\xi)}{\varepsilon t} \right) = O(T_0^{-d/2+1}),
$$

is a negligible term as T_0 is large enough.

This ends our analysis. $\qquad\square$

References

[Ag] S. Agmon, *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup. Pisa, Vol. 2, N. 4, pp. 151-218 (1975).

[AH] S. Agmon, L. Hörmander, *Asymptotic properties of solutions of differential equations with simple characteristics*, J. Anal. Math. Vol. 30, pp. 1-37 (1976).

[BCKP] J.D. Benamou, F. Castella, Th. Katsaounis, B. Perthame, *High frequency limit in the Helmholtz equation*, Rev. Mat. Iberoamericana, Vol. 18, N. 1, pp. 187-209 (2002), and, Séminaire E.D.P., École Polytechnique, exposé N. V, 27 pp., 1999-2000.

[BR] A. Bouzouina, D. Robert, *Uniform semiclassical estimates for the propagation of quantum observables*, Duke Math. J., Vol. 111, no. 2, pp. 223-252 (2002).

[Bu] N. Burq, *Semi-classical estimates for the resolvent in nontrapping geometries*, Int. Math. Res. Not., no. 5, pp. 221-241 (2002).

[Bt] J. Butler, *Global H Fourier Integral operators with complex valued phase functions*, preprint Bologna (2001).

[C] R. Carles, *Remarks on nonlinear Schrödinger equations with harmonic potential*, Ann. Henri Poincaré, Vol. 3, N. 4, pp 757-772 (2002).

[CPR] F. Castella, B. Perthame, O. Runborg, *High frequency limit of the Helmholtz equation II: source on a general smooth manifold*, Comm. P.D.E., Vol. 27, N. 3-4, pp. 607-651 (2002).

[Ca] F. Castella, *The radiation condition at infinity for the high-frequency Helmholtz equation with source term: a wave packet approach*, preprint.

[CRu] F. Castella, O. Runborg, In preparation.

[CRo] M. Combescure, D. Robert, *Semiclassical spreading of quantum wave packets and applications near unstable fixed points of the classical flow*, Asymptot. Anal., Vol. 14, no. 4, pp. 377-404 (1997).

[CRR] M. Combescure, J. Ralston, D. Robert, *A proof of the Gutzwiller semiclassical trace formula using coherent states decomposition*, Comm. Math. Phys., Vol. 202, no. 2, pp. 463-480 (1999).

[DG] J. Dereziński, C. Gérard, **Scattering theory of classical and quantum N-particle systems**, Texts and Monographs in Physics, Springer-Verlag, Berlin (1997).

[DS] M. Dimassi, J. Sjöstrand, **Spectral Asymptotics in the Semiclassical Limit**, London Math. Soc. Lecture Notes series 268, Cambridge Universuty Press (1999).

[Dsf] D. Dos Santos Ferreira, *Inégalités de Carleman L^p pour des indices critiques*, Thesis, Université de Rennes 1 (2002).

[Fo] G.B. Folland, **Harmonic analysis in phase space**, Annals of Mathematics Studies, Vol. 122. Princeton University Press, Princeton, NJ (1989).

[Fou] E. Fouassier, Thesis, University Rennes 1, in preparation.

[GM] C. Gérard, A. Martinez, *Principe d'absorption limite pour des opérateurs de Schrödinger à longue portée*, C. R. Acad. Sci. Paris Sér. I Math., Vol. 306, no. 3, pp. 121-123 (1988).

[HJ] G.A. Hagedorn, A. Joye, *Semiclassical dynamics with exponentially small error estimates*, Comm. Math. Phys., Vol. 207, pp. 439-465 (1999).

[H1] G.A. Hagedorn, *Semiclassical quantum mechanics III*, Ann. Phys., Vol. 135, pp. 58-70 (1981).

[H2] G.A. Hagedorn, *Semiclassical quantum mechanics IV*, Ann. IHP, Vol. 42, pp. 363-374 (1985).

[HR] B. Helffer, D. Robert, *Caclcul fonctionnel par la transformation de Mellin et opérateurs admissibles*, J. Funct. Anal., Vol. 53, N. 3, pp. 246-268 (1983).

[He] K. Hepp, *The classical limit of quantum mechanical correlation functions*, Comm. Math. Phys., Vol. 35, pp. 265-277 (1974).

[Ho] L. Hörmander, *Symplectic classification of quadratic forms, and general Mehler formulas*, Math. Z. Vol. 219, N. 3, pp. 413-449 (1995).

[J] T. Jecko, *From classical to semi-classical non-trapping behaviour: a new proof*, Preprint University of Rennes 1 (2002).

[Ma] A. Martinez, **An introduction to Semicassical and Microlocal Analysis**, Universitext, Springer-Verlag, New York (2002).

[PV1] B. Perthame, L. Vega, *Morrey-Campanato estimates for Helmholtz Equation*, J. Funct. Anal., Vol. 164, N. 2, pp. 340-355 (1999).

[PV2] B. Perthame, L. Vega, *Sommerfeld radiation condition for Helmholtz equation with variable index at infinity*, Preprint (2002).

[Ro] D. Robert, *Remarks on asymptotic solutions for time dependent Schrödinger equations*, to appear.

[Ro2] D. Robert, notes on lectures given at the University of Nantes (1999-2000).

[Rb] S. Robinson, *Semiclassical mechanics for time-dependent Wigner functions*, J. Math. Phys., Vol. 34, pp. 2150-2205 (1993).

[Wa] X.P. Wang, *Time decay of scattering solutions and resolvent estimates for Semiclassical Schrödinger operators*, J. Diff. Eq., Vol. 71, pp. 348-395 (1988).

[WZ] X.P. Wang, P. Zhang, *High frequency limit of the Helmholtz equation with variable refraction index*, Preprint (2004).

IRMAR - Université de Rennes 1
Campus Beaulieu - 35042 Rennes Cedex - France
francois.castella@univ-rennes1.fr