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SEMICLASSICAL RESOLVENT ESTIMATES FOR TWO AND THREE-BODY SCHRÖDINGER OPERATORS

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Introduction

The purpose of this work is to establish resolvent estimates in the semiclassical limit for generalized three-body Schrödinger operators $H = -h^2\Delta + V(x)$ on $\mathbf{R}^n$, where V is of the form :

$$V(x) = \sum_{a\in A} V_a(\pi^a x) \quad \text{for a family} \quad \{\pi^a\}_{a\in A}$$

of orthogonal projections on vector subspaces X^a of $\mathbf{R}^n$. (See Section I for the precise definition). We also consider two-body Schrödinger operators when the energy level tends to zero.

For two-body Schrödinger operators it is known since the works of Robert-Tamura [Ro-Ta1] and Wang [Wa1] that a necessary and sufficient condition for an estimate of the type

$$\|\langle x\rangle^{-s}R(\lambda \pm i0, h)\langle x\rangle^{-s}\| = 0(h^{-1}) \quad \text{for} \quad s > 1/2 \tag{0.1}$$

to hold is that λ is a non trapping energy level for the classical hamiltonian $p(x,\xi) = \xi^2 + V(x)$. Here $R(\lambda, h)$ is the resolvent $(H-\lambda)^{-1}$ and $\|\ \|$ is the operator norm. Robert and Tamura gave a proof of the sufficiency using h-parametrices and Wang proved the necessity by deriving uniform decay estimates for e^{-itH}.

We address here the problem of establishing similar estimates for generalized three-body operators anf for two-body operators at low energies.

We use the method of [Ge-Ma], which is based on Mourre 's commutator technique and on the construction of a conjuguate operator for H by quantizing a classical "escape function" which increases along the classical flow.

The main difference with the two-body case is that the classical behavior near infinity is governed by the behavior of the subsystems (see Section II) which correspond to the occurence of clusters of particles.

It is hence necessary to make also hypotheses on the classical behavior of the subsystems.

We will say that an energy level E is non trapping for an Hamiltonian p if : $\forall(x;\xi) \in p^{-1}(E)$, $\exp t\, H_p(x,\xi) \to \infty$ when t tends to $\pm\infty$. Here H_p is the Hamilton vector field of p.

We emphasize that this non trapping condition is more general and from our point of view more natural than the virial type conditions.

We will prove the following results :

Theorem 1.—
Let $H = -h^2\Delta + V(x)$ be a generalized three-body operator where V satisfies the hypothesis (H.1) (see Section I), and λ_0 an energy level which is non trapping for p and for all classical subsystems of p. Then there exist $\varepsilon, \delta, C > 0$ such that the following estimates hold :

$$(0.2) \qquad \|(|G(x,hD_x)|+1)^{-s}R(\lambda \pm i0,h)(|G(x,hD_x)|+1)^{-s}\| \leq Ch^{-1}\,.$$

$$(0.3) \qquad \|\langle x\rangle^{-s}R(\lambda \pm i0,h)\langle x\rangle^{-s}\| \leq Ch^{-1} \quad \text{for} \quad s > 1/2$$

uniformly for $|\lambda - \lambda_0| \leq \varepsilon, \quad 0 < h \leq \delta.$

Here $G(x,hD_x)$ is an h-pseudodifferential operator which is the Weyl quantization of an escape function and $\langle x\rangle = (1+x^2)^{1/2}$.

A similar estimate has been obtained by Jensen [Je] for generalized N-body operators under a virial condition which implies that λ_0 is non-trapping for all subsystems.

Concerning the necessity of the non trapping condition, we have the following partial result : (we refer to Section I for the notations).

We consider an h-pseudodifferential operator A which is the Weyl quantization (see Section I) of a function $G(x,\xi)$ in $S(\langle x\rangle\langle\xi\rangle, g)$ such that

$G(x,\xi) = G^a(x^a,\xi^a) + \langle x_a, \xi_a\rangle$ in the domain
$J_a = \{x \in X \,|\, \forall b \not\subset a |x^b| > \varepsilon_0|x|, |x| \geq R\}$ for some $\varepsilon_0, R > 0$,
and $G^a \in S(\langle x^2\rangle\langle\xi^2\rangle, g_0)$.

An example of such an A is the generator of dilations $\frac{1}{2}(x \cdot D_x + D_x \cdot x)$.
Then we have the following Theorem :

Theorem 2.—

Let H be a three-body generalized operator as in Theorem 1 and λ_0 an energy level such that the following estimate holds :

(0.4) $\quad \|(|A|+1)^{-s}|R(\lambda \pm i0, h)(|A|+1)^{-s}\| \leq Ch^{-1}$ *for some* $s > 1/2$

uniformly for h small enough.

Then λ_0 is non trapping for all subsystems of p.

We consider also the case of a two-body operator at low energies :
we assume that $H = -h^2\Delta + V(x)$ where $V(x)$ is a two-body potential satisfying (H.1) in Section I and :

(0.5) $\; V(x) \geq C_0\,\langle x\rangle^{-\rho} \quad$ for $C_0 > 0$

(0.6) the sojourn time of classical trajectories on $p^{-1}(\varepsilon)$ in the ball $B(0, C_1\varepsilon^{-1/\rho})$ is bounded by $C_2\varepsilon^{-1/2-1/\rho}$ for some $C_1 \gg 1$.

Let us remark that the condition (0.6) is implied by a virial condition at zero energy of the type $2V(x) + x \cdot \frac{\partial V}{\partial x}(x) \leq 0$.
We get then the following result :

Theorem 3.—

Let H be a two-body operator satisfying (0.5),(0,6). Then

There exist $C_0, \varepsilon_0 > 0$, such that :

$\forall s > 1/2, \forall 0 < \lambda < \varepsilon_0, \forall 0 < h < h(\lambda)$, *one has :*

$$\|(\lambda^{1/2}\langle x\rangle + 1)^{-s} R(\lambda \pm i0, h)(\lambda^{1/2}\langle x\rangle + 1)^{-s}\| \leq C_0 h^{-1}\lambda^{-1}.$$

For $h = 1$ a similar estimate has been obtained by Robert-Tamura [Ro-Ta2] for positive potentials.

I. Weyl calculus for N-body Schrödinger operators.

In this section we introduce a class of generalized N-body Schrödinger operators which has been first considered by Agmon [Ag].

From the point of view of pseudodifferential calculus, this class of Schrödinger operators falls very naturally into the framework of the Weyl calculus with non conformal metrics developped by Hörmander.

We first review the definition of generalized N-body Schrödinger operators (G.S.O.) introduced by Agmon (see also [De]) ;

One considers a finite dimensional vector space X, with a positive definite quadratic form g, and a finite family $\{X_a\}$ $a \in \mathcal{A}$ of linear vector subspaces of X. One denotes by X^a the space $X_a^{\perp}$, by π^a, π_a the orthogonal projections on X^a and X_a. We will denote by x^a, x_a the vectors $\pi^a x$, $\pi_a x$, and by ξ^a, ξ_a the vectors $\pi'^a\xi$, $\pi'_a\xi$, for the projections π'^a , π'_a associated to the dual spaces X'_a, X'^a and the dual quadratic form g' on X'.

One puts on $\mathcal{A}$ a partial ordering by saying that $b \subset a$ if $X^b \subset X^a$.

We assume that the family $\{X_a\}$ has the following properties :

- if $a_1, a_2 \in \mathcal{A}$, there exists $a_3 \in \mathcal{A}$ such that $X_{a_1} \cap X_{a_2} = X_{a_3}$. With the ordering introduced above, one has : $a_3 = a_1 \cup a_2$.
- if $a_{\max} = \bigcup\limits_{a\in\mathcal{A}} a$, $X_{a_{\max}} = \{0\}$.
  (This amounts to require that $\bigcap\limits_{a\in\mathcal{A}} X_a = \{0\}$).
- $X \in \{X_a\}$, $X = X_{a_{\min}}$.

Finally one denotes by $\sharp a$ the maximal number of distinct a_i such that :

$$a = a_n \subset a_{n-1} \cdots \subset a_1 = a_{\max} \quad \text{and} \quad a_n \neq a_{n-1} \cdots \neq a_1 \,,$$

and one puts $N(\mathcal{A}) = \sharp a_{\min}$.
If $N(\mathcal{A}) = N$, one defines a N-body G.S.O. as :

$$H = g'(hD_x, hD_x) + \sum_{a\in\mathcal{A}} V_a(x^a) = g'(hD_x, hD_x) + V(x)\,, \quad \text{where :}$$

- $D_x = \frac{1}{i}\frac{\partial}{\partial x}$, g' is the dual quadratic form of g. (In other words $g'(hD_x, hD_x)$ is the Laplace-Beltrami operator associated to g).
- (H.1) $V_a \in C^\infty(X^a, \mathbf{R})$ satisfies :

$|D^\alpha_{x^a} V_a| \leq C_\alpha \langle x^a\rangle^{-\rho-|\alpha|}$ for $\rho > 0$, $\alpha \in \mathbf{N}^{\dim X^a}$ and $\langle x^a\rangle = (1+g(x^a, x^a))^{1/2}$.

From now on, we can as well assume that (X,g) is $\mathbf{R}^n$ with the usual metric and $H = -h^2\Delta + V(x)$. We will denote by $p(x,\xi) = \xi^2 + V(x)$ the classical symbol of H.

Let us briefly recall how regular N-body Schrödinger operators with two-body forces (see for example [C-F-K-S]) enter into this class :

in this case we have :

- $X = \{x \in \mathbf{R}^{3N} \mid \sum_{i=1}^{N} m_i x_i = 0\}$, where m_i is the mass of the particle i, and

  $g(x,x) = 2\sum_{i=1}^{N} m_i x_i . x_i$.

- $\mathcal{A}$ is the set of partitions of $\{1,\cdots N\}$ (clusters) and if $a = \{A_1,\cdots A_k\}$, one has :

  $X_a = \{x \in X \mid x_i = x_j \text{ if } i \text{ and } j \text{ belong to the same } A_\ell\}$,

  $X^a = \{x \in X \mid R_{A_i}(x) = \sum_{j\in A_i} m_j x_j = 0 \quad i = 1,\cdots,k\}$,

  $\sharp a = k$.

The use of G.S.O. allows, for example, to add k-body forces for $k \leq N$ to a regular N-body Schrödinger operator.

We introduce now the following metrics on $\mathbf{R}^n$ and on $T^*(\mathbf{R}^n)$:

$$g_x(\delta x) = \sum_{a\in\mathcal{A}} \frac{(\pi^a \delta x)^2}{\langle x^a\rangle^2}$$

$$g(\delta x, \delta\xi) = g_x(\delta x) + \frac{(\delta\xi)^2}{\langle\xi\rangle^2}$$

$\tilde{g}_x$ will be the dual metric of g_x, g^σ the dual metric of g with respect to the symplectic form σ, and $g_0(\delta x, \delta\xi) = \frac{\delta x^2}{\langle x\rangle^2} + \frac{\delta\xi^2}{\langle\xi\rangle^2}$.

We will use the definitions of [Hö, chap.18]. In particular, we denote by $S(m_1, g)$ for a g temperate weight function m_1, the Frechet space of C^∞ functions on $T^*(\mathbf{R}^n)\times]0,1]$, such that :

$$\forall k \in \mathbf{N}, |u|_g^k(x,\xi,h) = \sup_{t_i\in T^*(\mathbf{R}^n)} |u^{(k)}(x,\xi,h,t_1\cdots t_k)| \prod_1^k g(t_i)^{1/2} \leq C_k m_1(x,\xi).$$

To a function $a \in S(m,g)$, for a σ, g temperate weight m, we associate the following h-pseudodifferential operator :

$$(1.3)\quad \mathrm{Op}^w a(x,hD_x)u(x) = (2\pi h)^{-n}\int e^{i(x-y)\cdot\xi/h} a\Big(\frac{x+y}{2},\xi,h\Big)u(y)\,dy\,d\xi\,.$$

The symbolic calculus of [Hö, chap.18] applies to this semiclassical Weyl quantization. The weight function $\tilde{h}(x,\xi)$ which represents the gain in the symbolic calculus is here : $\tilde{h}(x,\xi) = \frac{R^{1/2}(x)}{\langle\xi\rangle}$, where $R(x)$ is the largest eigenvalue of g_x. We will need slightly more general operators, called h admissible in [He-Ro].

One can prove quite easily that the functional calculus of [He-Ro], extends to our situation, and show that if $f \in C_0^\infty(\mathbf{R})$, $f(H)$ is an h-admissible operator with principal symbol $f(\xi^2 + V(x))$.

The classical flow The next step is to construct Lyapunov functions for the classical flow in two cases : the case of a G.S.O. with $N = 3$, and the case of a two-body operator near the zero energy level.

The case of a classical three-body Hamiltonian

The behavior of a classical N-body system near infinity is described by its subsystems which are defined as follows :

for $a \in \mathcal{A}$, $a \neq a_{\max}$, we set $p^a(x^a,\xi^a) = \xi^{a2} + \sum_{b\subset a} V_b(x^b) = \xi^{a2} + V^a(x^a)$.

Definition I.1.— *An energy level $E \in \mathbf{R}$ is non trapping for p^a, $a \in \mathcal{A}$ if $\forall (x^a,\xi^a) \in (p^a)^{-1}(E)$, $\exp tH_{p^a}(x^a,\xi^a) \to \infty$ when t tends to $+\infty$ or to $-\infty$.*

Here H_{p^a} is the Hamiltonian vector field of p^a. One can prove the following Proposition :

Proposition I.2.— *Let $p(x,\xi)$ be a classical three-body Hamiltonian, and let $E \in \mathbf{R}$ be an energy level such that :*

- *E is non trapping for all p^a $a \in \mathcal{A}$, and $E \neq 0$.*

Then there exists a function $G(x,\xi) \in C^\infty(T^*(\mathbf{R}^n))$ such that :

- *$H_p G(x,\xi) \geq C_0 > 0$ on $p^{-1}([E-\varepsilon_0, E+\varepsilon_0])$ for $C_0, \varepsilon_0 > 0$.*
- *$G \in S(\langle x\rangle\langle\xi\rangle, g_0) + S(R(x)^{-1/2}\langle\xi\rangle, g)$.*

Let us make some remarks on this Proposition.

- **Remark 1 :** It is easy to check that the hypotheses of the Proposition require that E is strictly positive. The proof of the Proposition shows that if $N(\mathcal{A}) = 3$, and E is non trapping for all p^a $a \neq a_{\max}$, then H_p cannot have trapped trajectories near infinity on $p^{-1}(E)$.

The trapping energy levels for subsystems of p can therefore be thought of as *classical thresholds* for p.

- **Remark 2 :** The non trapping condition is more natural than the virial condition or generalizations of it used for example in [B.C.D]. For example, there are "reasonable" non trapping potentials which violate the virial condition. One can even construct a C_0^∞ non trapping potential in $\mathbf{R}^3$ which cannot have any escape function linear in ξ ([Sj]), thus violating any generalization of the virial condition.

- **Remark 3 :** We conjecture that the above Proposition is true for N-body Hamiltonians with $N > 3$, since it is really a classical version of the Mourre estimate for N-body G.S.O. (See [Fr-He]). So far, we have not been able to prove it.

The case of a two-body Hamiltonian at low energy

We consider now a classical two-body Hamiltonian $p(x,\xi) = \xi^2 + V(x)$, where V satisfies (H.1) and the hypotheses (0.5) (0.6) which we recall :

(0.5) $V(x) \geq C_0 |x|^{-\rho}$ in $|x| \geq 1$

(0.6) The sojourn time of classical trajectories on $p = \varepsilon$ in $\{|x| \leq C_1\, \varepsilon^{-1/\rho}\}$ is bounded by $C_2 \varepsilon^{-1/2-1/\rho}$, for C_1 big enough.

We have the following Proposition :

Proposition I.3.— *Under the hypotheses above, there exits for ε small enough a function $G_\varepsilon(x,\xi)$ satisfying :*

i) $H_p G_\varepsilon \geq C_0 \varepsilon$ *on* $p^{-1}([(1-\varepsilon_0)\varepsilon, (1+\varepsilon_0)\varepsilon])$ *for some* $C_0, \varepsilon_0 > 0$

ii) $|\partial_x^\alpha \partial_\xi^\beta G_\varepsilon| \leq C_{\alpha,\beta}(1+|x|)^{1-|\alpha|}(\varepsilon^{1/2} + |\xi|)^{1-|\beta|}$.

II Proof of the results

To prove the resolvent estimates of Theorems 1 and 3, we use Mourre 's commutator method (see for example [M], [P-S-S], [C-F-K-S], [Je]).

We construct a conjuguate operator for H by quantizing the escape functions constructed in Propositions I.2 and I.3 .

For a three-body operator, we have to control the h-dependence of the constants, and for a two-body operator the energy dependence of the constants is also important.

The case of three-body operator

The function $G(x,\xi)$ constructed in Prop.I.2 is in $S(\langle x\rangle\langle\xi\rangle, g)$, which allows us to define $A = \mathrm{Op}^w G(x, hD_x)$ as :

$$Au(x) = (2\pi h)^{-n} \int e^{i(x-y)\cdot\xi/h} G(\frac{x+y}{2}, \xi) u(y) dy d\xi \,, \tag{3.1}$$

for u in the Schwartz space $S(\mathbf{R}^n)$.

Since A is bounded from $S(\mathbf{R}^n)$ to $S(\mathbf{R}^n)$ and from $S'(\mathbf{R}^n)$ to $S'(\mathbf{R}^n)$ and formally self adjoint , we know that A is self adjoint with domain $D(A) = \{u \in L^2(\mathbf{R}^n) \mid Au \in L^2(\mathbf{R}^n\}$.

We now state a Proposition which contains the local positivity of $[H, iA]$ and the technical hypotheses on $[H, iA]$, $[[H, iA], iA]$ needed to prove the resolvent estimates.

We will denote by $X(t) \in C_0^\infty(\mathbf{R})$ a cutoff function supported in $|\lambda - E| \leq \varepsilon_0$.

Proposition II.1.—

i) $[H, iA]$ *is bounded from* $H^2(\mathbf{R}^n)$ *into* $L^2(\mathbf{R}^n)$ *with norm* $0(h)$.

ii) $[[H, iA], iA]$ *is bounded from* $H^2(\mathbf{R}^n)$ *into* $L^2(\mathbf{R}^n)$ *with norm* $0(h^2)$.

iii) $\exists C_0 > 0$ *such that* $X(H)[H, iA]X(H) \geq C_0 h X^2(H)$ *for* h *small enough.*

The proof of Theorem I is now standard . (see for example [C.F.K.S.]) .

Let us now indicate the proof of Theorem 2 . Let $a \in \mathcal{A}$ with $\sharp a = 2$ a subsystem of H. We will show that :

$$\|\langle x^a\rangle^{-2}(H^a - (\lambda \pm i0))^{-1}\langle x^a\rangle^{-2}\|_{0,0} \leq C_1 h^{-1} \quad \text{for} \quad \lambda \quad \text{near} \quad E \,. \tag{2.1}$$

This can be proved by constructing suitable sequences of functions corresponding to separation of the system into the given cluster a. To prove the Theorem 2 we use the result of Wang [Wa 2], which shows that an estimate as (3.6) implies that λ is non trapping for two-body Schrödinger operators. ∎

The case of a two body-operator at low energy

As a conjugate operator for H near the energy level E, we take the operator $X((H-E)/E)\mathrm{Op}^w G_E(x,hD_x)X((H-E)/E) = A_E$, where G_E is the escape function constructed in Prop I.3, and $X \in C_0^\infty(\mathbf{R})$ is a cutoff function supported in $[-\frac{\varepsilon_0}{2}, \frac{\varepsilon_0}{2}]$.

As before A_E is selfadjoint with domain
$D(A_E) = \{u \in L^2(\mathbf{R}^n) | A_E u \in L^2(\mathbf{R}^n)\}$.

We prove now a Proposition similar to Prop. II.1.

Proposition II.2.—

i) *$[H, iA_\varepsilon]$ is bounded on $L^2(\mathbf{R}^n)$ with norm $0(Eh)$ for $h \le h(E)$, $E \le E_0$.*

ii) *$[[H, iA_E], iA_E]$ is bounded on $L^2(\mathbf{R}^n)$ with norm $0(Eh^2)$ for $h \le h(E)$, $E \le E_0$.*

iii) *$\exists\, C_0 > 0$ such that $X(2(H-E)/E)[H, iA_E]X(2(H-E)/E) \ge C_0 hEX^2(2(H-E)/E)$, for $h \le h(E), E \le E_0$.*

The proof of Theorem 3 is now similar to that of Theorem 1 . The only somewhat delicate point is to control the h and E dependence of the estimates .

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