Journées Équations aux dérivées partielles

L. Cattabriga

L. Zanghirati

Global analytic and Gevrey surjectivity of the Mizohata operator $D_2 + ix_2^{2k} D_1$

Journées Équations aux dérivées partielles (1989), p. 1-4

<http://www.numdam.org/item?id=JEDP_1989____A15_0>

© Journées Équations aux dérivées partielles, 1989, tous droits réservés.

L'accès aux archives de la revue « Journées Équations aux dérivées partielles » (http://www.math.sciences.univ-nantes.fr/edpa/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/legal.php). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

Numdam

Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques

http://www.numdam.org/

GLOBAL ANALYTIC AND GEVREY SURJECTIVITY OF THE MIZOHATA OPERATOR $D_2 + ix_2^{2k}D_1$ (*)

L. Cattabriga (**)

L. Zanghirati (***)

Global analytic surjectivity of all linear constant coefficient partial differential operators on $\mathbf{R}^2$ has been proved by E. De Giorgi and L. Cattabriga [5], where counterexamples for the case of operators on $\mathbf{R}^3$ are also indicated (see also E. De Giorgi [4] and for other results for operators on open sets of $\mathbf{R}^n$: T.Kawai [10] and L.Hörmander [8]).Global Gevrey surjectivity of all linear constant coefficient partial differential operators on $\mathbf{R}^2$ was subsequentely proved by L. Cattabriga [3] in the case of Gevrey spaces with rational index (see also L. Cattabriga [2], G. Zampieri [15], R.W. Braun-R. Meise-D. Vogt [1] for the case of operators on $\mathbf{R}^n$).As for the case of linear operators with variable coefficients results on global Gevrey surjectivity are given by L.Ehrenpreis [6] and in the analytic case by T.Kawai [11].

Here we consider,as a simple example of an operator on $\mathbf{R}^2$ with variable coefficients,the Mizohata operator $D_2 + ix_2^{2k}D_1$,where k is a positive integer and $D_j=-i\partial_j$, j = 1,2, and prove the following

Theorem. Let $\mathcal{E}^{\{s\}}(\mathbf{R}^n)$, $s \geq 1$, be the space of all C^∞ functions f on $\mathbf{R}^n$ such that for every compact subset K of $\mathbf{R}^n$ there exists a constant A > 0 such that

---

(*) Research carried out in partial fulfilment of the project M.P.I. 40% "Problemi per equazioni a derivate parziali lineari e non lineari" and of the C.E.E. contract n. ST2J-0096-2-I (EDB).

(**) Dipartimento Matematico, Università di Bologna, Piazza di Porta S. Donato 5, I-40127 Bologna.

(***) Dipartimento Matematico, Università di Ferrara, Via Machiavelli 35, I-44100 Ferrara.

$$\sup_{x\in K}\sup_{\alpha\in Z_+^n} A^{-|\alpha|}\alpha!^{-s}|\partial_x^\alpha f(x)| < +\infty.$$

Then for every $s \geq 1$

i) $(D_2 + ix_2^{2k}D_1)\,\mathcal{E}^{\{s\}}(R^2) = \mathcal{E}^{\{s\}}(R^2)$,

ii) $(D_2 + ix_2^{2k}D_1)\,\mathcal{E}^{\{s\}}(R^3) \subsetneq \mathcal{E}^{\{s\}}(R^3)$.

Proof. Let $f\in \mathcal{E}^{\{s\}}(R^2)$, $s\geq 1$, and let $\sigma > \max\{s,1\}$. Since $f\in \mathcal{E}^{\{\sigma\}}(R^2)$, from a result by H.Komatsu [12] it follows that there exists $v_1 \in \mathcal{E}^{\{\sigma\}}(R^2)$ such that

$$(1)\qquad \begin{cases} v_1(x_1,0) = 0 \\ \partial_x^\gamma\Big[(D_2 + ix_2^{2k}D_1)v_1 - f\Big](x_1,0) = 0 \qquad \forall\gamma\in Z_+^2,\ x_1\in R. \end{cases}$$

Let

$$(2)\qquad h(x_1,x_2) = f(x_1,x_2) - (D_2 + ix_2^{2k}D_1)v_1(x_1,x_2)\ ,$$

and with the new variables in R^2

$$\begin{cases} y_1 = x_1 \\ y_2 = x_2^{2k+1}/(2k+1), \end{cases}$$

set

$$g(y_1,y_2) = \begin{cases} h(y_1,[(2k+1)y_2]^{1/(2k+1)})[(2k+1)y_2]^{-2k/(2k+1)} & \text{for } y_2\neq 0 \\ 0 & \text{for } y_2=0. \end{cases}$$

In view of (1), $g\in C^\infty(R^2)$.Hence,by the global $C^\infty(R^2)$-surjectivity of all linear partial differential operators with constant coefficients,there exists $w\in C^\infty(R^2)$ such that

$$(D_2 + iD_1)w(y_1,y_2) = g(y_1,y_2) \qquad \text{on } R^2.$$

Letting

$$v_2(x_1,x_2) = w(x_1,x_2^{2k+1}/(2k+1))\quad ,\quad (x_1,x_2)\in R^2,$$

it follows that

$$v_2 \in C^\infty(R^2)$$

and

(3) $$(D_2 + ix_2^{2k}D_1)v_2(x_1,x_2) = x_2^{2k}g(x_1, x_2^{2k+1}/(2k+1)) = h(x_1,x_2)\,,\ (x_1,x_2) \in \mathbf{R}^2.$$

Thus from (2) and (3)

$$(D_2 + ix_2^{2k}D_1)(v_1+v_2) = f \qquad \text{on } \mathbf{R}^2,$$

where

$$v = v_1+v_2 \in C^\infty(\mathbf{R}^2).$$

Recalling now that the Mizohata operator $(D_2+ix_2^{2k}D_1)$ is (analytic and) s-Gevrey hypoelliptic on $\mathbf{R}^2$ for every $s \geq 1$ (see for example S.Mizohata [13], L.Rodino [14], and also L.Hörmander [9]), we conclude that v is in fact in the same space $\mathcal{E}^{\{s\}}(\mathbf{R}^2)$ as f is. This proves part i) of the theorem.

To prove part ii), let $f \in \mathcal{E}^{\{s\}}(\mathbf{R}^3)$, $s \geq 1$, and define

$$g(x_1,x_2,x_3) = f(x_1, x_2^{2k+1}/(2k+1), x_3)\ x_2^{2k} \qquad , \quad (x_1,x_2,x_3) \in \mathbf{R}^3.$$

Since $g \in \mathcal{E}^{\{s\}}(\mathbf{R}^3)$, if $(D_2 + ix_2^{2k}D_1)\ \mathcal{E}^{\{s\}}(\mathbf{R}^3) = \mathcal{E}^{\{s\}}(\mathbf{R}^3)$, there exists $w \in \mathcal{E}^{\{s\}}(\mathbf{R}^3)$ such that $(D_2+ ix_2^{2k}D_1)w(x_1,x_2,x_3) = g(x_1,x_2,x_3)$ on $\mathbf{R}^3$.

Hence $v(x_1,x_2,x_3) = w(x_1,[(2k+1)x_2]^{1/(2k+1)}, x_3)$ would be a $\mathcal{D}'(\mathbf{R}^3)$ solution of $(D_2+iD_1)v = f$. Since the operator D_2+iD_1 is partially hypoelliptic in $\mathbf{R}^3$ with respect to the (x_1,x_2) variables, from the corollary of Theorem 4.1.2 of [7], it follows that $v \in \mathcal{E}^{\{s\}}(\mathbf{R}^3)$ .So the operator $D_2 +iD_1$ on R^3 would be s-surjective,what is false as it has been proven in [3] and [4].

References

[1] R.W.Braun - R.Meise - D.Vogt, "Applications of the projiective limit functor to convolution and partial differential equations",Proceeding of the NATO ASW on "Fréchet spaces", Istanbul August 1988, D.Reidel Publishing Comp., (to appear).

[2] L.Cattabriga, "Solutions in Gevrey spaces of partial differential equations with constant coefficients" , Astérisque 89-90 (1981), 129-151.

[3] L.Cattabriga, "On the surjectivity of differential polynomials on Gevrey spaces", Rend.Sem.Mat.Univ.Politec.Torino,Fasc.speciale,Convegno "Linear partial and pseudo-differential operators",Torino, 30 Sett.-2 Ott. 1982,41 (1983), 81-89.

[4] E.De Giorgi,"Solutions analytiques des équations aux dérivées partielles à coefficients constants", Séminaire Goulaouic-Schwartz 1971-72, Equations aux dérivées partielles et analyse fonctionelle, Exp. N. 29, Ecole Polytech. Centre de Math. Paris 1972.

[5] E. De Giorgi - L. Cattabriga, "Una dimostrazone diretta dell'esistenza di soluzioni analitiche nel piano reale di equazioni a derivate parziali a coefficienti costanti", Boll.Un.Mat.Ital. (4) 4 (1971), 1015-1027.

[6] L. Ehrenpreis, "Lewy's operator and its ramifications", J .Funct.Analysis 68 (1986), 329-365.

[7] J. Friberg, "Estimates for partially hypoelliptic differential operators", Medd. Lund Univ. Mat. Sem. 17 (1963), 1-93.

[8] L.Hörmander, "On the existence of real analytic solutions of partial equations with constant coefficients", Inventiones Math. 21 (1973), 151-182.

[9] L.Hörmander, "The analysis of linear partial differential operators", Springer-Verlag, (1983), vol. IV.

[10] T.Kawai, "On the global existence of real analytic solutions of linear differential equations" I and II, J. Math. Soc. Japan 24 (1972), 481-517; 25 (1973), 644-647.

[11] T.Kawai, "On the global existence of real analytic solutions and hyperfunction solutions of linear differential equations", Publ. RIMS Kyoto University 555 (1986).

[12] H.Komatsu, "An analogue of the Cauchy-Kowalevsky theorem for ultradifferentiable functions and a divsion theorem of ultradistributions as its dual", J. Fac. Sci. Univ. Tokyo, Sec. IA 26 (1979), 239-254.

[13] S.Mizohata, "Solutions nulles et solutions non analytiques", J. Math. Kyoto Univ. 1 (1962), 271-302.

[14] L.Rodino, "On linear partial differential operators with multiple characteristics", Proceedings Conf. on partial Differential Equations, Holzhau (GDR), 1988, Teubner Text zur Mathematik.

[15] G.Zampieri, "An application of the fundamental principle of Ehrenpreis to the existence of global Gevrey solutions of linear differential equations", Boll. Un. Mat. Ital. (6) V B (1986), 361-392.