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LONG – RANGE SCATTERING OF TWO – AND THREE – BODY QUANTUM SYSTEMS

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Abstract. For two- and three-particle Schrödinger operators we give an elementary and essentially self-contained proof for existence and completeness of the Dollard wave operators. The gradient of the long-range part of the pair potentials has to decay like $(1+|x|)^{-\delta}$, $\delta > \sqrt{3}$ as $|x| \to \infty$. No implicit conditions are needed.

1 Introduction

In these lectures we prove existence and completeness of modified wave operators for two- and three-body systems. The potentials are of long and short range. Most of the results are already contained in [2] but the proof has been significantly simplified and shortened and we do not need implicit conditions. One major new tool is an absorbing phase space decomposition where each component has a positive total time derivative up to integrable corrections. This approach has been presented for short-range potentials in [4]. From the related method which Sigal and Soffer [7] gave for short-range N-body systems our method differs mainly in the phase space partition. Here it depends on position, velocity and *time*. The other simplification compared to [2] is the introduction of a better intermediate time evolution which is easier to control. As a new result all implicit conditions on bound states for two-body subsystems are eliminated by an observation of Wüller [8]. For simplicity of presentation we give the proof for bounded pair potentials, the inclusion of operator- or form-bounded potentials of short range is a straightforward technical exercise. As usual, the trivial free motion of the center of mass of the whole system is separated off. In order to present the method as clearly as possible we give a detailed exposition for two-body systems first. Then we proceed to three particles.

In the TWO-BODY case (potential scattering) we consider Schrödinger operators

$$H = H_0 + V = -\frac{1}{2m}\,\Delta_x + V(x) \tag{1.1}$$

on $\mathcal{H} = L^2(I\!R^\nu)$ where $x \in I\!R^\nu$ is the relative position of the particles, m their reduced mass, and Δ_x the Laplacian with respect to x.

The bounded potential function $V(x)$ is the sum $V(x) = V^s(x) + V^\ell(x)$ of short- and long-range parts. We assume

$$\sup_{|x|\geq R} |V^s(x)| \in L^1([0,\infty), dR) \tag{1.2}$$

and $V^\ell \in C^1(I\!R^\nu)$ with

$$|(\nabla V^\ell)(x)| \leq C\,(1+|x|)^{-1-\gamma}, \quad \gamma > 0. \tag{1.3}$$

The splitting into short- and long-range parts is not unique. Due to a lemma of Hörmander [5] it can be made without loss of generality such that in addition $V^\ell \in C^\infty(I\!R^\nu)$ and

$$|(\Delta V^\ell)(x)| \leq C(\varepsilon_0)\,(1+|x|)^{-1-2\gamma+\varepsilon_0} \quad \text{for any } \varepsilon_0 > 0. \tag{1.4}$$

Stronger decay assumptions like $\gamma > 1/2$ for the simplest proof of asymptotic completeness will be introduced where needed. The short-range part of the potential need not be a multiplication operator but it could be a pseudo-differential operator with suitable decay properties describing a velocity dependent force. Under our assumptions H is self-adjoint on $W^{2,2}(I\!R^\nu)$ and the unitary group of time evolutions $\exp\{-iHt\}$ is well defined.

Asymptotic Completeness is a complete classification theorem which distinguishes the states in $\mathcal{H}$ by their asymptotic time evolution. The state space $\mathcal{H}$ can be split into a direct sum of components such that on each component the asymptotic evolution in the future is well approximated by a simple explicitly known one (and similarly for the past). On these subspaces the generator H is unitarily equivalent to a free one. The equivalence can be established using the *modified wave operators* Ω^D as introduced by Dollard. For the two-body case let U^D be the *modified free time evolution* generated by the time-dependent Hamiltonian $H^D(t) := H_0 + V^\ell(Qt)$ where Q is the velocity operator

$$Q := p/m = (-i\nabla_x)/m = i[H_0, x]. \tag{1.5}$$

Then U^D can be calculated explicitly as a multiplication operator in velocity- or momentum-space

$$U^D(t,\tau) = \exp\left\{-iH_0(t-\tau) \;-\; i\int_\tau^t ds V^\ell(Qs)\right\}. \tag{1.6}$$

Here τ is the initial and $t \geq \tau$ the final time of the evolution which coincides with the free time evolution in the short-range case $V^\ell = 0$. For $\gamma > 1/2$ we give an elementary proof for *existence* of the modified wave operator

$$\Omega^D := \text{s-}\lim_{t\to\infty} \exp\{iHt\}\, U^D(t,0) \tag{1.7}$$

and of *completeness:* $\operatorname{Ran}\Omega^D = \mathcal{H}^{cont}(H)$ (the continuous spectral subspace of H). The latter is equivalent to existence of

$$\lim_{t\to\infty} U^D(t,0)^* \,\exp\{-iHt\}\,\Psi \tag{1.8}$$

for every Ψ in (a dense subset of) $\mathcal{H}^{cont}(H)$ (Section 2).

While our proof is simple and essentially self-contained one could also rely on known detailed microlocal estimates. As was pointed out to me by A. Martinez one shows easily existence and completeness for a larger class of long-range potentials (Section 3).

THREE-PARTICLE SYSTEMS interacting by pair potentials V^{ij} are treated in their center of mass frame as well.

$$H = H_0 + \sum_{i<j} V^{ij}(x^i - x^j) \tag{1.9}$$

is the Schrödinger operator acting on $\mathcal{H} = L^2(\mathbb{R}^{2\nu})$, $x^i - x^j \in \mathbb{R}^\nu$ is the relative coordinate for the pair (i,j) and each V^{ij} satisfies (1.2)-(1.4). Explicit expressions for H_0 will be given in Section 4.

For three particles one has to distinguish between decompositions d_k into k nonempty clusters. Thus there are the total decomposition d_3, three possibilities d_2, and the non-decomposition d_1. For each d_2 one has the cluster c representing the "pair" and the trivial cluster consisting of the "third particle" alone. The *decomposition Hamiltonians* $H(d_k)$ are those where the potentials are omitted which couple particles in different clusters. Thus $H(d_3) = H_0$, $H(d_2) = H_0 + V^{ij}$ if $(i,j) = c$ is the pair in d_2, $H(d_1) = H$.

If the particles i and m are in different clusters for d_k then $Q_{im}(d_k)$ denotes the relative velocity of the centers of mass of these two clusters. (See Section 4 for further details about the kinematics.) The modified Dollard time evolutions are generated by

$$H^D(t;d_k) := H(d_k) + \sum{}' \, V^{im,\ell}(Q_{im}(d_k)\,t) \tag{1.10}$$

where the decomposition-dependent $\sum'$ means the sum over those pairs $i < m$ lying in different clusters of d_k. Explicitly one obtains

$$U^D(t,\tau;d_k) = \exp\left\{-iH(d_k)(t-\tau) - i\int_\tau^t ds\ \sum{}' V^{im,\ell}(Q_{im}(d_k)\,s)\right\}. \quad (1.11)$$

Let $\mathbf{P}(d_2)$ be the bound state projection for the pair c in the decomposition d_2, i.e. $\operatorname{Ran}\mathbf{P}(d_2) = \mathcal{H}^{pp}(H(c))$ and $\mathbf{P}(d_3) = \mathbf{1}$. Then we show that the *modified Dollard wave operators* defined as

$$\Omega^D(d_k) := \mathrm{s}\!-\!\lim_{t\to\infty} \exp\{iHt\}\, U^D(t,0;d_k)\,\mathbf{P}(d_k) \quad (1.12)$$

exist if the decay rate of the long-range potentials satisfies $\gamma > 1/2$ and are complete:

$$\operatorname{Ran}\Omega^D(d_3) \bigoplus \bigoplus_{d_2} \operatorname{Ran}\Omega^D(d_2) = \mathcal{H}^{cont}(H) \quad (1.13)$$

if $\gamma > \sqrt{3}-1$. The physically relevant long-range potential of the Coulomb force with $\gamma = 1$ is included. Completeness (1.13) holds if every $\Psi \in \mathcal{H}^{cont}(H)$ can be split $\Psi = \Psi(d_3) + \sum_{d_2} \Psi(d_2)$ such that the following limits exist:

$$\lim_{t\to\infty} U^D(t,0;d_k)^* \exp\{-iHt\}\,\Psi(d_k), \quad (1.14)$$

and for every d_2

$$\lim_{t\to\infty} \left[\mathbf{1} - \mathbf{P}(d_2)\right] \exp\{-iHt\}\Psi(d_2) = 0. \quad (1.15)$$

This will be shown in Section 4. Clearly we have the same results with the same proofs for negative times $t \to -\infty$. Therefore we omit the usual indices $\pm$ at the wave operators.

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2 The Two-Body Problem

In this section we show existence and completeness of two-body wave operators (1.7), (1.8) under the assumptions made in the previous section. We use *asymptotic observables* to obtain a phase space localization of scattering states asymptotically in time. The operator Q is the velocity operator (1.5). The following theorem says that a scattering state eventually will move away from the scatterer into the region where the total and the kinetic energies coincide. Moreover, the instantaneous velocity Q and x/t, the average velocity up to time t, asymptotically coincide. In the long run a state will be localized there where it would be under the free time evolution starting near the origin at time zero. The choice if observables admits errors growing almost linearly in time. Since the statement is so rough it can be easily proved using mainly kinematics.

Theorem 2.1. *Let $\Psi \in \mathcal{H}^{cont}(H)$ be fixed. There is a sequence $\tau_n \to \infty$ such that*

(a) for any $g \in C_0^\infty(I\!R)$

$$\lim_{\tau_n \to \infty} \| [g(H) - g(H_0)] \exp\{-iH\tau_n\} \Psi\| = 0.$$

(b) For any $f \in C_0^\infty(I\!R^\nu)$

$$\lim_{\tau_n \to \infty} \| [f(Q) - f(x/\tau_n)] \exp\{-iH\tau_n\} \Psi\| = 0.$$

Idea of proof. Disregarding domain questions one has to show for (b) convergence to zero of

$$\| [(X/t) - Q] \exp\{-iHt\}\Psi\|^2$$

$$= t^{-2} \Big(\Psi, \exp\{iHt\} \left[x^2 - (xQ + Qx)t + Q^2t^2\right] \exp\{-iHt\}\Psi\Big). \tag{2.1}$$

The summands are related by differentiation:

$$(d/dt)^2 \Big(\Psi, \exp\{iHt\}\, x^2 \exp\{-iHt\}\Psi\Big)$$

$$= (d/dt)\ (\Psi, \exp\{iHt\}\ (xQ + Qx) \exp\{-iHt\}\Psi)$$

$$= \Big(\Psi, \exp\{iHt\}\ (2Q^2 + K) \exp\{-iHt\}\Psi\Big) =: 2\,h_1(t) + 2\,h_2(t),$$

where $K := iV^s xQ - ixQV^s + iV^sQx - iQxV^s - (2/m)(\nabla V^\ell)$ is relatively compact, i.e. $Kg(H) = \mathbf{C}$ is compact for any bounded decaying function g. Using Taylor's formula of second order one obtains for (2.1) with $(\Psi,\, x^2\, \Psi)/t^2 \to 0$

$$h_1(t) - t^{-2} \int_0^t 2s\, ds\, h_1(s) - t^{-2} \int_0^t 2s\, ds\, h_2(s).$$

As a consequence of Wiener's theorem for any compact $\mathbf{C}$ and any self-adjoint H

$$\left\| t^{-2} \int_0^t 2s\, ds\, \exp\{iHs\}\, \mathbf{C} \exp\{-iHs\} P^{cont}(H) \right\| \to 0. \tag{2.2}$$

Therefore the correction term with h_2 asymptotically vanishes. Since $[g(H) - g(H_0)]$ is compact as well one concludes similarly that part (a) holds. For suitable vectors Ψ the function h_1 is bounded and continuous. It cannot stay away from its weighted average forever and the difference has to approach zero for an infinite sequence of times. For the complete proof of a stronger statement see Section II of [2] or [1]. □

For each Ψ in a dense set of states in $\mathcal{H}^{cont}(H)$ there is a function $g \in C_0^\infty(I\!R)$ with $g(H)\,\Psi = \Psi$ and $\operatorname{supp} g \subset (0, \infty) \subset [0, \infty) = \sigma^{cont}(H)$. Then $g(H_0)$ characterizes a spherical shell **Sh** in velocity space which satisfies for some $v_0 > 0$

$$\mathbf{Sh} := \left\{ Q \in \mathbb{R}^\nu \,\middle|\, mQ^2/2 \in \operatorname{supp} g \right\} \subset \{Q \mid |Q| > 6v_0\}. \tag{2.3}$$

On this shell there is a finite positive smooth decomposition of the identity $\{f_j\}_{j=1,..,J}$ with $0 \le f_j(Q) \le 1$, $\sum_j f_j(Q) = 1$ for all $Q \in \mathbf{Sh}$, such that for suitable $v_j \in \mathbf{Sh}$ one has $\operatorname{supp} f_j \in B_{v_0}(v_j)$ (the ball of radius v_0 around v_j), $f_j^{1/2} \in C_0^\infty(\mathbb{R}^\nu)$. Corresponding to each f_j define a "covering function"

$$F_j(v) := \mathcal{F}\left(m|v - v_j|^2/2\right) \in C_0^\infty(\mathbb{R}^\nu), \tag{2.4}$$

where $\mathcal{F}$ satisfies $\mathcal{F}(mw^2/2) = 1$ (or 0) if $w \le (\sqrt{3}v_0)$ (or $w \ge (2v_0)$) and is monotone: $\mathcal{F}' \le 0$, $(-\mathcal{F}')^{1/2} \in C_0^\infty(\mathbb{R})$.

We define a shorthand for the *phase space localization operators*

$$L_j(t) := F_j(x/t)\, f_j(Q), \quad \|L_j(t)\| = 1. \tag{2.5}$$

Obviously one has with $L_j(t)^* = f_j(Q)\, F_j(x/t)$

$$\|L_j(t) - L_j(t)^*\| = \|\, [F_j(x/t),\, f_j(Q)]\, \| \le \text{const}/t \quad \text{for all } j. \tag{2.6}$$

For the next statement we will use only that the f_j are a decomposition of the identity on $\mathbf{Sh}$ and that $f_j\, F_j = f_j$. The other properties will be needed later.

Corollary 2.2. *With g, f_j, F_j as above and $\Psi = g(H)\,\Psi \in \mathcal{H}^{cont}(H)$*

$$\lim_{\tau_n \to \infty} \left\| \left\{ \mathbf{1} - \sum\nolimits_j L_j(\tau_n) \right\} \exp\{-iH\tau_n\}\,\Psi \right\| = 0, \tag{2.7}$$

and similarly for $L_j(\tau_n)^*$, τ_n as in Theorem 2.1.

Proof. With summation over finitely many j

$$\exp\{-iH\tau_n\}\,\Psi = g(H)\, \exp\{-iH\tau_n\}\,\Psi \approx g(H_0)\, \exp\{-iH\tau_n\}\,\Psi$$

$$= \sum f_j(Q)\, F_j(Q)\, g(H_0)\, \exp\{-iH\tau_n\}\,\Psi$$

$$\approx \sum f_j(Q)\, F_j(Q)\, \exp\{-iH\tau_n\}\,\Psi \approx \sum f_j(Q)\, F_j(x/\tau_n)\, \exp\{-iH\tau_n\}\,\Psi$$

$$= \sum\nolimits_j L_j(\tau_n)^*\, \exp\{-iH\tau_n\}\,\Psi \approx \sum\nolimits_j L_j(\tau_n)\, \exp\{-iH\tau_n\}\,\Psi.$$

In all occurrences of $\approx$ the error vanishes as $\tau_n \to \infty$ by Theorem 2.1 and (2.6). □

We use the notation for norm-integrable bounded operator valued functions

$$A(t) \in L^1 \quad \text{or} \quad A(t) \approx^1 0 \quad \text{if} \quad \|A(t)\| \in L^1([1,\infty), dt)$$

and analogously for $A(t) \approx^1 B(t)$. $A(t) = O(t^{-\ell})$ means that $t^\ell \cdot \|A(t)\|$ is bounded. It is convenient to introduce the time-dependent "tail part" V_t of the long-range potential V^ℓ as

$$V_t(x) := V^\ell(x)\, \varphi(|x/v_0 t|), \quad \|V_t\| \le C\,(1+|t|)^{-\gamma}, \tag{2.8}$$

where $(1-\varphi) \in C_0^\infty(\mathbb{R})$, $\varphi(q) = 0$ or 1 if $|q| \le 1/2$ or $|q| > 1$, $0 \le \varphi(q) \le 1$, $v_0 > 0$ as chosen in (2.3). One easily concludes from (1.3), (1.4)

$$\sup_z |(\nabla V_t)(z)| \le C\,(1+|t|)^{-1-\gamma} \in L^1 \text{ for all } \gamma > 0, \tag{2.9}$$

$$\sup_z |(\Delta V_t)(z)| \le C(\varepsilon_0)\,(1+|t|)^{-1-2\gamma+\varepsilon_0}, \text{ for any } \varepsilon_0 > 0. \tag{2.10}$$

The inner cutoff – here between the speeds v_0 and $v_0/2$ – can be adjusted to the kinematics. Energies below $m\,{v_0}^2/2$ must be excluded by $g(H)$. Then V^ℓ and V_t coincide in the region where the particle should be at time t according to its energy. The size of the cutoff only affects the constants in (2.9), (2.10). The time evolution corresonding to V_t is $U(t,\tau)$ with $U(\tau,\tau) = \mathbf{1}$ and

$$H(t) := H_0 + V_t(x), \quad i\,(d/dt)\,U(t,\tau) = H(t)\,U(t,\tau). \tag{2.11}$$

The Propagator U exists because V_t is a boundedly differentiable operator valued function of t. Evidently U is the free time evolution $\exp\{-iH_0(t-\tau)\}$ in the short-range case $V^\ell = 0$. U will play the role of an *intermediate* time evolution later on.

Lemma 2.3. *For all $j = 1, ..., J$ the following quantities are integrable in t.*

$$V^s(x)\,L_j(t), \quad L_j(t)\,V^s(x), \quad (V^\ell(x) - V_t(x))\,L_j(t),$$

$$L_j(t)\,(V^\ell(x) - V_t(x)), \quad [V^\ell(x),\, L_j(t)], \text{ and } [V_t(x),\, L_j(t)] \in L^1.$$

Proof. The first is evident from the decay assumption (1.2) of $V^s(x)$ and the fact that $\operatorname{supp} F_j(x/t) \subset \{x \in \mathbb{R}^\nu \mid |x| > 4v_0 t\}$ for each j. The third vanishes identically. Since $f_j(Q)$ in x-space acts as convolution with the rapidly decaying Fourier transform of f_j one has

$$\|F_j(x/t)\, f_j(Q)\, F(|x| \le 3\,v_0 t)\| \le C_N\, t^{-N} \tag{2.12}$$

for any $N \in \mathbb{N}$. As usual, F without an index denotes the multiplication operator with the characteristic function of the indicated region. With (2.12) also the second and fourth statements follow. For the remaining two terms

$$[V^\ell(x),\ L_j(t)] = F_j(x/t)\,[V^\ell(x),\ f_j(Q)] \approx^1 F_j(x/t)\ [V_t(x),\ f_j(Q)],$$

$$\|\,[V_t(x),\ f_j(Q)]\,\| \leq \text{const}(f_j)\cdot \sup|\nabla V_t| \in L^1$$

by (2.8) and (2.9). □

The next proposition is the crucial ingredient of this approach. Any positive operator can be written in the form A^*A, we need not know A explicitly. The essential positivity of the "total time derivatives" says that the phase space regions characterized by $L_j(t)$ are absorbing under all time evolutions, even certain mixed ones, up to harmless errors which are integrable in time.

Proposition 2.4. *For each $L_j(t)$ as chosen in* (2.5) *and U as given in* (2.11)

$$\begin{aligned}
&\exp\{-iHt\}\ d/dt\Big\{\exp\{iHt\}\ L_j(t)\ \exp\{-iHt\}\Big\}\ \exp\{iHt\}\\
&\approx^1 \exp\{-i(H_0+V^\ell)t\}\ d/dt\ \Big\{\exp\{i(H_0+V^\ell)t\}\ L_j(t)\ \exp\{-iHt\}\Big\}\exp\{iHt\}\\
&\approx^1 U(t,0)\ d/dt\Big\{U(t,0)^*\ L_j(t)\ \exp\{-iHt\}\Big\}\exp\{iHt\}\\
&\approx^1 \exp\{-iHt\}\ d/dt\Big\{\exp\{iHt\}\ L_j(t)\ U(t,0)\Big\}U(t,0)^*\\
&\approx^1 U(t,0)\ d/dt\Big\{U(t,0)^*\ L_j(t)\ U(t,0)\Big\}U(t,0)^*\\
&\approx^1 \exp\{-iH_0t\}\ d/dt\ \Big\{\exp\{iH_0t\}\ L_j(t)\ \exp\{-iH_0t\}\Big\}\exp\{iH_0t\}\\
&= i\,[H_0,\ L_j(t)] + \partial_t\,L_j(t) \approx^1 A_j(t)^*\ A_j(t)\ =\ O(t^{-1}).
\end{aligned} \tag{2.13}$$

The operators $A_j(t)$ can be chosen independent of the potentials.

Proof. $\exp\{-iHt\}\ d/dt\,\{\ \exp\{iHt\}\ L_j(t)\ \exp\{-iHt\}\}\ \exp\{iHt\}$

$$= \{i\,[H_0,\ L_j(t)] + \partial_t\,L_j(t)\} + i\,V^s\,L_j(t) - i\,L_j(t)\,V^s + i[V^\ell,\ L_j(t)]. \tag{2.14}$$

The three potential-terms are individually integrable by Lemma 2.3. Most other time derivatives in the proposition differ from (2.14) by the absence of some of these terms. For the third line one gets also $(V^\ell - V_t)\,L_j(t) \in L_1$, and similarly for the fourth. In the fifth line we have $i[V_t(x),\ L_j(t)]$. It remains to show the essential positivity of the terms which are independent of all potentials:

$$\{\, i\, [H_0,\; F_j(x/t)] + \partial_t\, F_j(x/t)\}\; f_j(Q)$$

$$= \left\{ \frac{mi}{2} \left[Q^2, \mathcal{F}\left(\frac{m}{2} \left| \frac{x}{t} - v_j \right|^2 \right) \right] - \mathcal{F}'\left(\frac{m}{2} \left| \frac{x}{t} - v_j \right|^2 \right) \left(\frac{x}{t} - v_j \right) \cdot \frac{x}{t} \frac{m}{t} \right\} f_j(Q)$$

$$= U_j(t)^* \left\{ \frac{mi}{2} \left[(Q + v_j)^2, \mathcal{F}\left(\frac{mx^2}{2t^2} \right) \right] - \mathcal{F}'\left(\frac{mx^2}{2t^2} \right) \frac{x}{t} \cdot \left(\frac{x}{t} + v_j \right) \frac{m}{t} \right\}$$

$$\times \quad f_j(Q + v_j) \quad U_j(t)$$

$$= U_j(t)^* \left\{ \frac{mi}{2} \left[Q^2, \mathcal{F}\left(\frac{mx^2}{2t^2} \right) \right] - \mathcal{F}'\left(\frac{mx^2}{2t^2} \right) \frac{x}{t} \cdot \frac{x}{t} \frac{m}{t} \right\} f_j(Q + v_j)\; U_j(t).$$

Here $U_j(t)$ is the unitary Galilei transformation mapping $Q \to Q + v_j$, $x \to x + v_j t$. In the new variables the velocity Q is restricted by the support of f_j to the ball $B_{v_0}(0)$ and x/t in the support of $\mathcal{F}'$ lies in $B_{2v_0}(0) \backslash B_{\sqrt{3}v_0}(0)$. The positivity is not affected by the factors $U_j(t)^* \cdots U_j(t)$ and we omit them to get

$$(m/t)\; \mathcal{F}'(mx^2/2t^2) \left\{ (x/t) \cdot Q - |x/t|^2 \right\} f_j(Q + v_j). \tag{2.15}$$

We have neglected the double commutator $[Q, [Q, \mathcal{F}]]$ because it is bounded in norm by const$/t^2$ and thus is integrable. Clearly (2.15) is $O(t^{-1})$. It is self-adjoint up to integrable corrections. Positivity follows because $f_j \geq 0$, $\mathcal{F}' \leq 0$, and $|Q| < |x/t|$ in the corresponding supports. More precisely, with the arguments of the functions as in (2.15):

$$(-\mathcal{F}')\; |x/t|^2 f_j \approx \sqrt{f_j}\; (-\mathcal{F}')\; |x/t|^2 \sqrt{f_j} \;\geq\; 3\, {v_0}^2 \sqrt{f_j}\, (-\mathcal{F}')\, \sqrt{f_j} \;\geq\; 0.$$

$$|(-\mathcal{F}')\; (x/t) \cdot Q\;\; f_j|$$

$$\lesssim \sqrt{-\mathcal{F}'\, |x/t|}\;\; |Q|\;\; f_j \sqrt{-\mathcal{F}'\, |x/t|} \;\leq\; v_0 \sqrt{-\mathcal{F}'\, |x/t|}\;\; f_j \sqrt{-\mathcal{F}'\, |x/t|}$$

$$\approx v_0 \sqrt{f_j}\, (-\mathcal{F}'\, |x/t|) \sqrt{f_j} \;\leq\; v_0 \cdot 2v_0\; \sqrt{f_j}\, (-\mathcal{F}')\, \sqrt{f_j}.$$

Reordering of the square roots yields correction terms which in addition to the factor (m/t) in (2.15) have one more inverse power of t and thus can be neglected. The difference of the two expressions is bounded below by ${v_0}^2 \sqrt{f_j}\, (-\mathcal{F}')\, \sqrt{f_j} \geq 0$. □

Instead of our explicit calculation one could have used that (2.15) is a pseudodifferential operator in Q and $x = im(d/dQ)$ with positive symbol. Thus the operator is positive up to corrections which are smaller by a factor of the small parameter $(1/t)$.

Now we are ready to verify the Cauchy criterion for the limit of

$$\Psi_j(t) := \exp\{iHt\}\ L_j(t)\ \exp\{-iHt\}\,\Psi \tag{2.16}$$

for every $\Psi \in \mathcal{H}$. The following quantity has to be smaller than $\varepsilon > 0$ for T_1 large enough uniformly in $T_2 \geq T_1$:

$$\begin{aligned}
&\left\| \int_{T_1}^{T_2} dt\ d/dt \Big\{ \exp\{iHt\}\ L_j(t) \exp\{-iHt\} \Big\} \Psi \right\| \\
&\leq \sup_{\|\Phi\|=1} \int_{T_1}^{T_2} dt\ \Big| \Big(\Phi,\ d/dt \Big\{ \exp\{iHt\}\, L_j(t)\ \exp\{-iHt\} \Big\} \Psi \Big) \Big| \\
&\approx \sup_{\|\Phi\|=1} \int_{T_1}^{T_2} dt\ |(A_j(t)\ \exp\{-iHt\}\,\Phi,\ A_j(t)\ \exp\{-iHt\}\,\Psi)| \\
&\leq \sup_{\|\Phi\|=1} \left\{ \int_{T_1}^{T_2} dt\ \|A_j(t)\ \exp\{-iHt\}\,\Phi\|^2 \right\}^{1/2} \times \\
&\qquad \times \left\{ \int_{T_1}^{T_2} dt\ \|A_j(t)\ \exp\{-iHt\}\,\Psi\|^2 \right\}^{1/2} .
\end{aligned} \tag{2.17}$$

In the approximation we have omitted integrable integrands. In the last step the Cauchy Schwarz inequality is used. With Propositon 2.4 the square of the second factor is

$$\begin{aligned}
&\int_{T_1}^{T_2} dt\ (\Psi,\ \exp\{iHt\}\ A_j(t)^*\ A_j(t)\ \exp\{-iHt\}\,\Psi) \\
&\approx \left| \int_{T_1}^{T_2} dt\ \frac{d}{dt} (\Psi,\ \ \exp\{iHt\}\ L_j(t)\ \exp\{-iHt\}\,\Psi) \right| \leq 2\ \|\Psi\|^2
\end{aligned}$$

Thus $\|A_j(t)\ \ \exp\{-iHt\}\,\Psi\|^2 \in L^1$ and the integral vanishes as $T_2 \geq T_1 \to \infty$. The same argument shows the boundedness of the first factor uniformly in $\|\Phi\| = 1,\ T_2 \geq T_1 \geq 1$. Existence of the limit (2.16) has been verified.

Our convergence proof of (2.16) is a Hilbert space version of the fact that if $h(t) \leq M < \infty$, $h'(t) = a_+(t) + a_1(t)$ with $a_+(t) \geq 0$, $a_1 \in L^1$, then $\lim_{t\to\infty} h(t)$ exists and also $a_+(t) \in L^1$. We know about the positive term only that it decays like $(1/t)$, the integrability which follows from the convergence was not obvious. The estimate is entirely "differential". In contrast to our older treatments one needs not control an integrated time evolution for a long time, not even the stationary phase estimates for the free one. Therefore we got rid of the asymmetry between free and interacting evolutions, the full time evolution can be estimated as easily as the free one.

We can use (2.16) to replace the sequence of times $\tau_n \to \infty$ in Corollary 2.2 by a limit $t \to \infty$, i.e.

$$\lim_{t\to\infty}\left\{\mathbf{1}-\sum\nolimits_j L_j(t)\right\}\exp\{-iHt\}\,\Psi=0 \tag{2.18}$$

for any $\Psi=g(H)\Psi\in\mathcal{H}^{cont}(H)$. The norm of (2.18) is

$$\left\|\Psi-\sum\nolimits_j \exp\{iHt\}\,L_j(t)\exp\{-iHt\}\,\Psi\right\|$$

$$\leq\left\|\Psi-\sum\exp\{iH\tau_n\}\,L_j(\tau_n)\exp\{-iH\tau_n\}\,\Psi\right\|$$

$$+\sum\nolimits_j\|\exp\{iH\tau_n\}\,L_j(\tau_n)\exp\{-iH\tau_n\}\,\Psi-\exp\{iHt\}\,L_j(t)\exp\{-iHt\}\,\Psi\|$$

This tends to zero uniformly for $t\geq\tau_n$ as $\tau_n\to\infty$. Actually, Theorem 2.1 holds for two-body system also for $t\to\infty$ by an extra argument [1] which is not necessary in our application. Obviously convergence of

$$\lim_{t\to\infty}U(t,0)^*\,L_j(t)\exp\{-iHt\}\,\Psi=:\Phi_j(\infty) \tag{2.19}$$

for each $j=1,...,J$ is sufficient for convergence of $U(t,0)^*\exp\{-iHt\}\,\Psi=:\Phi(\infty)$. However, the proof of (2.19) is the same as that of (2.16), only the integrable correction terms are different, see Proposition 2.4. Similarly one has that

$$\lim_{t\to\infty}\exp\{iHt\}\,L_j(t)\,U(t,0)\Phi \tag{2.20}$$

exists for any $\Phi\in\mathcal{H}$. In particular we conclude from (2.19) that for large enough $\tau\geq\tau(\varepsilon)$ uniformly in $t\geq\tau$ for all j

$$\|L_j(t)\exp\{-iHt\}\,\Psi-U(t,\tau)L_j(\tau)\exp\{-iH\tau\}\,\Psi\|<\varepsilon. \tag{2.21}$$

This allows to show convergence (1.8) needed for completeness

$$\lim_{t\to\infty}U^D(t,0)^*\,\exp\{-iHt\}\,\Psi=\lim_{t\to\infty}g(H_0)\,U^D(t,0)^*\,\exp\{-iHt\}\,\Psi \tag{2.22}$$

by reducing it to convergence of the simpler

$$\lim_{t\to\infty}g(H_0)\,U^D(t,0)^*\,U(t,\tau)\,L_j(\tau)\exp\{-iH\tau\}\,\Psi \tag{2.23}$$

for arbitrarily large fixed τ. As a preparation we derive for the intermediate time evolution U a better asymptotic correlation between Q and x/t than expressed in Theorem 2.1 or (2.18) which corresponds to $\gamma'=0$ below.

Lemma 2.5. *(a) For $t\geq\tau\geq1$ and each j (and similarly for U^D)*

$$\|(Qt-x)\,U(t,\tau)\,L_j(\tau)\|\leq\text{const}(\tau)\,(1+t^{1-\gamma}). \tag{2.24}$$

(b) Let $h\in C_0^\infty(\mathbb{R})$ satisfy $h(0)=1$. Then for any $0\leq\gamma'<\gamma$

$$\lim_{t\to\infty} \left\| \left[h(t^{\gamma'}(Q - x/t)) - \mathbf{1}\right] \; U(t,0)\, \Phi(\infty)\right\| = 0, \tag{2.25}$$

$$\lim_{t\to\infty} \left\|\exp\{-iHt\}\Psi - h(t^{\gamma'}(Q - x/t))\, U(t,0)\, \Phi(\infty)\right\| = 0. \tag{2.26}$$

Proof. (a) The norm is bounded by

$$\|U(T,\tau)^* \; [Q\,T - x] \; U(T,\tau)\, L_j(\tau)\|$$

$$\leq \| \, [Q\tau - x] \; L_j(\tau)\| + \int_\tau^T dt \, \|Q + i\,[(H_0 + V_t(x)), \; (Qt - x)] \, \|$$

$$\leq \; \mathrm{const}(\tau) + \int_\tau^T dt \; (t/m) \; \sup|\nabla V_t| \; \leq \; \mathrm{const}\,(\tau) \; \left(1 + T^{1-\gamma}\right). \tag{2.27}$$

For U^D we get correspondingly

$$\|Q + i\,[(H_0 + V_t(Qt)), \; (Qt - x)] \,\| \leq C\, t \; \sup|\nabla V_t|. \tag{2.28}$$

(b) is an easy consequence of (a) and (2.21). □

Note that the potential may depend on additonal variables which commute with Q. Then the proof of the estimate (2.24) is not affected. This situation occurs for three-body systems e.g. in the proof of Lemma 4.6. To show (2.23) it is sufficient to estimate for $t_1 \geq \tau$

$$\left\|\int_{t_1}^\infty dt \; g(H_0)\,(d/dt)\,U^D(t,0)^*\; U(t,\tau)\; L_j(\tau)\right\|$$

$$\leq \int_{t_1}^\infty dt \; \| \, [V_t\,(Qt) - V_t(x)] \, U(t+\tau,\tau)\; L_j(\tau)\|. \tag{2.29}$$

Here we have used that $g(H_0)\, V^\ell\,(Qt) = g(H_0)\; V_t\,(Qt)$ because $g(H_0)$ equals zero in a neighborhood of the origin in velocity space. With the Baker-Campbell-Hausdorff formula one computes

$$V_t\,(Qt) - V_t(x) =$$

$$= \int_0^1 d\lambda \Big\{ (\nabla V_t)(x(1-\lambda) + \lambda\, Qt) \cdot (Qt - x)$$

$$+ \,(i/2m)\; t\; (\Delta V_t)\,(\,x(1-\lambda) + \lambda\, Qt)\Big\}. \tag{2.30}$$

The first term would be there for commuting arguments as well, the second is the correction due to $i\,[Q_it,\ x_k] = (t/m)\,\delta_{ik}$. Thanks to the cutoff in the potential V_t we need not consider the complicated arguments of the derivatives of the potential and obtain

$$\|\,[V_t(Qt) - V_t(x)]\ U(t,\tau)\ L_j(\tau)\|$$

$$\leq \sup_z |(\nabla V_t)\,(z)|\ \|\,(Qt - x)\ U(t,\tau)\ L_j(\tau)\|$$

$$+ \text{ const } t\ \sup_z |\,(\Delta V_t)\,(z)|. \tag{2.31}$$

For $\gamma > 1/2$ the second summand decays integrably in t by (2.10) with $\varepsilon_0 < 2\gamma - 1$. With (2.9) and (2.24) the first is bounded by const $(1+t)^{-2\gamma}$ and it is integrable for $\gamma > 1/2$. The integral (2.29) vanishes as $t_1 \to \infty$ and the convergence (1.8) for completeness is established.

The existence (1.7) is well known. It also follows easily from the estimates above. They show that the norm limit

$$\lim_{t\to\infty} U(t,0)^*\ U^D(t,0)\,g(H_0) \text{ exists.} \tag{2.32}$$

For both U^D and U the results on asymptotic phase space localization of Corollary 2.2 and (2.18) hold as well. This implies existence. Thus we have proved

Theorem 2.6. *Let the Hamiltonian H satisfy* (1.1) - (1.3) *with $\gamma > 1/2$. Then the modified Dollard wave operator Ω^D exists* (1.7) *and is complete, i.e. it is unitary from $\mathcal{H}$ to $\mathcal{H}^{cont}(H)$. In particular H has no singular continuous spectrum and its continuous part is unitarily equivalent to H_0, i.e. $H \restriction \mathcal{H}^{cont}(H) = \Omega^D\, H_0\, (\Omega^D)^*$.*

Remark 2.7. *So far we did not use part (b) of Lemma 2.5. Instead of the proof above one could have studied*

$$(d/dt)\,g(H_0)\ U^D(t,0)^*\ h(t^{\gamma'}(Q - x/t))\ U(t,0)$$

$$= g\ (U^D)^*\ \left\{i\,V_t(Qt)\,h(..) - i\,h(..)\ V_t(x) + i\,[H_0,\ h(..)] + \partial_t\,h(..)\right\}\ U.$$

The difference of the potential terms is integrable by similar estimates as given above for $\gamma > \gamma' \geq 1/2$. The total free time derivative of $h(..)$ is positive without any corrections for $\gamma' \leq 1$ if h decays monotone: $y \cdot \nabla h(y) \leq 0$.

$$i\,[H_0,\ h(t^{\gamma'}(Q - x/t))] + \partial_t\,h(t^{\gamma'}(Q - x/t)) =$$

$$= -\Big[(1-\gamma')/t\Big]\,(t^{\gamma'}(Q-x/t))\cdot(\nabla h)(t^{\gamma'}(Q-x/t)) = O(t^{-1}) \geq 0 \qquad (2.33)$$

This last statement is not surprising because $(Qt - x)$ is constant under the free evolution.

While we could avoid to treat a threshold contribution with energy close to zero by a density argument this is no longer possible if the two-body system is part of a larger system. The corresponding phase space propagation estimate uses

$$L_0(t) := g(H)\, G\left(\frac{mx^2}{2t^2}\right) \qquad (2.34)$$

where $g \in C_0^\infty(I\!R)$, $\operatorname{supp} g \subset \{\lambda \in I\!R |\ \lambda < m\, v_0{}^2/2\}$ and G is the same as $\mathcal{F}$ in the construction (2.4) of the other $L_j(t)$. The interesting functions g equal one on the bound state energies of H and on a neighborhood of zero or on a tiny neighborhood of zero only. The localization is not spoiled by the factor $g(H)$. We have shown in Proposition 6.2 of [2]

$$\|F(|x| < \rho)\ g(H)\ F(|x| > \rho + r)\| \leq C_n\,(1+r)^{-N} \quad \text{for all } N \in I\!N \qquad (2.35)$$

uniformly in ρ. Integrable decay (which is sufficient here) is easier to prove. This implies

$$L_0(t)\ F(|x| > 2v_0 t) = 0, \quad \|F(|x| > 3v_0 t)\ L_0(t)\| \leq C_N\,(1+t)^{-N}. \qquad (2.36)$$

Also $L_0(t)$ is absorbing.

Lemma 2.8. *For $\gamma > 0$*

$$D_t\, L_0(t) = i\,[H,\ L_0(t)] + \partial_t\, L_0(t) \approx^1 A_0(t)^*\, A_0(t) = O(1/t). \qquad (2.37)$$

$A_0(t)$ can be chosen independent of the potential.

$$\lim_{t\to\infty}\ \exp\{iHt\}\ L_0(t)\ \exp\{-iHt\}\Psi \qquad (2.38)$$

exists for any $\Psi \in \mathcal{H}$.

Proof.

$$D_t\, L_0(t) = g(H)\ \Big\{i\,\Big[H_0,\ G\left(m\,x^2/2t^2\right)\Big] + \partial_t\, G\left(m\,x^2/2t^2\right)\Big\}$$

$$\approx^1 g(H)\ \frac{m}{t}\left\{\frac{x^2}{t^2} - Q\cdot\frac{x}{t}\right\}(-G')\left(\frac{mx^2}{2t^2}\right) \approx^1$$

$$\approx^1 g(H_0)\,\frac{m}{t}\left\{\frac{x^2}{t^2} - Q\cdot\frac{x}{t}\right\}(-G')\left(\frac{mx^2}{2t^2}\right) \tag{2.39}$$

In the first approximation we omitted $O(1/t^2)$ terms from reordering the non commuting operators. In the second we used decay

$$\|\,[g(H) - g(H_0)]\,F(|x| > v_0|t|)\,\| \leq C\,(1+|t|)^{-\gamma} \tag{2.40}$$

i.e. the slowest decay of the potentials. (2.39) is of the same essentially positive type as (2.15) with $g(H_0)$ playing the role of $f_j(Q+v_j)$. The second statement is proved as (2.16). □

Remark 2.9. *If* $g \in C_0^\infty((-\infty, 0))$ *one gets the stronger* $D_t\,L_0(t) \approx^1 0$ *because* $g(H_0) = 0$.

Note that in (2.37), (2.38) H cannot be replaced on the left or right by an approximation. One does not show more than stability of that component. Nevertheless this simple lemma is very useful for the treatment of three-body systems, it replaces e.g. the whole Section VI in [2]!

3 More on Long-Range Forces

For the completeness proof in this section we assume about the long-range potential instead of (1.2) - (1.3) with $\gamma > 1/2$ a slower decay of V^ℓ but additional requirements on its higher derivatives. For multiindices α let V^ℓ satisfy

$$|(D^\alpha\,V^\ell)(x)| \leq C_\alpha\,(1+|x|)^{-|\alpha|-\gamma},\ \gamma > 0,\ 0 \leq |\alpha| \leq A(\nu), \tag{3.1}$$

with a constant $A(\nu)$ depending on the dimension of space. For this class Isozaki und Kitada [6] have constructed a Fourier integral operator J which has a solution of the eikonal equation for the classical purely long-range problem as phase function. Its construction involves subtle estimates of classical trajectories. It is a time-independent modifier in the definition of modified wave operators

$$\Omega := \text{s-}\lim_{t\to\infty}\ \exp\{iHt\}\,J\,\exp\{-iH_0t\}\Psi. \tag{3.2}$$

By the considerations of the previous section it is sufficient for existence and completeness of Ω to show existence of the limits

$$\exp\{iHt\}\ J\ L_j(t)\,\exp\{-iH_0t\}\Phi, \tag{3.3}$$

$$\exp\{iH_0t\}\ L_j(t)^*\ J^*\ \exp\{-iHt\}\Psi \tag{3.4}$$

as $t \to \infty$. J is constructed such that it satisfies on outgoing states of stricly positive kinetic energy

$$(H\,J - J\,H_0)\;L_j(t) \in L^1. \tag{3.5}$$

This follows easily from the eikonal equation, see (4.1), (2.33) in [6]. It implies in particular

$$i\,[H,\; J\,L_j(t)\,J^*] + \partial_t\,J\,L_j(t)\,J^* \approx^1 J\;A_j(t)^*\;A_j(t)\;J^* \geq 0. \tag{3.6}$$

The remaining arguments are much easier than in [6]. Only a minor modification of our estimate (2.17) is needed.

$$\begin{aligned}
&|\,(\Psi,\;\exp\{iHT_2\}\;J\;L_j(T_2)\;\exp\{-iH_0T_2\}\Phi) \\
&\quad -\;(\Psi,\;\exp\{iHT_1\}\;J\;L_j(T_1)\;\exp\{-iH_0T_1\}\Phi)\,|^2 \\
&\approx \left|\int_{T_1}^{T_2} dt\;(\Psi,\;\exp\{iHt\}\;J\;A_j(t)^*\;A_j(t)\;\exp\{-iH_0t\}\Phi)\right|^2 \\
&\lesssim \int_{T_1}^{T_2} dt\;(d/dt)\,(\Phi,\;\exp\{iH_0t\}\;L_j(t)\;\exp\{-iH_0t\}\Phi) \\
&\qquad \times \int_{T_1}^{T_2} dt\;(d/dt)\,(\Psi,\;\exp\{iHt\}\;J\;L_j(t)\,J^*\;\exp\{-iHt\}\Psi).
\end{aligned}$$

Since $J\,L_j(t)\,J^*$ is uniformly bounded the proof is the same as above.

4 Three-Body Systems

We proceed in close analogy to our treatment of the two-body case above. We construct an absorbing decomposition in phase space of scattering states at late times such that on each component a simpler comparison dynamics approximates the time evolution. The new feature is that we have to deal with several *channels* due to the possibility of having asymptótically either all particles moving freely relative to each other or a bounded pair moving freely relative to the third particle. They are indexed by d_3 for the total decomposition or by d_2 labelling the three possibilities for pairings, respectively. For each two cluster decomposition d_2 we denote by c the non-trivial cluster of two particles. It is convenient to use Jacobi coordinates. The internal coordinate of the cluster or "pair" is

$$X(c) := x^i - x^j \in \mathbb{R}^\nu \quad \text{if} \;\; c = (i,j) \in d_2,\; i < j, \tag{4.1}$$

and the coordinate of the third particle relative to the center of mass of the pair c is

$$Y(d_2) := x^k - (m_i x^i + m_j x^j) / (m_i + m_j), \quad k \notin c = (i,j) \in d_2. \tag{4.2}$$

Relative coordinates $Y(d_3)$ for all three particles can be represented by a pair $(X(c), Y(d_2)) \in \mathbb{R}^{2\nu}$ for any d_2. The reduced masses $\mu(c) := m_i m_j/(m_i + m_j)$ and $\mu(d_2) := m_k\, \mu(c)/(m_k + \mu(c))$ can be used to form invariant inner products with norms

$$\|X(c)\|^2 := \mu(c)\, |X(c)|^2, \quad \|Y(d_2)\|^2 := \mu(d_2)\, |Y(d_2)|^2, \tag{4.3}$$

$$\|Y(d_3)\|^2 := \|X(c)\|^2 + \|Y(d_2)\|^2, \quad \text{any } d_2, \tag{4.4}$$

where $|\cdot|$ denotes Euclidean norm in $\mathbb{R}^\nu$. (4.4) turns out to be independent of d_2. The corresponding velocity operators are

$$Q(c) := -i\nabla_{X(c)} / \mu(c), \quad Q(d_2) := -i\nabla_{Y(d_2)} / \mu(d_2) \tag{4.5}$$

and $Q(d_3)$ may again be represented as a pair. The free Hamiltonian H_0 is with the above norm

$$H_0 = \|Q(d_3)\|^2/2 = \|Q(c)\|^2/2 + \|Q(d_2)\|^2/2$$

$$=: H_0(c) + T(d_2) \equiv H(d_3) \tag{4.6}$$

independent of d_2. Instead of using explicit coordinates one could have defined $2H_0$ equivalently as the Laplace Beltrami operator on $\mathbb{R}^{2\nu}$ with metric (4.3), (4.4). It satisfies the familiar kinematical relations

$$i[H_0, X(c)] = i[H_0(c), X(c)] = Q(c), \tag{4.7}$$

$$i[H_0, Y(d_2)] = i[T(d_2), Y(d_2)] = Q(d_2), \tag{4.8}$$

$$i[H_0, Y(d_3)] = Q(d_3). \tag{4.9}$$

With $H(c) := H_0(c) + V^{ij}$ if $(i,j) = c$ one can express the non-trivial decomposition Hamiltonians as

$$H(d_2) := H(c) + T(d_2) = H - \sum{}' V^{im}. \tag{4.10}$$

Recall that $\sum'$ depending on the decomposition d_k denotes summation over all pairs (i,m) which are *not* in a cluster of d_k.

The closed countable set of thresholds $\mathcal{T} := \{0\} \cup \bigcup_c \sigma^{pp}(H(c))$ is the closure of all subsystem eigenvalues. $\mathcal{T} \subset (-\infty, 0]$ because our class of potentials is known not to have positive energy bound states. (The inclusion of such bound states would not be difficult.) We will consider only scattering states with bounded energy away from thresholds, i.e. for Ψ there is a $\tilde{g} \in C_0^\infty(\mathbb{R})$ with

$$\tilde{g}(H)\Psi = \Psi \in \mathcal{H}^{cont}(H), \quad \mathrm{supp}\tilde{g} \cap \mathcal{T} = \emptyset. \tag{4.11}$$

Such states are dense in $\mathcal{H}^{cont}(H)$, the orthogonal complement of the bound states.

Experience from physics suggests that asymptotically in time either all particles move away from each other, or a pair is bounded and it separates from the third particle, or a pair is in a scattering state with very small energy and the third particle runs away. We have excluded the possibility that all three particles stay together by avoiding threshold energies. Thus in each of the cases the system decays into two or more clusters. The short-range interaction between the clusters can be neglected asymptotically and the long-range part results only in a simple explicitly tractable modification of the free relative motion of the clusters. A proof of *asymptotic completeness* (1.12), (1.13) shows that this intuitition is correct in the mathematical model. We construct our phase space decomposition accordingly.

Corresponding to Theorem 2.1. in the two-body case we can use again asymptotic observables to describe propagation in phase space and the correlations between position and velocity for a sequence of times $\tau_n \to \infty$. The splitting for two body subsystems into their bound state part $\mathcal{H}^{pp}(H(c))$ and scattering part $\mathcal{H}^{cont}(H(c))$ is not time-invariant due to the interaction with the third particle. Therefore it is advantageous to split off only a finite number of bound states depending on an error bound ε. We denote by

$$\mathbf{P}^N(d_2) \quad \text{with} \quad \lim_{N\to\infty} \mathbf{P}^N(d_2) = \mathbf{P}(d_2) \tag{4.12}$$

the orthogonal projection corresponding to the first N eigenvectors of the subsystem Hamiltonian $H(c)$. In the internal subsystem Hilbert space $\mathcal{H}(c)$ this is an N-dimensional projection. Since the state of the third particle relative to the cluster is unrestricted $\mathbf{P}^N(d_2)$ is infinite dimensional on $\mathcal{H}$. The numbering of the eigenvectors does not matter.

Theorem 4.1. *Let $\Psi \in \mathcal{H}^{cont}(H)$ be given. There is a sequence of times $\tau_n \to \infty$ and for every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that the following holds.*

(a) For any g, $\tilde{g} \in C_0^\infty(\mathbb{R})$, $\varepsilon > 0$, and all large enough τ_n

$$\left\|\mathbf{P}^N(d_2)\, [\tilde{g}(H) - \tilde{g}(H(d_2))]\, \exp\{-iH\tau_n\}\Psi\right\| < \varepsilon, \tag{4.13}$$

$$\left\|\left[\tilde{g}(H(d_3))\left\{\mathbf{1} - \sum\nolimits_{d_2} \mathbf{P}^N(d_2)\right\} - \left\{\mathbf{1} - \sum\nolimits_{d_2} \mathbf{P}^N(d_2)\right\}\tilde{g}(H)\right]\right.$$

$$\left. \times \exp\{-iH\tau_n\}\Psi\right\| < \varepsilon, \tag{4.14}$$

$$\left\| [g(H(c)) - g(H_0(c))] \left\{ \mathbf{1} - \sum\nolimits_{d_2} \mathbf{P}^N(d_2) \right\} \exp\{-iH\tau_n\}\Psi \right\| < \varepsilon \qquad (4.15)$$

for every pair c. In (4.14) *one can replace* $g(H(d_3))$ *by* $g(H(d_2))$ *for any* d_2.

(b) For any d_2, $f \in C_0^\infty(\mathbb{R}^\nu)$, $\varepsilon > 0$, *and all large enough* τ_n

$$\left\| \left[f\left(\frac{Y(d_2)}{\tau_n}\right) - f(Q(d_2)) \right] \mathbf{P}^N(d_2) \exp\{-iH\tau_n\}\Psi \right\| < \varepsilon. \qquad (4.16)$$

(c) For any $f \in C_0^\infty(\mathbb{R}^{(k-1)\nu})$, $\varepsilon > 0$, *and all large enough* τ_n

$$\left\| \left[f\left(\frac{Y(d_k)}{\tau_n}\right) - f(Q(d_k)) \right] \left\{ \mathbf{1} - \sum\nolimits_{d_2} \mathbf{P}^N(d_2) \right\} \exp\{-iH\tau_n\}\Psi \right\| < \varepsilon \qquad (4.17)$$

and similarly for $G(\|X(c)\|^2/2\tau_n^2) - G(\|Q(c)\|^2/2)$, G *as in* (2.34).

With suitable adjustments this theorem holds for any particle number, see [3] for a proof. The bound states of the pairs not covered by $\mathbf{P}^N(d_2)$ for large N are weakly bounded. For them $H(c) \approx H_0(c)$, $Q(c)$, and $X(c)/t$ are all very small. Therefore (4.14), (4.15), and (4.17) can hold for components with weakly bounded pairs as well. Clearly $N(\varepsilon)$ grows as $\varepsilon \to 0$. Part (a) says that particles or clusters of bounded particles eventually separate and relative energies are relative kinetic energies. Parts (b) and (c) state that the particles or clusters approximately will be localized there where they would be as classical particles starting with the given velocity at time zero near the origin. The statements are relatively easy to prove because they involve only $Y(d_k)/t$, i.e. not a very precise localization for large t.

As in the two-body case we use this information to construct a finite phase space decomposition into absorbing outgoing components. Let $\tilde{g} \in C_0^\infty(\mathbb{R})$ be given with

$$\operatorname{dist}(\operatorname{supp}\tilde{g}, \mathcal{T}) \geq v_0^2 \left[18 \max_{d_2}(\mu(d_2)) + \min_c \mu(c)/2 \right] > 0. \qquad (4.18)$$

The speed $v_0 > 0$ will be chosen later. Consider a function g with $\sqrt{g} \in C_0^\infty(\mathbb{R})$ which equals one on a neighborhood of the threshold set $\mathcal{T}$ and satisfies for $\lambda_0 := v_0^2 \min_c \mu(c)/2$: $\operatorname{supp} g \subset \{\lambda \in \mathbb{R} \mid \lambda < \lambda_0\}$. The covering function G satisfies $g(\lambda)\, G(\lambda) = g(\lambda)$, $G(\lambda) = 0$ for $\lambda \geq 4\lambda_0$, $G'(\lambda) \leq 0$, $\{-G'\}^{1/2} \in C_0^\infty(\mathbb{R})$, and $\operatorname{supp} G' \subset \{3\lambda_0 < \lambda < 4\lambda_0\}$. For any $v_0 > 0$ the two functions g and G are as in (2.34) and satisfy the assumptions of Lemma 2.8. The maximal speed $|X(c)/t|$ in the range of $G(\|X(c)\|^2/2t^2)$ is for every pair c bounded by $2v_0$.

Let $\mathbf{Sh}_2$ be a bounded shell of velocities of the third particle relative to the center of mass of the pair which contains all $Q(d_2)$ in the ranges of $g(H(c))\, \tilde{g}(H(d_2))$ for every d_2, i.e.

$$\mathbf{Sh}_2 \supset \bigcup_{d_2} \bigcup_{\lambda \in \mathcal{T}} \bigcup_{0 \leq \lambda \leq \lambda_0} \left\{ Q \in \mathbb{R}^\nu \mid (\lambda + \mu(d_2)\, |Q|^2/2) \in \operatorname{supp}\tilde{g} \right\}. \qquad (4.19)$$

Due to the choices made above we may choose $\mathbf{Sh}_2 \subset \{Q \mid |Q| \geq 6\, v_0\}$ as for $\mathbf{Sh}$ in (2.3). On this shell we introduce again a decomposition fo the identity $\{f_j\}$ with covering functions F_j with the same properties as in Section 2. We define the phase space localization operators for each d_2

$$L_j(t; d_2) := g(H(c))\, G(\|X(c)\|^2/2t^2)\, F_j(Y(d_2)/t)\, f_j(Q(d_2)). \tag{4.20}$$

On the range of $G(\|X(c)\|^2/2t^2)\, F_j(Y(d_2)/t)$ we have $|x^i - x^m| \geq |Y(d_2)| - |X(c)| \geq 2v_0 t$ if the particles i and m do not form the cluster c. With the analogon to (2.12) we have

$$\|L_j(t; d_2)\, F(|x^i - x^m| < v_0 t)\|,\ \|F(|x^i - x^m| < v_0 t)\, L_j(t; d_2)\| \leq C_N\, t^{-N} \tag{4.21}$$

for any $N \in I\!N$. For a justification of the second part see (2.35).

Thus $L_j(t, d_2)$ characterize states where the third particle is well separated from a small pair. Short-range intercluster potentials yield integrable terms when multiplied with $L_j(t; d_2)$, see Lemma 4.2 below.

It remains to construct a phase space decomposition for the channel d_3. The support of $\tilde{g}$ characterizes an (elliptic) shell in some Jacobi coordinate system

$$\mathbf{Sh}_3 := \left\{Q \in I\!R^{2\nu} \mid \|Q\|^2/2 \in \operatorname{supp}\tilde{g}\right\} \tag{4.22}$$

which is disjoint from a neighborhood of the origin. The three functions $g(H_0(c)) \sum_j f_j(Q(d_2))$ interpreted as functions $\mathcal{G}(Q; d_2)$, $Q \in I\!R^{2\nu}$ equal one on a neighborhood of the intersection of $\mathbf{Sh}_3$ with the ν-dimensional subspaces $Q(c) = 0$, respectively. For small enough $v_0 > 0$ their supports are disjoint. Denote by $\mathbf{Sh}_3''$ the subset of $\mathbf{Sh}_3$ where all three functions differ from one. Pick $v_0 > 0$ so small that it satisfies (4.18) and, in addition,

$$|Q(c)| \geq 6v_0 \text{ for all } Q \in \mathbf{Sh}_3'', \text{ all } c. \tag{4.23}$$

With $\lambda_0 = v_0^2 \min_c \mu(c)/2$ there is a finite collection of $Q_\ell \in \mathbf{Sh}_3''$ and functions $\tilde{f}_\ell \geq 0$, $\{\tilde{f}_\ell\}^{1/2} \in C_0^\infty(I\!R^{2\nu})$, $\operatorname{supp}\tilde{f}_\ell \subset \{Q \in I\!R^{2\nu} \mid \|Q - Q_\ell\|^2/2 \leq \lambda_0\}$ such that

$$\sum_{d_2} \mathcal{G}(Q, d_2) + \sum_\ell \tilde{f}_\ell(Q) = 1 \text{ for } Q \in \mathbf{Sh}_3. \tag{4.24}$$

The corresponding covering functions $\tilde{F}_\ell(Q) = \tilde{\mathcal{F}}(\|Q - Q_\ell\|^2/2)$ satisfy $\tilde{f}_\ell \tilde{F}_\ell = \tilde{f}_\ell$, $\tilde{\mathcal{F}}(\lambda) = 1$ (or 0) if $\lambda \leq 3\lambda_0$ (or $\geq 4\lambda_0$), $\tilde{\mathcal{F}}' \leq 0$, and $(-\tilde{\mathcal{F}}')^{1/2} \in C_0^\infty([3\lambda_0, 4\lambda_0])$, just as for $\mathcal{F}$ and F_j in Section 2. On the range of $\tilde{F}_\ell(Y(d_3)/t)$ one has $|X(c)| \geq 4v_0 t$ for all pairs c. The phase space localization operators are defined as

$$L_\ell(t; d_3) := \tilde{F}_\ell(Y(d_3)/t)\, \tilde{f}_\ell(Q(d_3)). \tag{4.25}$$

They satisfy for all pairs (i,m)

$$\|L_\ell(t;d_3)\, F(|x^i - x^m| < 3v_0 t)\| \le C_N\, t^{-N}, \text{ all } N \in I\!N. \tag{4.26}$$

Thus all short-range potentials give integrable contributions on their range. We define as in (2.8)

$$V_t^{im}(x^i - x^m) := V^{im,\ell}(x^i - x^m)\, \varphi(|x^i - x^m|/v_0 t) \tag{4.27}$$

which has the same properties (2.9), (2.10). As in (1.10), (4.10) $\sum'$ means summation over those pairs which are not in a cluster for the given decomposition d_k.

Lemma 4.2. *For all j and $\gamma > 0$ the following quantities are integrable*

$$\sum{}' V^{im,s}\, L_j(t;d_k), \quad L_j(t;d_k) \sum{}' V^{im,s},$$

$$\sum{}' (V^{im,\ell} - V_t^{im})\, L_j(t;d_k), \quad L_j(t;d_k) \sum{}' (V^{im,\ell} - V_t^{im}),$$

$$\sum{}' [V^{im,\ell},\ L_j(t;d_k)], \text{ and } \sum{}' [V_t^{im},\ L_j(t;d_k)] \in L^1.$$

Proof. The first four terms are integrable following the decay assumptions of the potentials and (4.21), (4.26). The fifth follows with these and the last. For d_3 one uses $\|\, [V_t^{im}(x^i - x^m),\ \tilde{f}_\ell(Q(d_3))]\, \| \le \text{const}(\tilde{f}_\ell) \sup |\nabla V_t^{im}| \in L^1$. For d_2 and $(i,m) \neq c$ one has $x^i - x^m =: Y_{im}(d_2) + \lambda_{im} X(c)$ where $Y_{im}(d_2) = \pm Y(d_2)$, λ_{im} a constant depending only on the masses in the pair. Then $\|\, [V_t^{im}(x^i - x^m),\ f_j(Q(d_3))]\, \| \le \text{const}(f_j) \sup |\nabla V_t^{im}| \in L^1$. The remaining contribution is $\|\, [V_t^{im}(x^i - x^m),\ g(H(c))]\, \| \le \text{const}(g) \sup |\nabla V_t^{im}| \in L^1$. This last claim follows because we have for perturbed Hamiltonians as well $\|\, [h(X(c)),\ g(H(c))]\, \| \le \text{const}(g)\ \|\nabla h\|$. See e.g. the proof of Proposition 6.2 in [2]. □

The next step is to show that the phase space localization operators (4.20), (4.25) form an approximate decomposition of the identity on scattering states at late times.

Proposition 4.3. *For $\Psi = \tilde{g}(H)\Psi \in \mathcal{H}^{cont}(H)$ let $\tau_n \to \infty$ be the sequence of times as in Theorem 4.1. Then*

$$\lim_{\tau_n \to \infty} \left\| \left\{ \mathbf{1} - \sum_\ell L_\ell(\tau_n, d_3) + \sum_{d_2} \sum_j L_j(\tau_n, d_2) \right\} \exp\{-iH\tau_n\}\Psi \right\| = 0. \tag{4.28}$$

Proof. For given $\varepsilon > 0$ choose $N = N(\varepsilon)$ as in Thereom 4.1.

$$\exp\{-iH\tau_n\}\Psi = \left\{ \sum{}_{d_2} \mathbf{P}^N(d_2) + \mathbf{1} - \sum{}_{d_2'} \mathbf{P}^N(d_2') \right\} \tilde{g}(H)\, \exp\{-iH\tau_n\}\Psi$$

$$\approx \sum_{d_2} \mathbf{P}^N(d_2)\, \tilde{g}(H(d_2))\, \exp\{-iH\tau_n\}\Psi$$

$$+\ \tilde{g}(H(d_3)) \left\{\mathbf{1} - \sum\nolimits_{d_2'} \mathbf{P}^N(d_2')\right\} \exp\{-iH\tau_n\}\Psi$$

$$= \sum_{d_2} \sum_{j} g(H(c))\, f_j(Q(d_2))\, \mathbf{P}^N(d_2)\, \tilde{g}(H(d_2))\, \exp\{-iH\tau_n\}\Psi$$

$$+ \left\{ \sum_{d_2, j} g(H_0(c))\, f_j(Q(d_2)) + \sum_{\ell} \tilde{f}_\ell(Q(d_3)) \right\} \tilde{g}(H(d_3))$$

$$\times \left\{\mathbf{1} - \sum\nolimits_{d_2'} \mathbf{P}^N(d_2')\right\} \exp\{-iH\tau_n\}\Psi$$

$$\approx \sum_{d_2, j} g(H(c))\, f_j(Q(d_2))\, \mathbf{P}^N(d_2)\, \exp\{-iH\tau_n\}\Psi$$

$$+ \sum_{d_2, j} g(H_0(c))\, f_j(Q(d_2)) \left\{\mathbf{1} - \sum\nolimits_{d_2'} \mathbf{P}^N(d_2')\right\} \exp\{-iH\tau_n\}\Psi$$

$$+ \sum_{\ell} \tilde{f}_\ell(Q(d_3)) \left\{\mathbf{1} - \sum\nolimits_{d_2} \mathbf{P}^N(d_2)\right\} \exp\{-iH\tau_n\}\Psi. \tag{4.29}$$

The errors are smaller than const ε for all large τ_n by Theorem 4.1 (a). In the next steps we use parts (b) and (c) as well. The summands in the second line are

$$g(H_0(c))\, G(\|Q(c)\|^2/2)\, f_j(Q(d_2))\, F_j(Q(d_2)) \left\{\mathbf{1} - \sum\nolimits_{d_2'} \mathbf{P}^N(d_2')\right\} \exp\{-iH\tau_n\}\Psi$$

$$\approx g(H_0(c))\, G(\|X(c)\|^2/2\tau_n^2)\, f_j(Q(d_2))\, F_j(Y(d_2)/\tau_n) \left\{\mathbf{1} - \sum\nolimits_{d_2'} \mathbf{P}^N(d_2')\right\} \exp\{-iH\tau_n\}\Psi$$

$$\approx g(H(c))\, G(\ldots)\, f_j(\ldots)\, F_j(\ldots) \left\{\mathbf{1} - \mathbf{P}^N(d_2)\right\} \exp\{-iH\tau_n\}\Psi \tag{4.30}$$

In the last step we have used (4.15) and

$$\left\| G(\|X(c)\|^2/2t^2)\, F_j(Y(d_2)/t)\, \mathbf{P}^N(d_2') \right\| \to 0 \quad \text{for } d_2 \neq d_2',\ t \to \infty. \tag{4.31}$$

The latter holds for any finite N. With $\|\{\mathbf{1} - G(\|X(c)\|^2/2t^2)\}\, \mathbf{P}^N(d_2)\| \to 0$ as $t \to \infty$ for finite N the summands in the first line of (4.29) are approximated by

$$g(H(c))\, G(\|X(c)\|^2/2\tau_n^2)\, f_j(Q(d_2))\, F_j(Y(d_2)/\tau_n)\, \mathbf{P}^N(d_2)\, \exp\{-iH\tau_n\}\Psi.$$

Thus the contributions from the first two lines in (4.29) asymptotically add up to

$$\sum_{d_2}\sum_{j} L_j(\tau_n; d_2)\, \exp\{-iH\tau_n\}\Psi.$$

The summands in the last line are

$$\tilde{f}_\ell(Q(d_3))\, \tilde{F}_\ell(Q(d_3)) \left\{\mathbf{1} - \sum_{d_2} \mathbf{P}^N(d_2)\right\} \exp\{-iH\tau_n\}\Psi$$

$$\approx \tilde{f}_\ell(Q(d_3))\, \tilde{F}_\ell(Y(d_3)/\tau_n) \left\{\mathbf{1} - \sum_{d_2} \mathbf{P}^N(d_2)\right\} \exp\{-iH\tau_n\}\Psi$$

$$\approx L_\ell(\tau_n, d_3)\, \exp\{-iH\tau_n\}\Psi \tag{4.32}$$

because of Theorem 4.1 (c) and $\|\tilde{F}_\ell(Y(d_3)/t)\, \mathbf{P}^N(d_2)\| \to 0$ as $t \to \infty$ for finite N and every d_2. This completes the verification of (4.28) because all errors are bounded by a finite multiple of ε uniformly for large τ_n. □

We shall show next that the decomposition of an old scattering state is asymptotically stable and that the components are absorbing. This follows from positivity of the total time derivatives.

Proposition 4.4. *For $L_j(t; d_k)$ as defined in* (4.20), (4.25)

$$i\,[H,\ L_j(t; d_k)] + \partial_t L_j(t; d_k)\ \approx^1\ i\,[H(d_k),\ L_j(t; d_k)] + \partial_t L_j(t; d_k)$$

$$\approx^1\ A_j(t; d_k)^*\, A_j(t; d_k) = O(t^{-1}). \tag{4.33}$$

Moreover, for pairs of generators $H_1(t)$, $H_2(t)$

$$i\, H_1(t)\, L_j(t; d_k) - i\, L_j(t; d_k)\, H_2(t) + \partial_t L_j(t; d_k) \tag{4.34}$$

$$\approx^1 A_j(t; d_k)^*\, A_j(t; d_k) \tag{4.35}$$

if for given d_k the generators differ by intercluster potentials of short range, i.e.

$$H_1(t) - H_2(t) = \sum{}' V^{im}(t) \tag{4.36}$$

where $V^{im}(t)$ may be $V^{im,s}$ or $(V^{im,\ell} - V_t^{im})$, etc..

Proof. That intercluster potentials of effective short range as well as commutators with long-range potentials yield integrable correction terms was shown in Lemma 4.2. For d_3 the essential positivity of

$$i\,[H_0,\ L_\ell(t;d_3)] + \partial_t\, L_\ell(t;d_3)$$

is exactly Proposition 2.4 in higher dimension. In the two-cluster case we reduce it to "two-body" estimates.

$$i[H(d_2),\ L_j(t;d_2)] + \partial_t\, L_j(t;d_2)$$

$$= G(H(c))\left\{i\left[H(c),\ G(\|X(c)\|^2/2t^2)\right] + \partial_t\, G(\|X(c)\|^2/2t^2)\right\}\ F_j\, f_j$$

$$+\, g\, G\left\{i\,[T(d_2),\ F_j(Y(d_2)/t)] + \partial_t\, F_j(Y(d_2)/t)\right\}\ f_j(Q(d_2)).$$

By Lemma 2.8 and Proposition 2.4 the factors $g\,\{i\,[H(c),\ G] + \partial_t\, G\}$ and $\{i\,[T(d_2),\ F_j]\ +\partial_t\, F_j\}\, f_j$ are positive and $O(t^{-1})$. The other factors $F_j\, f_j$ and $g\, G$, respectively, are positive up to $O(t^{-1})$ corrections. Consequently the sum of products is positive up to an integrable correction. □

Now we are ready to show stability of the decomposition and approximation properties of the time evolution for these components. In the short-range case $V^{ij,\ell} = 0$ this would finish the completeness proof. We define $U(t,\tau;d_k)$ as the propagator generated by $H(t;d_k) := H(d_k) + \sum{}'\, V_t^{im}(x)$

$$U(\tau,\tau) = \mathbf{1},\quad i\,(d/dt)\, U(t,\tau;d_k) = H(t;d_k)\; U(t,\tau;d_k). \tag{4.37}$$

in analogy to (2.11). $U(t,\tau;d_k) = \exp\{-iH(d_k)(t-\tau)\}$ if the intercluster potentials are of short range.

Proposition 4.5. *The following limits exist for every* $\Psi \in \mathcal{H}$

$$\lim_{t\to\infty}\ \exp\{iHt\}\ L_j(t;d_k)\ \exp\{-iHt\}\Psi \tag{4.38}$$

$$\lim_{t\to\infty}\ U(t,0;d_k)^*\ L_j(t;d_k)\ \exp\{-iHt\}\Psi \tag{4.39}$$

and similarly whenever the generators of the two time evolutions differ at most by an effective short-range potential. In particular Proposition 4.3 *holds in the time limit* $t \to \infty$.

Proof. This is shown exactly as (2.16), (2.19), and (2.18) using Proposition 4.4. □

So far we have used for the long-range potentials only the decay assumption $\gamma > 0$ in (1.3). From now on we will require $\gamma > 1/2$ (and in the final step even $\gamma > \sqrt{3} - 1$) to approximate U by the Dollard time evolution U^D as defined in (1.11).

Lemma 4.6. *For* $\gamma > 1/2$ *and* τ *arbitrarily large*

$$\lim_{t\to\infty} U^D(t,0;d_3)^*\, U(t,\tau;d_3)\, L_j(\tau;d_3)\, \exp\{-iH\tau\}\Psi \tag{4.40}$$

exists for the total decomposition. Moreover, for

$$\Psi'(d_3) := \sum_{\ell} \lim_{t\to\infty} \exp\{iHt\}\, L_\ell(t;d_3)\, \exp\{-iHt\}\Psi \tag{4.41}$$

the limit $\lim U^D(t,0;d_3)^*\, \exp\{-iHt\}\, \Psi'(d_3)$ *exists, i.e.* (1.14) *is satisfied for the part* $\Psi'(d_3)$ *of* $\Psi(d_3)$.

Proof. The estimates (2.24), (2.29) - (2.31) carry over because $H(t;d_3)$ contains only the tail parts of the long-range potentials. See the comment following the proof of Lemma 2.5. In particular the estimate for $t \geq \tau$

$$\|\, [Q(c)\, t - X(c)]\; U(t,\tau;d_3)\; L_\ell(\tau;d_3)\| \leq \text{const}(\tau)\; t^{\,1-\gamma} \tag{4.42}$$

corresponding to (2.24) holds for every pair c. For (4.41) see (2.21). □

The three components

$$\Psi'(d_2) := \sum_{j} \lim_{t\to\infty} \exp\{iHt\}\, L_j(t;d_2)\, \exp\{-iHt\}\Psi$$

require a finer analysis because each contains two physically different parts: either asymptotically the pair is bounded with $U^D(\cdot,\cdot;d_2)$ as asymptotic evolution or the pair ends up in a low energy scattering state with evolution $U^D(\cdot,\cdot;d_3)$. In the short-range case $\exp\{-iH(d_2)t\}$ can be used for both parts. However, when there are effective long-range intercluster potentials then $U(\cdot,\cdot;d_k)$ has to be used with the correct k, one does not have convergence for both. In the presence of infinitely namy bound states the distinction between the parts cannot be made using a smooth cutoff function $g(H(c))$. In addition, states with energy close to a threshold approach their asymptotic behaviour very slowly. For them we will need the extra decay assumption later. Accordingly we decompose the function g (which was chosen below equation (4.18)) as follows

$$g = g_< + g_0 + g_>,\;\; 0 \leq g_\cdot \leq 1,\; g_< \in C_0^\infty((-\infty,0)),\; g_> \in C_0^\infty((0,\infty)). \tag{4.43}$$

The function g_0 should have very small support around zero. The part with $g_>$ need not be treated separately because it corresponds to the case with three pairwise separating particles. The transition $g \to g_< + g_0$ only would mean a redefinition of $L_\ell(t;d_3)$ with steeper cutoff functions getting closer to the critical sets and with smaller $v_0 > 0$. The convergences shown above remain valid, but the approximations as e.g. in Proposition 4.3 will be good only for much later times.

No matter how small the support of g_0 is chosen there is only a finite number of bound states for $H(c)$ with energy in the range of $g_<$. It is well known that the eigenfunctions decay exponentially in $\|X(c)\|$. We define

$$L_<(t;d_2) := \sum_j g_<(H(c))\, G(\|X(c)\|^2/2t^2)\, F_j(Y(d_2)/t)\, f_j(Q(d_2)). \tag{4.44}$$

By Remark 2.9 the convergence as in Proposition 4.5 holds here as well. We may set

$$\Psi_<(d_2) := \lim_{t\to\infty} \exp\{iHt\}\, L_<(t;d_2)\, \exp\{-iHt\}\Psi \tag{4.45}$$

Proposition 4.7. *For any $g_< \in C_0^\infty((-\infty,0))$ and N large enough*

$$\lim_{t\to\infty} U^D(t,0;d_2)^*\, \exp\{-iHt\}\, \Psi_<(d_2) \in \operatorname{Ran} \mathbf{P}^N(d_2) \tag{4.46}$$

exists, i.e. (1.14), (1.15) *hold for the part $\Psi_<(d_2)$ of $\Psi(d_2)$.*

Proof. Again it is sufficient to show convergence of

$$U^D(t,\tau;d_2)^*\, U(t,\tau;d_2)\, L_<(\tau;d_2) \tag{4.47}$$

as $t\to\infty$ for arbitrarily late τ. Choose $\bar g_< \in C_0^\infty((-\infty,0))$ such that $\bar g_<\, g_< = g_<$. Then

$$\|(d/dt)\, U(t,\tau;d_2)^*\, \bar g_<(H(c))\, U(t,\tau;d_2)\|$$

$$\le {\sum}' \left\| [\bar g_<(H(c)),\ V_t^{im}(Y_{im}(d_2)+\lambda_{im}X(c))] \right\|$$

$$\le {\sum}' \operatorname{const}(g_<)\ \sup |\nabla V_t^{im}| \in L^1. \tag{4.48}$$

Since $L(t;d_2) = \bar g_<(H(c))\, L_<(t;d_2)$ we can study instead of (4.47) for large τ

$$U^D(t,\tau;d_2)^*\, \bar g_<(H(c))\, U(t,\tau;d_2)\, L_<(\tau;d_2). \tag{4.49}$$

The propagator U^D commutes with $\bar g_<(H(c))$, so we get in particular that the limit (if it exists) lies in $\operatorname{Ran}\bar g_<(H(c)) \subset \operatorname{Ran}\mathbf{P}^N(d_2) \subset \operatorname{Ran}\mathbf{P}(d_2)$ for sufficiently large N. Denote by $\bar U$ the evolution generated by $\bar H(t;d_2) = H(d_2) + \sum' V_t^{im}(Y_{im}(d_2))$. Then the time derivative

$$(d/dt)\, \bar U(t,\tau;d_2)^*\, \bar g_<(H(c))\, U(t,\tau;d_2) \tag{4.50}$$

is bounded by terms

$$\left\|\bar g_<(H(c)) \left\{V_t^{im}(Y_{im}(d_2)) - V_t^{im}(Y_{im}(d_2)+\lambda_{im}X(c))\right\} \right\|$$

$$\le \|\bar g_<(H(c))\, X(c)\|\ \sup |\nabla V_t^{im}| \in L^1. \tag{4.51}$$

where the particles i and m, $i < m$ lie in different clusters, $x^i - x^m = Y_{im}(d_2) + \lambda_{im} X(c)$. Now $\bar{g}_<(H(c))$ has done its duty and can be taken away again. We have shown for $t \geq \tau$ large

$$\exp\{-iHt\}\Psi_<(d_2) \approx \bar{U}(t,\tau;d_2)\, L_<(\tau;d_2)\, \exp\{-iH\tau\}\Psi. \tag{4.52}$$

With the velocity $Q_{im}(d_2) = i\,[H(d_2),\ Y_{im}(d_2)]$ we conclude convergence (4.46) from (4.52) and integrability of

$$\left\| \left\{ V_t^{im}(Q_{im}(d_2)\,t) - V_t^{im}(Y_{im}(d_2)) \right\} \bar{U}(t,\tau;d_2)\, L_<(\tau;d_2) \right\| .$$

This holds as in the two-body case for (2.29). □

Summing up we have shown so far that for

$$L_j^0(t;d_2) := \sum_j g_0(H(c))\, G(\|X(c)\|^2/2t^2)\, F_j(Y(d_2)/t)\, f_j(Q(d_2)) \tag{4.53}$$

and for their sum $L^0(t;d_2) := \sum_j L_j^0(t;d_2)$ the following limits exist for any $\Psi \in \mathcal{H}^{cont}(H)$ and d_2 if $\gamma > 0$.

$$\lim_{t\to\infty} \exp\{iHt\}\ L^0(t;d_2)\ \exp\{-iHt\}\Psi =: \Psi^0(d_2), \tag{4.54}$$

$$\lim_{t\to\infty} U(t,0;d_2)^*\ L^0(t;d_2)\ \exp\{-iHt\}\Psi =: \Phi^0(d_2). \tag{4.55}$$

On $\mathcal{H}^{pp}(H)$ the limit is zero. Moreover, for all other components in the decomposition of the scattering state Ψ one has asymptotic approximation of the time evolution by the corresponding Dollard evolution if $\gamma > 1/2$ or with the methods of Section 3 for slower decay. This holds whenever g_0 equals one in a neighborhood of the origin. We pick it such that $\lambda\, g_0'(\lambda) \leq 0$ and $g_0(\lambda) = 1$ (or 0) if $|\lambda| \leq \lambda_0$ (or $\geq 2\lambda_0$) for some $\lambda_0 > 0$. Let g_n, $n \in \mathbb{N}$, be a sequence of such functions with shrinking supports, e.g. $g_n(\lambda) = g_0(n\lambda)$, and denote correspondingly $L^n(t;d_2)$, $\Psi^n(d_2)$, $\Phi^n(d_2)$. Clearly,

$$\lim_{t\to\infty}\ \|\exp\{-iHt\}\Psi^n(d_2) - U(t,0;d_2)\,\Phi^n(d_2)\| = 0. \tag{4.56}$$

If $\lim_{n\to\infty}\Psi^n(d_2) = 0$ for all d_2 then the completeness proof is finished. Otherwise, there is a component for some d_2 where the cluster asymptotically has zero internal energy. We will show that this corresponds to the channel with the pair in a zero energy bound state asymptotically.

Lemma 4.8. *For* $z \leq \inf \sigma(H(c)) - 1$

$$\lim_{n\to\infty} \Phi^n(d_2) =: \Phi^\infty(d_2),\ \lim_{n\to\infty} \Psi^n(d_2) =: \Psi^\infty(d_2)\ \text{ exist and} \tag{4.57}$$

$$\lim_{t\to\infty} H(c)\,(H(c)-z)^{-1}\,U(t,0;d_2)\,\Phi^\infty(d_2) = 0. \tag{4.58}$$

Proof. Observe that for $n \geq N(\varepsilon)$

$$\|\Psi^0(d_2) - \Psi^n(d_2)\|^2 \geq \sup_m \|\Psi^0(d_2) - \Psi^m(d_2)\|^2 - \varepsilon^2$$

because it is bounded and monotonically growing: Assume there were an $m > n$ with

$$\|\Psi^0(d_2) - \Psi^n(d_2)\|^2 - \|\Psi^0(d_2) - \Psi^m(d_2)\|^2 \geq 2\alpha > 0.$$

Then we would have for all $t \geq T(\alpha)$

$$\|(g_0 - g_n)(H(c))\; G\; \sum_j F_j\; f_j\; \exp\{-iHt\}\Psi\|^2$$

$$- \|(g_0 - g_m)(H(c))\; G\; \sum_j F_j\; f_j\; \exp\{-iHt\}\Psi\|^2 \geq \alpha > 0.$$

This is impossible because $g_0 - g_m \geq g_0 - g_n$. Similarly we see that for all $k \geq m \geq 2n$

$$\|(\Psi^m(d_2) - \Psi^k(d_2)) + (\Psi^0(d_2) - \Psi^n(d_2))\|^2 \leq \sup_m \|\Psi^0(d_2) - \Psi^m(d_2)\|^2$$

and that $(\Psi^m(d_2) - \Psi^k(d_2))$ and $(\Psi^0(d_2) - \Psi^n(d_2))$ are orthogonal because $(g_m - g_k)\,(g_0 - g_n) = 0$. Consequently $\|\Psi^m(d_2) - \Psi^k(d_2))\| \leq \varepsilon$ for all $k \geq m \geq 2\,N(\varepsilon)$. To show the second statement

$$\|H(c)\,(H(c)-z)^{-1}\,U(t,0;d_2)\,\Phi^\infty(d_2)\|$$

$$\stackrel{\varepsilon}{\approx} \|H(c)\,(H(c)-z)^{-1}\,U(t,0;d_2)\,\Phi^n(d_2)\|$$

$$\stackrel{\varepsilon}{\approx} \|H(c)\,(H(c)-z)^{-1}\,L^n(t;d_2)\,\exp\{-iHt\}\Psi\| < \varepsilon$$

for $n \geq n(\varepsilon)$ and all $t \geq T(n,\varepsilon)$. □

The internal energy of the pair is changed only by the tail part of the long-range intercluster potentials which decays in time. To reach zero in the limit this energy must be rather small for finite times, i.e. one has qualified decay in (4.58). To improve the decay we introduce another approximation which exploits the fact that the internal energy can be changed effectively only if $\|X(c)\|$ is large. Let

$$\tilde{V}_t^{im} := \varphi(\|X(c)\|/t^{\gamma'})\; V_t^{im}(x^i - x^m) + (1-\varphi)(\|X(c)\|/t^{\gamma'})\; V_t^{im}(Y_{im}(d_2)) =$$

$$= V_t^{im}(Y_{im}(d_2)) + \varphi(...) \left\{ V_t^{im}(x^i - x^m) - V_t^{im}(Y_{im}(d_2)) \right\} \tag{4.59}$$

with an outside cutoff function φ as in (2.8) and an exponent $0 < \gamma' < \gamma$.

$$\|\tilde{V}_t^{im} - V_t^{im}(x^i - x^m)\| \le C\, t^{\gamma'} (1+t)^{-1-\gamma} \in L^1. \tag{4.60}$$

If $\tilde{U}(t,0;d_2)$ is generated by $\tilde{H}(t;d_2) := H(d_2) + \sum' \tilde{V}_t^{im}$ then it follows from (4.60) that there is a $\tilde{\Phi}(d_2)$ with

$$\lim_{t\to\infty} \| U(t,0;d_2)\, \Phi^\infty(d_2) - \tilde{U}(t,0;d_2)\, \tilde{\Phi}(d_2)\| = 0$$

and (4.58) holds correspondingly. Moreover,

$$\lim_{t\to\infty} \| [g_0(H(c)) - \mathbf{1}]\, \tilde{U}(t,0;d_2)\, \tilde{\Phi}(d_2)\| = 0$$

and we will use this cutoff to avoid problems with the unbounded $H(c)$.

Lemma 4.9. *For* $\gamma > \gamma' \ge (\sqrt{3} - 1)$

$$\|H(c)\, g_0(H(c))\, \tilde{U}(t,0;d_2)\, \Phi(d_2)\| \in L^1. \tag{4.61}$$

Proof. Setting $\mathbf{h} := H(c)\, g_0(H(c))$

$$\|(d/dt)\, \tilde{U}(t,0;d_2)^*\, \mathbf{h}\, \tilde{U}(t,0;d_2)\, \Phi(d_2)\| \le \sum{}' \, \| [\tilde{V}_t^{im},\, \mathbf{h}]\, \tilde{U}(t,0;d_2)\, \Phi(d_2)\|.$$

$$[\tilde{V}_t^{im},\, \mathbf{h}] = [\varphi(\|X(c)\|/t^{\gamma'}),\, \mathbf{h}]\, \left\{ V_t^{im}(Y_{im}(d_2) + \lambda_{im}\, X(c)) - V_t^{im}(Y_{im}(d_2)) \right\}$$

$$+ \varphi(\|X(c)\|/t^{\gamma'})\, [V_t^{im}(Y_{im}(d_2) + \lambda_{im}\, X(c)),\, \mathbf{h}]. \tag{4.62}$$

Calculating the commutators and rearranging the factors one gets as leading terms up to constants

$$t^{-\gamma'} \left\| \varphi'(X(c)\|/t^{\gamma'})\, \left\{ V_t^{im}(Y_{im}(d_2) + \lambda_{im}\, X(c)) - V_t^{im}(Y_{im}(d_2)) \right\} \right\|$$

$$\times \|Q(c)\, \varphi(2\|X(c)\|/t^{\gamma'})\, (H(c) - z)^{-1}\, \tilde{U}(t,0;d_2)\, \tilde{\Phi}(d_2)\|$$

$$+ \|\nabla V_t^{im}\| \cdot \|Q(c)\, \varphi(\|X(c)\|/t^{\gamma'})\, (H(c) - z)^{-1}\, \tilde{U}(t,0;d_2)\, \tilde{\Phi}(d_2)\|$$

$$\le C\, (1+t)^{-1-\gamma}\, \|Q(c)\, \varphi(\|X(c)\|/t^{\gamma'})\, (H(c) - z)^{-1}\, \tilde{U}(t,0;d_2)\, \tilde{\Phi}(d_2)\| \tag{4.63}$$

(and a similar term). Since the norm in (4.63) is bounded we get as a first estimate for (4.61) the bound $C\,(1+t)^{-\gamma}$. In the next step we use this information to show decay of the norm involving the velocity. Its square

$$(\mu(c)/2)\,\|Q(c)\,\varphi(\|X(c)\|/t^{\gamma'})\,(H(c)-z)^{-1}\,\tilde U(t;0;d_2)\,\tilde{\Phi}(d_2)\|^2$$

$$\leq \left|\left(\tilde{\Phi}(d_2),\ \tilde U^*\ (H(c)-z)^{-1}\,\varphi\,H(c)\,\varphi\ (H(c)-z)^{-1}\,\tilde U\,\tilde{\Phi}(d_2)\right)\right|$$

$$+\ \|V^{ij}(X(c))\,\varphi(\|X(c)\|/t^{\gamma'})\|.$$

The last term is bounded by $(1+t)^{-\gamma\gamma'}$ and has slowest decay. For the first we get a bound proportional to

$$\|\,[Q(c),\ \varphi(\|X(c)\|/t^{\gamma'})]\,\| + \|H(c)\,(H(c)-z)^{-1}\,\tilde U\,\tilde{\Phi}(d_2)\|$$

$$\leq C\,(t^{-\gamma'} + (1+t)^{-\gamma}).$$

Summing up, the norm in (4.61) decays with an exponent $\gamma + (\gamma\gamma')/2 > 1$ if $\gamma > \gamma' \geq \sqrt{3}-1$. □

The remaining steps are simple. The last lemma permits to replace the generator $\tilde H(t)$ by $\tilde H_0(t)$ where $H(c)$ is set zero, with propagator $\tilde U_0$. One has existence of

$$\tilde{\Phi}_0(d_2) := \lim_{t\to\infty} \tilde U_0(t,0;d_2)\,\tilde U(t,0;d_2)\,\tilde{\Phi}(d_2).$$

With respect to the variable $X(c)$ the evolution $\tilde U_0$ acts trivially as a phase factor. Thus for $\varepsilon > 0$ there is an R such that for all $t \geq 0$

$$\|\,F(\|X(c)\| > R)\,\tilde U_0(t,0;d_2)\,\tilde{\Phi}(d_2)\| < \varepsilon.$$

Since $\|F(\|X(c)\| < R)\,\{V_t^{im}(Y_{im}(d_2) + \lambda_{im}\,X(c)) - V_t^{im}(Y_{im}(d_2))\}\| \leq C\,(1+t)^{-1-\gamma} \in L^1$ we can eliminate the dependence on any internal variables of the pair altogether and conclude that the state asymptotically is a zero energy bound state of the pair, because it is in the ranges of *all* $g_n(H(c))$. Finally one can replace $V_t^{im}(Y_{im}(d_2))$ by $V_t^{im}(Q_{im}(d_2)t)$ as above in the proof of Proposition 4.7 to have the Dollard time evolution also for this term. We have shown existence of the limit

$$\lim_{t\to\infty} U^D(t,0;d_2)\,\exp\{-iHt\}\,\Psi^\infty(d_2) \in \mathbf{P}(d_2).$$

Thus also $\Psi^\infty(d_2)$ is part of $\Psi(d_2)$ in (1.14) and (1.15). If the subsystem does not have zero energy bound states then such a term must vanish. This finishes the proof of asymptotic completeness.

As for the two body case one can use essentially the same estimates for proving existence of the Dollard modified wave operators for $\gamma > 1/2$. The faster decay is not needed because the Dollard time evolutions conserve the internal energy of subsystems. We omit the obvious details. Thus we have shown using the well known intertwining properties

Theorem 4.10. *Let the Hamiltonian H of the three-body system* (1.9) *have pair potentials which satisfy* (1.2), (1.3) *for $\gamma > 1/2$. Then the modified Dollard wave operators $\Omega^D(d_k)$ exist* (1.12). *For $\gamma > \sqrt{3} - 1$ they are complete* (1.13). *On the subspaces $\mathcal{H}(d_k) := \operatorname{Ran} \Omega^D(d_k)$ the Hamiltonian is unitarily equivalent to a free three-body Hamiltonian: $H \upharpoonright \mathcal{H}(d_3) = \Omega^D(d_3)\ H_0\ (\Omega^D(d_3))^*$, or to a direct sum of shifted free two-body Hamiltonians: $H \upharpoonright \mathcal{H}(d_2) = \Omega^D(d_2)\ H(d_2)\ (\Omega^D(d_2))^*$. In particular H has no singular continuous spectrum.*

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