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A CLASS OF WEIGHTED FUNCTION SPACES,

AND INTERMEDIATE CACCIOPPOLI-SCHAUDER ESTIMATES

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1 - A THEOREM OF D. GILBARG AND L. HORMANDER

Consider the Dirichlet problem

(1) $$Lu = f \text{ in } \Omega \,, \; u\big|_{\partial\Omega} = \varphi \,,$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $\partial\Omega$ its boundary, and L a linear second order uniformly elliptic differential operator with coefficients defined on $\overline{\Omega}$. The classical Caccioppoli-Schauder approach to (1) provides, under suitable regularity assumptions about $\partial\Omega$ and the coefficients of L, a priori bounds on norms

$$|u|_{C^{k,\delta}(\Omega)} \,,\; k = 2, 3, \ldots \quad \text{and} \quad \delta \in \,]0,1[\;;$$

this of course requires, to start with, the membership of f in $C^{k-2,\delta}(\overline{\Omega})$ and of φ in $C^{k,\delta}(\partial\Omega)$.

What happens now if we weaken our assumption about φ by requiring that it belong to $C^{k',\delta'}(\partial\Omega)$ for some $k' = 0, 1, \ldots$ and some $\delta' \in \,]0,1[$ such that $k' + \delta' < k + \delta$? An answer to this question was given by Gilbarg and Hörmander [4] : they provided weighted $C^{k,\delta}$ norm estimates for solutions of (1), the weight consisting of the α–th power of the distance from $\partial\Omega$ with $\alpha \equiv k + \delta - (k' + \delta')$. Note that, for what correspondingly concerns f, the natural regularity requirement is now only that its weighted $C^{k-2,\delta}$ norm be finite.

In order to illustrate the key point of [4] we introduce some notations. Letting

$$B_r(x^o) \equiv \{x \in \mathbb{R}^N \mid |x-x^o| < r\}$$

$$B_r^+(x^o) \equiv \{x \in B_r(x^o) \mid x_N > x_N^o\}$$

$$S_r^+(x^o) \equiv \{x \in \partial B_r(x^o) \mid x_N > x_N^o\}$$

$$S_r^o(x^o) \equiv \partial B_r^+(x^o) \setminus \overline{S_r^+(x^o)}.$$

(under the convention that the dependence on x^o, r be depressed if $x^o = O$, $r = 1$), we define $C_\alpha^{k,\delta}(B_R^+)$ as the space of functions $u = u(x)$, $x \in B_R^+$, having finite norms

$$|u|_{C_\alpha^{k,\delta}(B_R^+)} \equiv \sup_{S>O} S^\alpha |u|_{C^{k,\delta}(B_{R[S]}^+)}$$

here, $k = O, 1, \ldots$, $O < \delta \le 1$, $\alpha \ge O$, and $B_{R[S]}^+ \equiv \{x \in B_R^+ \mid x_N > S\}$. (When $\alpha < O$ the right-hand side in the above definition of norm is finite only for $u = O$). Through direct investigation of Green's function for the Laplace operator in the upper half space Gilbarg and Hörmander proved the following result (Theorem 3.1 of their paper) : let $k = 2, 3, \ldots$, $O < \delta < 1$, $O \le \alpha < k + \delta$ and $k + \delta - \alpha \notin \mathbb{N}$; then there exists a constant C such that

$$(2)_k \qquad |u|_{C_\alpha^{k,\delta}(B^+)} \le C\, |f|_{C_\alpha^{k-2,\delta}(B^+)}$$

whenever u is a function from $C_\alpha^{k,\delta}(B^+)$ which vanishes near S^+ and satisfies (in the pointwise sense)

$$(3) \qquad u\big|_{S^o} = O \quad , \quad \Delta u = f \text{ in } B^+.$$

What we are going to describe in the present article is an alternative approach to (3), which yields a slightly more general result than the bounds $(2)_k$. Notice that the passage from Δ to more general variable coefficient operators L can be achieved through a perturbation argument as in [4 , prop. 4.3] ; the case of nonvanishing Dirichlet data φ on S^o can be handled through suitable extensions of the φ's to the upper half space [4 , lemma 2.3] ; finally, partitions of unity and changes of variables near boundary points lead to the general setting of (1) [4 , theorem 5.1]. This procedure exhibits rather delicate technical features, if one wants to adopt the "natural" generality for what concerns regularity assumptions about the coefficients of L as well as $\partial\,\Omega$. The crux of the matter lies, however, within the study of (3).

2 - THE MAIN RESULTS OF THIS ARTICLE

We are going to deal with weak solutions to a problem such as

(4) $$u\big|_{S^o} = O \ , \quad \Delta u = f + f^i_{x_i} \quad \text{in } B^+$$

i.e., for some $p \in \,]1, \infty[$,

$$u \in H^{1,p}(B^+) \ , \ u\big|_{S^o} = O \ ,$$

(5) $$\int_{B^+} u_{x_i} \varphi_{x_i} \, dx = \int_{B^+} (-f\varphi + f^i \varphi_{x_i}) \, dx \qquad \forall \ \varphi \in C^\infty_o(B^+)$$

(summation convention of repeated indices). Here and throughout, $H^{k,p}$ and $H^{k,p}_o$ are the standard notations for Sobolev spaces.

For our study of regularity we find it convenient to introduce new (norms and) function spaces. Namely, for $1 \leq p < \infty$, $\alpha \in \mathbb{R}$ and $O \leq \lambda \leq N+p$ let

$$[u]_{L^{p,\lambda}_\alpha(B^+_R)} \equiv \sup_{x^o \in B^+_R, \rho > o} \rho^{-\lambda} \inf_{c \in \mathbb{R}} \int_{B^+_R \cap B_\rho(x^o)} x_N^{p\alpha} \, |u - c|^p \, dx$$

and denote by $L^{p,\lambda}_\alpha(B^+_R)$ the space of functions $u = u(x)$, $x \in B^+_R$, having finite norms

$$|u|_{L^{p,\lambda}_\alpha(B^+_R)} \equiv \Big(\int_{B^+_R} x_N^{p\alpha} \, |u|^p \, dx + [u]^p_{L^{p,\lambda}_\alpha(B^+_R)} \Big)^{1/p} .$$

It is clear that, for any value of α, $L^{p,\lambda}_\alpha(B^+_R)$ at least contains $C^\infty_o(B^+_R)$.

$L^{p,\lambda}_o(B^+_R)$ is the by now classical campanato space, and $L^{p,\lambda}_o(B^+_R) \sim C^{o,(\lambda-N)/p}(\overline{B^+_R})$ if $N < \lambda \leq N+p$ [2]. But we have more :

<u>Lemma 1</u>

For $\alpha \geq O$ and $N < \lambda \leq N+p$ the spaces $L^{p,\lambda}_\alpha(B^+_R)$ and $C^{o,(\lambda-N)/p}_\alpha(B^+_R)$ are isomorphic.

$L^{p,N}_o(B^+_R)$ is a BMO ($\equiv$ Bounded Mean Oscillation) space [6]. The importance of BMO spaces as "good substitutes" for C^o and L^∞ has since long been acknowledged in PDE's (and Harmonic Analysis ...). Take for instance our initial considerations about the classical Caccioppoli-Schauder approach to (1) :

BMO spaces are known to fill the gaps left over by the exclusion of the two values $\delta = O$ and $\delta = 1$ [3]. But weighted norms lead to another example. Precisely, consider the continuous imbedding

$$(6) \qquad C^{o,\delta+\beta}_{\alpha+\beta}(B^+_R) \subset C^{o,\delta}_{\alpha}(B^+_R)$$

which is proven in [4] for $\alpha \geq O$, $O \leq \delta < 1$ and $\beta > O$ with $\delta + \beta \leq 1$, under the restriction $\alpha \neq \delta$. This restriction has far-reaching consequences, such as the above-mentioned requirement $k + \delta - \alpha \notin \mathbb{N}$ for the validity of $(2)_k$. But, why cannot $\alpha = \delta$ be allowed? For sure, (6) is false when $\alpha = \delta = O$, as the one-dimensional example given in [4], that is, $u(x) \equiv log\ x$, $O < x < 1$, clearly shows. But, as it happens, this function u belongs to $L^{p,N}_{o}(\,]O,1[\,)$ ... We can indeed prove the following result, which contains (6) in all cases except $\alpha \neq O = \delta$.

Lemma 2

For $\alpha \geq O$, $O \leq \delta < 1$ and $\beta > O$ with $\delta + \beta \leq 1$, the continuous imbedding

$$L^{p,N+p(\delta+\beta)}_{\alpha+\beta}(B^+_R) \subset L^{p,N+p\delta}_{\alpha}(B^+_R)$$

is valid.

We can now arrive at our results about solutions to (5). Adopting the symbol $L^{\infty}_{\beta}(B^+)$ to denote the space of measurable functions $h = h(x)$, $x \in B^+$, such that

$$|h|_{L^{\infty}_{\beta}(B^+)} \equiv |x^{\beta}_N h|_{L^{\infty}(B^+)}$$

is finite, we begin with first derivatives.

Theorem 1

Let $O \leq \delta < 1$, $O \leq \alpha < 1+\delta$. If, for a suitable value of $p > 1$, u satisfies (5) with $f \in L^{\infty}_{1+\alpha-\delta}(B^+)$ and $f^1, \dots, f^N \in C^{o,\delta}_{\alpha}(B^+)$, then all its first derivatives belong to $L^{p,N+p\delta}_{\alpha}(B^+_R)$, $0 < R < 1$, and satisfy

$$\sum_{i=1}^{N} |u_{x_i}|_{L^{p,N+p\delta}_{\alpha}(B^+_R)} \leq C\Big(|f|_{L^{\infty}_{1+\alpha-\delta}(B^+)} + \sum_{i=1}^{N} |f^i|_{C^{o,\delta}_{\alpha}(B^+)} + |u|_{H^{1,p}(B^+)}\Big)$$

with C independent of $u, f, f^1, \dots, f^N$.

The passage to second derivatives is performed, so to speak, through "differentiation" of (5) with respect to $x_1, \dots, x_{N-1}$. Without loss of generality, it can be assumed that $f^1 = \dots = f^N = O$; as for f, the "natural" requirement becomes

$$f \in C^{o,\delta}_{\alpha}(B^+)$$

for $O \le \alpha < 2+\delta$. It is the range $1+\delta \le \alpha < 2+\delta$, of course, that poses new difficulties : no longer is then f in some $L^p(B^+)$, so that the $H^{2,p}$ regularity theory does apply to (5), and the above results about u are not inherited by u_{x_S}, $S = 1, \dots, N-1$. But $H^{2,p}$ regularity does apply to $x_N u$, and $U \equiv x_N u_{x_S}$ satisfies, in the weak sense,

$$U\big|_{S^o_{R_1}} = O\,, \ \Delta U = -x_N f_{x_S} + 2\, u_{x_S x_N} \quad \text{in} \quad B^+_{R_1}$$

for any $R_1 \in \,]O, 1[$. We can thus arrive at.

<u>Theorem 2</u>

Let $O \le \delta < 1$, $O \le \alpha < 2+\delta$. If, for a suitable value of $p > 1$, u satisfies (5) with $f \in C^{o,\delta}_{\alpha}(B^+)$ and $f^1 = \dots = f^N = O$, then all its second derivatives belong to $L^{p,N+p\delta}_{\alpha}(B^+_R)$ when restricted to B^+_R, $O < R < 1$, and satisfy

$$(7) \qquad \sum_{i,j=1}^{N} |u_{x_i x_j}|_{L^{p,N+p\delta}_{\alpha}(B^+_R)} \le C\left(|f|_{C^{o,\delta}_{\alpha}(B^+)} + |u|_{H^{1,p}(B^+)}\right)$$

with C independent of u, f.

(If we want to be more specific in the choice of p, we take $p = 2$ for $O \le \alpha < \frac{1}{2} + \delta$ and $1 < p < \frac{1}{\alpha-\delta}$ for $\frac{1}{2} + \delta \le \alpha < 1+\delta$ in both Theorems 1 and 2, $p = 2$ for $1+\delta \le \alpha < \frac{3}{2} + \delta$ and $1 < p < \frac{1}{\alpha-1-\delta}$ for $\frac{3}{2} + \delta \le \alpha < 2 + \delta$ in Theorem 2).

When supp $u \cap S^+ = \emptyset$, (7) holds for $R = 1$ without the term $|u|_{H^{1,p}(B^+)}$ on its right hand side. This means that $(2)_2$ holds for all values of α in the range $[O, 2+\delta[$, $O < \delta < 1$, that is, without exception for $\alpha = \delta$ and $\alpha = 1+\delta$. Since the procedure leading to Theorem 2 can be repeated for all higher order derivatives, $(2)_k$ holds whenever $k = 2, 3, \dots$ and $O \le \alpha < k+\delta$, $O < \delta < 1$, <u>no exception being made for</u> $k+\delta-\alpha \in \mathbb{N}$.

As for $\delta = O$, we simply mention that $C^{o,o}_{\alpha}(B^+)$ could safely be replaced by $L^\infty_\alpha(B^+)$ throughout. The above results can therefore be said to contain "weighted versions of the $L^\infty \to BMO$ type of regularity".

A few words about our techniques. The main tools are estimates such as

(8) $$\int_{B_\rho(x^o)} |\nabla w|^p\, dx \le C(p)\Big[\Big(\frac{\rho}{r}\Big)^N \int_{B_r(x^o)} |\nabla w|^p dx + \sum_{i=1}^{N} \int_{B_r(x^o)} |h^i|^p\, dx\Big]$$

and

(9) $$\int_{B_\rho(x^o)} |\nabla w - (\nabla w)_{\rho;\alpha}|^p\, dx \le C(p,\alpha)\Big[\Big(\frac{\rho}{r}\Big)^{N+p} \int_{B_r(x^o)} |\nabla w - (\nabla w)_{r,\alpha}|^p\, dx + \sum_{i=1}^{N} \int_{B_r(x^o)} |h^i - (h^i)_{r,\alpha}|^p\, dx\Big],$$

which hold whenever w satisfies

$$w \in H^{1,p}(B_r(x^o)),$$

$$\int_{B_r(x^o)} w_{x_i}\, \varphi_{x_i}\, dx = \int_{B_r(x^o)} h^i\, \varphi_{x_i}\, dx \quad \forall\, \varphi \in C^\infty_o(B_r(x^o))$$

where $O < \rho \le r < \infty$, $x^o \in \mathbb{R}^N$; in (9), the symbol $(.)_{\rho\,;\,\alpha}$ denotes average over $B_\rho(x^o)$ with respect to $x_N^\alpha\, dx$, $\alpha \ge O$. We need p from $]1,2]$. For $p = 2$, (8) and (9) are obtained [3] through typical techniques of the Hilbert space theory of elliptic PDE's. The passage to $1 < p < 2$ requires some preliminary results from the corresponding $H^{k,p}$ theory which can be found, for instance, in [7].

If spheres $B_\rho(x^o)$ are replaced throughout by hemispheres $B^+_\rho(x^o)$ - and w is required to vanish on $S^o_r(x^o)$ - the counterpart of (8) is obviously valid for $1 < p \le 2$, while the counterpart of (9) is only needed here for $p = 2$ as in [3].

Detailed proofs will appear in a forthcoming article.

The results mentioned here could be compared with those of [1] , [5] , where the perturbing role of the boundary appears through degeneration of operators rather than explosion of some norms of free terms (and boundary data).

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