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SINGULAR PERTURBATIONS AND PARAMETER-DEPENDENT PSEUDO-DIFFERENTIAL BOUNDARY PROBLEMS

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The purpose of this talk is to show how the theory of parameter-dependent pseudo-differential boundary problems developed in Grubb [7-12] can be used to treat some important problems in singular perturbation theory, for differential operators as well as pseudo-differential operators.

The parameter-dependent ps.d.o. calculus was developed in particular for the study of resolvents of elliptic boundary value problems. Let $\bar{\Omega}$ be a compact n-dimensional C^∞ manifold with boundary $\partial\Omega$ and interior Ω ; let E be an N-dimensional C^∞ vector bundle over $\bar{\Omega}$ and F an M-dimensional C^∞ vector bundle over $\partial\Omega$. For a positive integer d we consider the system of operators defined within the Boutet de Monvel calculus [2]

$$\text{(1)} \qquad \begin{pmatrix} P_\Omega + G \\ T \end{pmatrix} : \ C^\infty(E) \to \begin{matrix} C^\infty(E) \\ \times \\ C^\infty(F) \end{matrix} \quad ,$$

where P is a pseudo-differential operator in a bundle $\tilde{E}$ extending E to an open neighborhood of $\bar{\Omega}$ and P_Ω is its restriction to Ω , P has the transmission property at $\partial\Omega$, G is a singular Green operator in E of order d and class $\leq d$, and $T = \{T_0,\dots,T_{d-1}\}$ is a system of trace operators from E to F_k of order k and class $\leq k+1$; here $F = F_0 \oplus\dots\oplus F_{d-1}$ and $\dim F_k = M_k \geq 0$, so that $M = M_0 +\dots+ M_{d-1}$. (When $M_k=0$, T_k is void.) Let B denote the L^2-realization defined by $\{P_\Omega+G,T\}$, that is, B <u>acts like</u> $P_\Omega+G$ in $L^2(E)$, <u>with domain</u>

$$\text{(2)} \qquad D(B) = \{u \in H^d(E) \mid Tu = 0\} \ .$$

The resolvent $R_\lambda = (B-\lambda)^{-1}$ is constructed in our work under the following hypothesis (expressed in local coordinates, with $x = (x',x_n) \in \bar{\mathbb{R}}^n_+$, $\xi \in \mathbb{R}^n$, $\mu \geq 0$):

<u>Hypothesis of parameter-ellipticity on the ray</u> $\lambda = re^{i\theta_0}$. *Let* $\theta_0 \in [0,2\pi]$, *and let* $\omega = \exp[i(\theta_0-\pi)]$.

(I) *The principal, strictly homogeneous, interior symbol* $p^h(x,\xi)+\omega\mu^d$ *is bijective in* $\mathbb{C}^N$ *for all* $x \in \overline{\mathbb{R}}^n_+$, $\xi \in \mathbb{R}^n\setminus 0$, $\mu \geq 0$.

(II) *The principal, strictly homogeneous, boundary symbol operator*

$$\text{(3)}\qquad a^h(x',\xi',\mu,D_n) = \begin{pmatrix} p^h(x',0,\xi',D_n)_\Omega + g^h(x',\xi',D_n) + \omega\mu^d \\ t^h(x',\xi',D_n) \end{pmatrix} : H^d(\overline{\mathbb{R}}_+)^N \to \begin{matrix} L^2(\mathbb{R}_+) \\ \times \\ \mathbb{C}^M \end{matrix}$$

is bijective for all $x' \in \mathbb{R}^{n-1}$, $\xi' \in \mathbb{R}^{n-1}\setminus 0$, $\mu \geq 0$.

(III) *For* $\xi' \to 0$, $a^h(x',\xi',\mu,D_n)$ *converges in symbol norm to an operator* $a^h(x',0,\mu,D_n)$ *that is bijective from* $H^d(\overline{\mathbb{R}}_+)^N$ *to* $L^2(\mathbb{R}_+)^N \times \mathbb{C}^M$ *for all* x', *all* $\mu > 0$.

Conditions (I) and (II) are straightforward ellipticity hypotheses, whereas (III) plays a more special rôle. Condition (III) implies that the boundary condition is <u>normal</u>; i.e. the trace operator is of the form

$$\text{(4)}\qquad T = \begin{pmatrix} s_{00}(x')\gamma_0 + T_0' \\ s_{11}(x')\gamma_1 + s_{10}(x',D')\gamma_0 + T_1' \\ \vdots \\ s_{d-1,d-1}(x')\gamma_{d-1} + \sum_{j<d-1} s_{d-1,j}(x',D')\gamma_j + T_{d-1}' \end{pmatrix} ,$$

where the T_j' are of class 0 , and the coefficient matrix

$$\text{(5)}\qquad \begin{pmatrix} s_{00}(x') \\ s_{11}(x') \\ \vdots \\ s_{d-1,d-1}(x') \end{pmatrix} : \mathbb{C}^N \to \mathbb{C}^M$$

is <u>surjective</u> at each x' . In fact, if $N = 1$ (and in certain cases where $N > 1$, e.g. when $\dim F_k = N$ or 0 for all k), condition (III) is <u>equivalent with normality</u>. (In the most general cases, normality plus invertibility of a certain matrix function imply (III).)

(III) is usually naturally assumed (more or less tacitly) in the resolvent constructions for differential operator boundary problems, cf. Agmon [1], Seeley [17]. It is of interest to discuss cases where (III) does not hold (cf. Rempel and Schulze [16] and Grubb [10-12]), but our main positive results are shown under this assumption.

We show that when (I)-(III) hold for a θ_0 , then the resolvent $R_\lambda = (B-\lambda)^{-1}$ exists, as a bounded operator in $L^2(E)$, for $\lambda = \mu^d \exp(i\theta_0)$, $\mu \geq \mu_0$ (with a sufficiently large μ_0). It satisfies the estimates

$$(6) \qquad \langle\mu\rangle^{s+d} \| R_\lambda f \|_0 + \| R_\lambda f \|_{d+s} \leq c_s (\langle\mu\rangle^s \| f \|_0 + \| f \|_s) \quad \text{for any} \quad s \geq 0 ,$$

and the precise structure is described in a generalized Boutet de Monvel calculus. This is used in [12] to discuss: complex powers B^z (when the Hypothesis holds for $\theta_0 = \pi$); the "heat operator" $\exp(-tB)$ and its trace, cf. also [10] (when the Hypothesis holds for all $\theta_0 \in [\pi/2, 3\pi/2]$, i.e. parabolicity holds); the index of any normal elliptic problem; spectral asymptotics; and singular perturbations (see also [11]). We describe here the last application.

Consider a boundary problem (all orders in the following are integers)

$$(7) \qquad \begin{aligned} A_0 u &= f \quad \text{in } E , \\ T_{0,j} u &= \varphi_{0,j} \quad \text{at } \partial\Omega , \qquad 0 \leq j < k_0' , \end{aligned}$$

where $A_0 = P_{0,\Omega} + G_0$ is of order $r \geq 0$ and class 0 , and $T_{0,j}: C^\infty(E) \to C^\infty(F_j^0)$ is of order j and class $\leq j+1$ for each j ; here we denote $\{T_{0,j}\}_{0 \leq j < k_0'} = T_0$, $\oplus_{0 \leq j < k_0'} F_j^0 = F^0$ and $\dim F^0 = M^0$. Consider also an associated "perturbed problem"

$$(8) \qquad \begin{aligned} (\varepsilon^d A_1 + A_0) u_\varepsilon &= f \quad \text{in } E , \\ T_{0,j} u_\varepsilon &= \varphi_{0,j} \quad \text{at } \partial\Omega \qquad \text{for } 0 \leq j < k_0' , \\ T_{1,j} u_\varepsilon &= \varphi_{1,j} \quad \text{at } \partial\Omega \qquad \text{for } k_1 \leq j < k_1 , \end{aligned}$$

where $A_1 = P_{1,\Omega} + G_1$ is of order $r+d$ and class 0 , $d > 0$, and $T_{1,j}: C^\infty(E) \to C^\infty(F_j^1)$ is of order j and class $\leq j+1$ for each j . Here we denote $\{T_{1,j}\}_{k_1 \leq j < k_1'} = T_1$, $\oplus_{k_1 \leq j < k_1'} F_j^1 = F^1$ and $\dim F^1 = M^1$. We assume:

1^0 $0 \leq k_0' < r$, *and* $\begin{pmatrix} A_0 \\ T_0 \end{pmatrix}$ *is <u>elliptic</u> and bijective, with inverse* $(R_0 \quad K_0)$.

2^0 $k_0' < k_1 \leq r$ *and* $0 \leq k_1' - k_1 < d$, T_1 *is <u>normal</u>, and* $\{A_1, T_0, T_1\}$ *is <u>elliptic</u>.*

3^0 *The parameter-dependent principal symbol* $\varepsilon^d p_1^h(x,\xi) + p_0^h(x,\xi)$ *is bijective in* $\mathbb{C}^N$ *for all* x , *all* $\xi \neq 0$ *and all* $\varepsilon > 0$; *and the parameter-dependent principal boundary symbol operator*

$$(9) \qquad \begin{pmatrix} \varepsilon^d a_1^h(x',\xi',D_n) + a_0^h(x',\xi',D_n) \\ t_0^h(x',\xi',D_n) \\ t_1^h(x',\xi',D_n) \end{pmatrix} : H^{r+d}(\overline{\mathbb{R}}_+)^N \to \begin{matrix} L^2(\mathbb{R}_+)^N \\ \times \\ \mathbb{C}^{M^0} \\ \times \\ \mathbb{C}^{M^1} \end{matrix}$$

is bijective for all x' , *all* $\xi' \neq 0$ *and all* $\varepsilon > 0$.

(An extra hypothesis is added if $N > 1$ and not all $\dim F_j^1$ are either N or 0 ; otherwise the above conditions suffice for our purposes.)

The problem (8) will be solvable for sufficiently small ε , and one is interested in the behavior of u_ε for $\varepsilon \to 0$. In fact, u_ε converges to u in certain spaces, and one wants to determine this behavior, and more precisely to get explicit formulas for the operator mapping $\{f,\varphi_0,\varphi_1\}$ into $u_\varepsilon - u$.

The problem has been studied for differential operators by numerous authors; let us in particular mention Vishik and Lyusternik [18], Huet [13-15] and Frank [3], and the generalizations of Frank and Wendt [4-6] to certain pseudo-differential cases with rational symbols. See also their references.

Example. Consider the problems

$$(10) \qquad \begin{aligned} \varepsilon^2\Delta^2 u_\varepsilon - \Delta u_\varepsilon &= f \quad \text{in } \Omega , \\ \gamma_0 u_\varepsilon &= 0 \quad \text{at } \partial\Omega , \\ \gamma_1 u_\varepsilon &= 0 \quad \text{at } \partial\Omega , \end{aligned}$$

and

$$(11) \qquad \begin{aligned} \varepsilon^2\Delta^2 u'_\varepsilon - \Delta u'_\varepsilon &= f \quad \text{in } \Omega , \\ \gamma_0 u'_\varepsilon &= 0 \quad \text{at } \partial\Omega , \\ \gamma_2 u'_\varepsilon &= 0 \quad \text{at } \partial\Omega , \end{aligned}$$

that are singular perturbations of the Dirichlet problem

$$(12) \qquad -\Delta u = f \quad \text{in } \Omega , \qquad \gamma_0 u = 0 \quad \text{at } \partial\Omega .$$

An obvious reduction of (10) and (11) is to insert $v_\varepsilon = -\Delta u_\varepsilon$ (so that $u_\varepsilon = R_0 v_\varepsilon$) . Multiplying by $\mu^2 = \varepsilon^{-2}$ we reduce (10) to the problem

(13) $$(-\Delta+\mu^2)v_\varepsilon = g \quad \text{in } \Omega , \qquad \gamma_1 R_0 v_\varepsilon = 0 \quad \text{at } \partial\Omega ,$$

and we reduce (11) to the problem

(14) $$(-\Delta+\mu^2)v'_\varepsilon = g \quad \text{in } \Omega , \qquad \gamma_2 R_0 v'_\varepsilon = 0 \quad \text{at } \partial\Omega ,$$

where $g = \mu^2 f$. Here

(15) $$\gamma_2 R_0 = s(x')\gamma_0 + T'_0$$

is a normal trace operator with $s(x') \neq 0$ for all x' , whereas $\gamma_1 R_0$ is of order -1 and not normal. Our resolvent theory applies easily to (14), but not to (13) where condition (III) is violated.

Completely similar considerations hold in the general case: If we insert $u_\varepsilon = R_0 v_\varepsilon + K_0\varphi_0$, we can reduce problem (8) to a resolvent problem, where the boundary condition is normal if the lowest order in T_1 is $\geq r$; in which case the resolvent construction is directly applicable. But if T_1 contains nontrivial entries of order $< r$, the boundary condition resulting from this reduction is not normal, so condition (III) is violated.

The violation of condition (III) need not be prohibitive in constant coefficient cases on $\overline{\mathbb{R}}^n_+$, where it is sometimes possible to get solvability results and resolvent formulas e.g. by working in other (parameter-dependent) Sobolev spaces than the most usual ones; see Frank and Wendt [4 - 6] for a treatment of certain types of ps.d.o. singular perturbations, and see Rempel and Schulze [16] Part I for some further observations.

However, for variable coefficient cases, the failure of condition (III) is a severe handicap if one wants to use pseudo-differential techniques, for it means (in the symbol classes we consider) that the terms in the inversion formulas arizing from differentiation and lower order symbols do not have a better μ-behavior than the principal part (in fact they generally blow up in μ for $\mu \to \infty$), so the usual series expansion of the resolvent does not converge.*) (Indeed, in Part II of Rempel-Schulze [16] on variable coefficient cases, there are made hypotheses in order to avoid this phenomenon, which imply condition (III) for Boutet de Monvel type operators. Frank and Wendt [4-6] give very few details concerning this point for boundary problems, and we have not been able to (re)construct their argument.)

Now it turns out that the problem of the violation of condition (III) (the so-called negative regularity) can be avoided altogether if we use a little more

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*) When (I)-(III) hold, the ξ-derivatives and lower order terms are at least $\mu^{-\frac{1}{2}}$ better than the principal term.

of the Boutet de Monvel calculus before going on to the resolvent. By the hypotheses on the various orders, there exists an integer ℓ such that

(16) $$k_0' < r-\ell \leq k_1$$

(for example, $\ell = 1$ in (10)). Moreover, there exists an elliptic ps.d.o. $\Lambda_{-\ell}$ (restricted to Ω) such that $\Lambda_{-\ell}$ maps $H^s(E)$ isomorphically onto $H^{s+\ell}(E)$ for all $s \geq 0$. (The symbol of $\Lambda_{-\ell}$ is a modified version of $(\langle\xi'\rangle - i\xi_n)^{-\ell}$.) Instead of inserting $u_\varepsilon = R_0 v_\varepsilon + K_0\varphi_0$, we now insert $u_\varepsilon = R_0(\Lambda_{-\ell})^{-1}w_\varepsilon + K_0\varphi_0$ in (8) , and arrive by composition with $\Lambda_{-\ell}$ and multiplication by $\mu^d = \varepsilon^{-d}$ at the problem

(17) $$\Lambda_{-\ell}A_1R_0(\Lambda_{-\ell})^{-1}w_\varepsilon + \mu^d w_\varepsilon = g \text{ in } E , \qquad T_1R_0(\Lambda_{-\ell})^{-1}w_\varepsilon = \varphi_1 \text{ at } \partial\Omega ,$$

$g = \Lambda_{-\ell}\mu^d f$, where the new interior operator

(18) $$A = \Lambda_{-\ell}A_1R_0(\Lambda_{-\ell})^{-1} = [\Lambda_{-\ell}P_1P_0^{-1}(\Lambda_{-\ell})^{-1}]_\Omega + G$$

is of order d and class 0 , and the new trace operator

(19) $$T = T_1R_0(\Lambda_{-\ell})^{-1}$$

is <u>normal</u>. The hypotheses 1^0-3^0 then assure that (17) satisfies conditions (I)-(III) if $N = 1$, or $N > 1$ and $\dim F_j^1 = N$ or 0 for all j (in the remaining cases, and extra matrix condition is added).

Now the theorems of Grubb [7 -12] can be applied to (17), which gives an explicit solution operator for $\mu \geq \mu_0$ (μ_0 sufficiently large), i.e. for $\varepsilon \in]0,\mu_0^{-1}]$. This is then carried back to give the solution operator for (8), and the various resolvent and operator estimates can be used to obtain estimates of $u_\varepsilon - u$ for $\varepsilon \to 0$, as well as expansions in powers of $\varepsilon^{\frac{1}{2}}$.

Further details are given in Grubb [11] and [12].

Also problems with a more complicated ε-dependence can be treated by use of the general μ-dependent calculus in Grubb [12].

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