JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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On the wellposed singular boundary value problems for heat operator

ON THE WELLPOSED SINGULAR BOUNDARY VALUE

PROBLEMS FOR HEAT OPERATOR

by S. MIZOHATA

§ 1 - INTRODUCTION -

In [2], S. Itô treated the following initial-boundary value problem for heat operator :

(I.B.P.)

(1) $\dfrac{\partial u}{\partial t} = \Delta u$, in $\Omega \subset \mathbb{R}^n$,

(2) $Bu = a(x) \dfrac{\partial u}{\partial n} + b(x)u = 0$, $x \in \partial\Omega$,

where $\dfrac{\partial u}{\partial n}$ is the derivative in the direction of outer-normal, and,

$a(x) \geqslant 0$, $b(x) \geqslant 0$, $a(x) + b(x) = 1$,

(3) $u\big|_{t=o} = u_o(x)$

He proved that this problem is well-posed by constructing explicitly fundamental solution. His method is fairly complicated. Later several authors (cf. [1], [3], [4], [6]), treated the elliptic boundary value problem with the boundary condition (2), using the methods of functional analysis. Fairly recently, M. Terakado pointed out that in the paper of S. Itô, a crucial lemma is not clear. From that time, the author discussed often with him about this problem. Our purpose is to consider what conditions should be imposed on $a(x)$ and $b(x)$ in (2), in order that the problem (I.B.P.) to be wellposed.

To be more precise, we consider the problem under the following situation.

$\Omega = \{(x,y) \in \mathbb{R}^2 \,|\, y < 0$, $a(x) \dfrac{\partial u}{\partial y} + b(x)u\big|_{y=o} = 0$,

$a(x)$ and $b(x)$ are real-valued ; $a(x)^2 + b(x)^2 = 1$;

$a(x)$, $b(x) \in C^\infty$, and bounded with all their derivatives.

We are concerned with $H^\infty(\Omega)$-wellposedness for (I.B.P.), imposing the compatibility conditions. Namely, denoting

(4)
$$Bu \equiv a(x) \frac{\partial u}{\partial y} + b(x)u\Big|_{y=0} \, ,$$

(5)
$$B(\Delta^j u_0(x,y)) = 0, \text{ for } j = 0,1,2,\ldots$$

The result is the following.

Theorem. For the above problem to be H^∞-wellposed, the following conditions are necessary and sufficient :

(i)   a(x) does not change the sign. We assume therefore $a(x) \geqslant 0$,

(ii)   On the set $\{x; \; a(x) = 0\}$, $b(x) > 0$.

We are concerned here with the necessity. The sufficiency is proved fairly easily. First we observe that, if we put

$$u(.,t) = T_t u_0 \, ,$$

T_t is a semi-group. T. Komura obtained the necessary and sufficient condition to the infinitesimal generator in Fréchet spaces [5]. However it seems difficult to apply her method to the actual problem. Instead of that, suggested by this article, we use the truncated Laplace transform :

$$\hat{u}(.,\lambda) = \int_0^1 e^{-\lambda t} u(.,t) dt.$$

§ 2 - PRELIMINARIES -

We assume the problem (I.B.P.) is H^∞-wellposed, and from this assumption some basic facts.

1)- Continuity :

By the assumption of the wellposedness, by Banach, there exist an integer q and the constant C(T) such that

$$(2.1) \qquad \underset{t \in [o,T]}{Sup} \ ||u(.,t)||_{H^2} \leqslant C(T)||u(.,o)||_{H^{2q}} .$$

In this inequality, the initial data should satisfy the compatibility conditions :

$$(2.2) \qquad B(\Delta^j u(.,o)) = 0 \ , \ 0 \leqslant j \leqslant q-1.$$

2)- Truncated Laplace transform :

Let :

$$(2.3) \qquad \hat{u}(.,\lambda) = \int_o^1 e^{-\lambda t} u(.,t) dt \ ,$$

we obtain

$$(2.4) \qquad (\lambda - \Delta) \ \hat{u}(.,\lambda) = u(.,o) - e^{-\lambda} u(.,1).$$

We take $u(.,o)$ in the following form: Let $f_o(x) \in C_o^\infty$, and put

$$u_1(x,y) = (\sum_{j=o}^q (\lambda - \frac{d^2}{dx^2})^j \ f_o(x) . \frac{y^{2j}}{(2j)!}) \alpha(g),$$

where $\alpha(y) \in C_o^\infty$, which takes the value 1 in a neighborhood ot the origin.

Then :

$$g(x,g) = (\lambda - \Delta) u_1 = (\lambda - \frac{d^2}{dx^2})^{q+1} f_o(x) . \frac{y^{2q}}{(2q)!},$$

in a neighborhood of the origin. g satisfies

$$B(\Delta^j g) = 0 \ (0 \leqslant j \leqslant q-1) \ , \qquad \text{and}$$

$$(2.5) \qquad B \ u_1 = - \ b(x) \ f_o(x).$$

Observe that g is determined by $f_o(x)$. We take

$$f_o(x) \in H^{4q+2}.$$

We put $u(.,o) = g(x,y)$. Then by hypothesis, there exists a unique

solution $u(.,t)$ with initial data g. Thus $\hat{u}(.,\lambda)$ is defined by (2.3). Since $(\lambda-\Delta)u_1 = u(.,o)$, from (2.4) we get

$$(\lambda-\Delta)\ (\hat{u}(.,\lambda) - u_1) = -\ e^{-\lambda}\ u(.,1).$$

Moreover, since $B\hat{u}(.,\lambda) = 0$, we obtain

$$B(\hat{u}-u_1) = -\ Bu_1 = -\ b(x)\ f_o(x).$$

We proceed further. Let $u_D(x,y)$ be the solution of

$$\begin{cases} (\lambda-\Delta)u_D = u(.,1) \\ \\ u_D\Big|_{y=o} = 0. \end{cases}$$

Then denoting

(2.6) $$v(x,y) = \hat{u}(.,\lambda) - u_1(x,y) - e^{-\lambda}u_D$$

(2.7) $$\begin{cases} (\lambda-\Delta)\ v(x,y) = 0 \\ \\ B\ v = b(x)\ f_o(x) + e^{-\lambda}\ \tilde{f}(x)\ , \end{cases}$$

where $\tilde{f}(x) = a(x)\ \dfrac{\partial}{\partial y}\ u_D\Big|_{y=o}$.

3) - Functional equation on the boundary :

If we use the Poisson representation of $v(x,y)$,

$$v(x,y) = \frac{1}{2\Pi} \int_{-\infty}^{\infty} e^{ix\xi}\ e^{\sqrt{\lambda+\xi^2}\ y}\hat{v}(\xi,o)d\xi\ ,$$

putting $v(x,o) = v(x)$, the above equation can be written

(2.8) $$A(x,D)\ v(x) = (a(x)\ \sqrt{\lambda+D^2} + b(x))\ v(x) = f(x) + e^{-\lambda}\ \tilde{f}(x)$$

where $f(x) = b(x)\ f_o(x).$

We assume hereafter $b(x) \neq 0$, in a neighborhood of the origin, and $f_0(x)$ has its support in a small neighborhood of the origin. We obtain :

Proposition 1 :

(2.9) $\qquad \tilde{A}(x,D) = a(x) + b(x) \sqrt{\lambda+D^2}^{-1} \equiv a(x) + b(x) \Lambda^{-1}.$

Then for any $f(x) \in H^{4q+2}$, there exists a solution $w(x) \in H^{\frac{1}{2}}$

(2.10) $\qquad \tilde{A}(x,D) \ w(x) = f(x) + e^{-\lambda} \ \tilde{f}(x) \ ,$

satisfying the following type inequalities.

$$||w(x)||_{\frac{1}{2}} \ , \ ||\tilde{f}(x)||_{\frac{3}{2}} \leqslant \text{const.} \ ||f(x)||_{4q+2}.$$

Remark :

Hereafter we denote $||\cdot||_s = ||\cdot||_{H^s}$, and

$$||f(x)||_s^2 = \int (\lambda+\xi^2)^s \ |\hat{f}(\xi)|^2 \ d\xi.$$

§ 3 - ALMOST NULL SOLUTIONS -

For simplicity, we consider the problem under the following assumptions :

(3.1) $\qquad \begin{cases} \text{(i)} \quad a(x) = x^2 \text{ in a neighborhood } V \text{ of the origin,} \\ \\ \text{(ii)} \quad b(x) = -1, \ x \in \mathbb{R}. \end{cases}$

We consider a solution $\psi_0(x)$ of the following equation

(3.2) $\qquad (x^2 - \Lambda^{-1}) \ \psi_0(x) = 0.$

By taking the Fourier transformation,

(3.3) $\qquad (\dfrac{d^2}{d\xi^2} + (\lambda + \xi^2)^{-\frac{1}{2}}) \ \hat{\psi}_0(\xi) = 0.$

We see that, there exists the solution satisfying

$$(3.4) \quad \begin{cases} |\hat{\psi}_0(\xi)| \sim \lambda^{-\frac{1}{8}} (\lambda + \xi^2)^{-\frac{1}{8}}, \quad (\lambda \to \infty) \\ |(\frac{d}{d\xi})^p \hat{\psi}_0(\xi)| \leqslant c_p (\lambda + \xi^2)^{-\frac{1}{8} - \frac{1}{8}p} \lambda^{-\frac{1}{8}} \end{cases}$$

Observe that :

$$(3.5) \quad \begin{cases} \psi_0(x) \in H^{-\delta} \quad (\delta > \frac{3}{4}) \\ ||\Lambda^{-2} \psi_0||_1 \sim C_1 \lambda^{-\frac{1}{4}} \quad (C_1 : \text{positive constant}) \end{cases}$$

Take a $\beta(x) \in C_0^\infty(V)$, which takes the value 1 in a neighborhood of the origin. From (3.2), it follows

$$(a(x)-\Lambda^{-1})(\beta\psi_0) - [\beta, \Lambda^{-1}]\psi_0 = 0.$$

Let us introduce the symbol "≈ 0". $f \approx 0$ means that, for any k and any p,

$$||\Lambda^k f||_0 \leqslant C_{kp}/\lambda^p \text{ when } \lambda \text{ is large.}$$

From the property (ii) of (3.4), it follows

$$[\beta, \Lambda^{-1}]\psi_0 \approx 0 \ , \ [a(x)-\Lambda^{-1}](1-\beta(x))\psi_0 \approx 0.$$

Therefore, it holds

$$(3.6) \quad (a(x)-\Lambda^{-1})\psi_0(x) \approx 0.$$

Following the notation (2.9), we denote $\tilde{A} = a(x)-\Lambda^{-1}$. Using the results of proposition 1, we get.

Proposition 2 :

For $f (=\Lambda^{-2}\psi_0) \in H^1$, there exists an element $w \in H^1$, satisfying

(i) $||\tilde{A}w - f||_1 \leqslant \frac{1}{2} ||f||_1$,

(ii) There exist positive constant k_o and C such that

$$||w||_1 \leqslant C \lambda^{k_o} \quad (\text{if } \lambda \text{ is large}).$$

Now it is easy to see that these properties are <u>not compatible</u> with (3.6), which shows, in the case (3.1). The problem (I.B.P.) is not well-posed.

Infact, in view of (3.5) :

$$(3.7) \qquad \text{Re} (\tilde{A}w, \Lambda^{-2}\psi_o)_1 \geqslant \frac{1}{2} ||\Lambda^{-2}\psi_o||_1^2 \sim \frac{1}{2} c_1^2 \lambda^{-\frac{1}{2}}$$

On the other hand, the left-hand side is equal to

$$\text{Re} <\tilde{A}w, \overline{\psi}_o> = \text{Re} <w, \overline{\tilde{A}\psi_o}>.$$

This is estimated from above by $||w||_1 ||\tilde{A}\psi_o||_{-1}$. By the property of w, and (3.6), we see that, this last quantity is estimated of the form $C'_p \lambda^{-p}$ for any p. This is not compatible with (3.7).

§ 4 – PROOF OF PROPOSITION 2 –

We define $\tilde{f}, w_o$, w in the following order

$$f \rightarrow \tilde{f} \rightarrow w_o \rightarrow w$$

$$(4.1) \qquad \left\lbrace \begin{array}{ll} (i) & \tilde{f} = \rho_\delta * f \\[2ex] (ii) & \tilde{A} w_o = \tilde{f} + e^{-\lambda}f_1 \\[2ex] (iii) & w = \rho_\epsilon * w_o \end{array} \right.$$

where ρ_δ, ρ_ϵ are mollifiers.

(i) $\qquad ||f - \tilde{f}||_1 \leqslant C\delta^s ||f||_{1+s}$ $(o<s<\frac{1}{4})$. We suppose here $\delta \leqslant \lambda^{-\frac{1}{2}}$;

C is independent of δ. Since $||f||_{1+s} \sim C \lambda^{\frac{s}{2}} ||f||_1$, we take $\delta = \varepsilon_0 \lambda^{-\frac{1}{2}}$ (ε_0 being a small constant). This, are have

(4.2) $$||f - \tilde{f}||_1 \leqslant \frac{1}{6} ||f||_1.$$

(ii) Observe $\tilde{f} \in H^\infty$, and

$$||\tilde{f}||_{4q+2} \leqslant \text{cont. } \delta^{-(4q+2)} ||f||_1 \leqslant \text{cont. } \lambda^{2q+1} ||f||_1,$$

(since $\delta = \varepsilon_0 \lambda^{-\frac{1}{2}}$).

By proposition 1, there exists $w_0 \in H^{\frac{1}{2}}$ such that

$$\overset{\curlyvee}{A} w_0 = \tilde{f} + e^{-\lambda} f_1,$$

where

$$||w_0||_{\frac{1}{2}} \leqslant \text{const. } ||\tilde{f}||_{4q+2} \leqslant \text{const. } \lambda^{2q+1} ||f||_1,$$

$$||f_1||_{\frac{s}{2}} \leqslant \text{const. } ||\tilde{f}||_{4q+2} \leqslant \text{const. } \lambda^{2q+1} ||f||_1.$$

Observe that $||\overset{\curlyvee}{A} w_0 - \tilde{f}||_1 = e^{-\lambda} ||f_1||_1 \leqslant \text{const. } e^{-\lambda} \lambda^{2q+1} ||f||_1$, which is negligible.

(iii) $f - \overset{\curlyvee}{A} w = (f - \tilde{f}) + (\tilde{f} - \overset{\curlyvee}{A} w_0) + (\overset{\curlyvee}{A} w_0 - \overset{\curlyvee}{A} w).$

We are concerned with the last term. Observe $\overset{\curlyvee}{A} w_0 \in H^{\frac{3}{2}}$, and

(4.3) $$||\overset{\curlyvee}{A} w_0||_{\frac{3}{2}} \leqslant ||\tilde{f}||_{\frac{3}{2}} + e^{-\lambda} ||f_1||_{\frac{3}{2}},$$

where the last term is negligible when $\lambda \to \infty$. Now,

$$||\tilde{f}||_{\frac{3}{2}} \leqslant \text{const. } \frac{1}{\sqrt{\delta}} ||f||_1 \leqslant \text{const. } \lambda^{\frac{1}{4}} ||f||_1.$$

$$\tilde{A}\, w_o - \tilde{A}(\rho_\varepsilon * w_o) = (\tilde{A}\, w_o - \rho_\varepsilon * \tilde{A}\, w_o) + (\rho_\varepsilon * \tilde{A}\, w_o - \tilde{A}(\rho_\varepsilon * w_o)).$$

The first term is estimated by

$$||\tilde{A}\, w_o - \rho_\varepsilon * (\tilde{A}\, w_o)||_1 \leqslant \text{const.} \; ||\tilde{A}\, w_o||_{\frac{3}{2}} \sqrt{\varepsilon}.$$

By virtue of (4.5), this is estimated, when λ is large, by

$$\text{const.} \; (\lambda^{\frac{1}{4}} \sqrt{\varepsilon}) \; ||f||_1.$$

Now put, $\varepsilon = \varepsilon_1' \; \lambda^{-(4q+2)}$, thus

$$||[\rho_\varepsilon * , \tilde{A}]\, w_o||_1 \leqslant \frac{1}{6} ||f||_1 ,$$

which implies

$$||\tilde{A}\, w_o - \tilde{A}\, w||_1 \leqslant (\frac{1}{6} + \text{const.} \; \lambda^{-2q})||f||_1,$$

which completes the proof of proposition 2.

§ 5 - FINAL REMARKS -

In the case when $a(x)$ has a finite order of zero at the origin, we can argue essentially in the same way. However the technical complication arises when $a(x)$, and $b(x)$ are general. In the case when $a(x)$ has a infinite order of zero at the origin, we can argue in a fairly different way.

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