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**ÉQUATIONS AUX DÉRIVÉES PARTIELLES**

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## 1. Introduction

### 1.1. Physical context

The study of laser-plasma interactions is a field of intense activity, with large-scale experiments being led in the US at the NIF and in France by the CEA. Energy losses due to Raman scatterring and Landau damping are key issues. Models based on the fundamental equations of physics, such as the Euler-Maxwell or the Vlasov-Maxwell equations, are too complex to allow efficient numerical simulations of these phenomena. Hence a need for simpler models. We study here the mathematical validity of a model system of equations introduced by V. Zakharov in the seventies, starting from the Euler-Maxwell equations.

### 1.2. The equations

We work on the non-dimensional form of the Euler-Maxwell equations introduced in [12], in a "cold ions" regime. For small amplitudes, the system has the form:

$$
\text{(EM)}\quad
\begin{cases}
\partial_t B + \nabla \times E = 0, \\
\partial_t E - \nabla \times B = \dfrac{1}{\varepsilon} e^{n_e} v_e - \dfrac{1}{\theta_e} e^{n_i} v_i, \\
\partial_t v_e + \theta_e (v_e \cdot \nabla) v_e = -\theta_e \nabla n_e - \dfrac{1}{\varepsilon}(E + \theta_e v_e \times B), \\
\partial_t n_e + \theta_e \nabla \cdot v_e + \theta_e (v_e \cdot \nabla) n_e = 0, \\
\partial_t v_i + \varepsilon (v_i \cdot \nabla) v_i = -\alpha^2 \varepsilon \nabla n_i + \dfrac{1}{\theta_e}(E + \varepsilon v_i \times B), \\
\partial_t n_i + \varepsilon \nabla \cdot v_i + \varepsilon v_i \cdot \nabla n_i = 0.
\end{cases}
$$

The variable is $U = (B, E, v_e, n_e, v_i, n_i)$, where $(B, E) \in \mathbb{R}^{3+3}$ is the electromagnetic field, $(v_e, v_i) \in \mathbb{R}^{3+3}$ are the velocities of the electrons and of the ions and $(n_e, n_i) \in \mathbb{R}^{1+1}$ are the fluctuations of densities of both species. The variable $u$ depends on

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time $t \in \mathbb{R}_+$ and space $x \in \mathbb{R}^3$. A brief discussion of the relevance of this model is given in [12]. The small parameter $\varepsilon$ is $\varepsilon := \frac{1}{\omega_p t_0}$, where $t_0$ is the duration of the laser pulse and $\omega_p$ is the plasma frequency: $\omega_p := \sqrt{\frac{4\pi e^2 n_0}{m_e}}$, depending on $m_e$ the mass of the electrons and $n_0$ the background density of the plasma. A typical value for $\varepsilon$ in realistic physical applications is $\varepsilon \simeq 10^{-6}$. The fixed parameter $\theta_e$ is the electronic thermal velocity, while $\alpha$ is the ratio of the electronic and ionic temperatures. Typically, $\alpha \ll \theta_e \simeq 10^{-3}$. The "cold ions" hypothesis consists here in setting the ionic thermal velocity $\theta_i$ to be equal to $\varepsilon$. We investigate the high-frequency limit $\varepsilon \to 0$, as $\theta_e$ and $\alpha$ are held fixed.

Formal computations led in [12] show that in some specific regime, the high-frequency limit of (EM) is the following Zakharov system:

$$(\mathrm{Z})_c \begin{cases} i(\partial_t + c\partial_z)\tilde{E} + \Delta_\perp \tilde{E} = \bar{n}\tilde{E}, \\ \qquad\qquad \partial_t^2 \bar{n} - \Delta_\perp \bar{n} = \Delta_\perp |\tilde{E}|^2, \end{cases}$$

where $\tilde{E}$ is the envelope of the electric field and $\bar{n}$ is the mean mode of the ionic fluctuations of density in the plasma, $z$ is the direction of propagation of the laser pulse and $\Delta_\perp$ is the Laplace operator in the directions transverse to $z$. This model was derived by V. Zakharov and his collaborators in the seventies [14].

## 1.3. Mathematical context

Formal WKB expansions led in [12] showed how the weakly nonlinear limit of (EM) fails to describe nonlinear interactions; such a phenomenon had been observed by Joly, Métivier and Rauch in the context of the Maxwell-Bloch equations. This explains why we consider large-amplitude solutions in this paper. In [2], Klein-Gordon-waves systems were formally derived from Euler-Maxwell, and the Zakharov system $(\mathrm{Z})_0$ was rigorously derived as a high-frequency limit of these Klein-Gordon-waves systems. We extend here the results of [12, 2], as we rigorously derive $(\mathrm{Z})_0$ from (EM).

The (local in time) initial value problem for the Zakharov model $(\mathrm{Z})_0$ has received much attention. Ozawa and Tsutsumi [9] and Schochet and Weinstein [10] proved existence of smooth solutions. Many authors studied weak solutions, see [13] for references.

Following the formal derivation in [12] of the systems $(\mathrm{Z})_c$ with a non-zero group velocity $c$, Linares, Ponce and Saut showed its well-posedness in $H^s(\mathbb{R}^d)$, for large $s$, and Colin and Métivier showed ill-posedness in a periodic setting.

The study of highly nonlinear geometric optics was originated in [5] for semilinear Maxwell-Bloch equations. Stability and instability results for systems of conservation laws in this regime are established in [1]. We deal here with quasi-linear *dispersive* equations.

Our proof relies on pseudo-differential and para-differential changes of variables. As usual in nonlinear problems, we are led to deal with symbols with limited regularity. Such classes of symbols were studied by Taylor[11] and Grenier [4]. We use here the precise bounds of Lannes [6].

## 2. Statement of the result

We are interested in solutions of (EM) in the form

$$u(t,x) = \varepsilon U(\varepsilon t, x). \tag{2.1}$$

That is, we study large-amplitude solutions in a diffractive regime. For solutions in the form (2.1), the system (EM) takes the form

$$\partial_t u + \frac{1}{\varepsilon^2}\mathcal{A}(\varepsilon, \varepsilon u, \varepsilon\partial_x)u = \frac{1}{\varepsilon}\mathcal{B}(u,u) + \mathcal{G}^\varepsilon(u), \tag{2.2}$$

where $\mathcal{A}$ has the form

$$\mathcal{A}(\varepsilon, \varepsilon v, \varepsilon\xi) = \mathcal{A}^{(0)}(\varepsilon\xi) + \varepsilon\mathcal{A}^{(1)}(\varepsilon, v, \varepsilon\xi), \tag{2.3}$$

where $\mathcal{A}^{(0)}$ is affine in $\xi$ (*not* homogeneous, as the equations modelling light-matter interactions are dispersive), and where $\mathcal{A}^{(1)}$ is linear in $\xi$ and encodes in particular the convective terms in Euler. We work with the classical Sobolev spaces $H^s(\mathbb{R}^3)$ for profiles with values in $\mathbb{R}^{14}$, endowed with the norms

$$\|v\|^2_{\varepsilon',s} := \sum_{|\beta|\le s} \|(\varepsilon'\partial_x)^\beta v\|^2_{L^2(\mathbb{R}^3)}.$$

We will use $\varepsilon' = 1$ and $\varepsilon' = \varepsilon$. We say that a family of profiles $v^\varepsilon$ is bounded in $H^s_\varepsilon(\mathbb{R}^3)$ when it is bounded with respect to the semi-classical norm $\|\cdot\|_{\varepsilon,s}$, uniformly in $\varepsilon \in (0,\varepsilon_0)$, for some $\varepsilon_0 \in (0,1)$. Note that the injection $H^s_\varepsilon \hookrightarrow W^{k,\infty}$ has norm $O(\varepsilon^{-k-3/2})$.

Consider an initial datum $a(x)$ of the form $a = (0, E^0, v^0_e, 0, 0, 0) \in H^\sigma$, for some large $\sigma$.

**Proposition 1.** *For any $l$ such that $\sigma - l > \sigma_0 + 3/2$, for some $\sigma_0 \in \mathbb{N}$, the system* (EM) *has a unique approximate solution $u^\varepsilon_a$ of the form* (2.1) *satisfying the initial condition $u^\varepsilon_a(0,x) = a(x)$ and such that:*

- *$u^\varepsilon_a$ is defined over a time interval $[0,t^*)$ independent of $\varepsilon$;*
- *$u^\varepsilon_a$ satisfies* (EM) *up to a residual of the form $\varepsilon^l R^\varepsilon_a$, where $R^\varepsilon_a$ is a bounded family in $H^{\sigma-l}_\varepsilon$, locally uniformly in $t \in [0,t^*)$;*
- *$u^\varepsilon_a$ decomposes as $u^\varepsilon_a = (u_{0,-1}e^{-it/\varepsilon^2} + u_{0,1}e^{it/\varepsilon^2} + \varepsilon u_{1,0}) + \varepsilon v^\varepsilon_a$, where $v^\varepsilon_a, \varepsilon^2\partial_t v^\varepsilon_a$ are bounded families in $H^{\sigma-l-1}_\varepsilon$, locally uniformly in $t \in [0,t^*)$, and where the components of $u_{0,1}, u_{0,-1}$ and $u_{1,0}$ satisfy a system of the form $(\mathrm{Z})_0$.*

Consider now a perturbation $a^\varepsilon := a + \varepsilon^k\varphi^\varepsilon$, where $\varphi^\varepsilon$ is a bounded family in $H^{\sigma-l}_\varepsilon$.

**Theorem 2.** *If $k > 3 + \frac{3}{2}$, then the system* (EM) *has a unique solution $u^\varepsilon$ of the form* (2.1) *satisfying the initial condition $u^\varepsilon(0,x) = a^\varepsilon(x)$ and defined over $[0,t^*)$. Moreover, for all $0 < t_0 < t^*$, for $s$ smaller than $\sigma - l$ but large enough, there holds*

$$\sup_{0\le t\le t_0} \|u^\varepsilon - u^\varepsilon_a\|_{\varepsilon,s} \le C\varepsilon^{k-1}. \tag{2.4}$$

That $u^\varepsilon$ is defined over a time interval independent of $\varepsilon$ is not trivial. A direct $H^s$ estimate for solutions of (2.2) with data of size $O(1)$ would indeed give a bound in $e^{Ct/\varepsilon}$, hence an existence time $O(\varepsilon)$ only, precisely because of the large-source term $\mathcal{B}/\varepsilon$ in the right-hand side. Note that in contrast, the contribution of the propagator $\mathcal{A}/\varepsilon^2$ in the $H^s$ estimate is not singular with respect to $\varepsilon$, because of the specific scaling that we chose – see (2.3).

The above theorem asserts that the approximate solution $u_a^\varepsilon$ is stable under perturbations of the form $\varepsilon^k \varphi^\varepsilon$, where $\varphi^\varepsilon$ belongs to $H_\varepsilon^s$, that is may contain fast oscillations of the form $e^{ikx/\varepsilon}$. Remark that the initial datum $a$ is not oscillating. This amounts to a polarization condition at $k = 0$ for the initial datum and implies in particular that the group velocity is $c = 0$; see Figures 1 and 2. It would be interesting to consider more general initial data of the form $a(x)e^{ikx/\varepsilon}$, with $k \neq 0$. Then, as shown in [12], the limit system is $(\mathrm{Z})_c$, with $c \neq 0$. The result of Linares, Ponce and Saut [7] asserts that an approximate solution can be constructed. Is is stable, in the sense of Theorem 2 ? This is an interesting direction for future work.

In the lower bound for $k$, $2 + 3/2$ powers of $\varepsilon$ account for the embedding $H_\varepsilon^s \hookrightarrow W^{2,\infty}$, and the extra power of $\varepsilon$ comes from the rescaling of section 6.2.

Note finally that the estimate (2.4) and the condition $k > 3 + \frac{3}{2}$ imply

$$\sup_{0 \leq t \leq t_0} \sup_{x \in \mathbb{R}^3} \left( |E^\varepsilon - (E_{0,1} e^{i\omega t/\varepsilon^2} + c.c.)| + |n^\varepsilon - \varepsilon n_{e1,0}| \right) \leq C\varepsilon^2. \tag{2.5}$$

(2.5) is the estimate that validates the Zahkarov model, as it actually gives a description of the electric field $E^\varepsilon$ and the fluctuation of density $n^\varepsilon$ in (EM) by means of the solution $(E_{0,1}, n_{e1,0})$ of $(\mathrm{Z})_0$.

## 3. WKB approximate solution

We sketch the proof of Proposition 1 in this section. We look for $u_a^\varepsilon$ in the form of a WKB expansion: $u_a^\varepsilon = \sum_{m=0}^{M} \varepsilon^m u_m$, where for all $m$, $u_m$ has a finite decomposition in oscillating terms: $u_m = \sum_{p \in \mathcal{R}_m} e^{ip\omega t/\varepsilon^2} u_{m,p}(t,x)$, where the profiles $u_{m,p}$ are sought in $W^{1,\infty}([0, t^*(\varepsilon)], L^\infty(\mathbb{R}^3))$, for some $t^*(\varepsilon) > 0$ and where the sets $\mathcal{R}_m \subset \mathbb{Z}$ are finite. Plugging this ansatz in (EM), one finds a cascade of WKB equations. We also use below the notation $(v)_p$ to denote the $p-$th harmonic of a profile $v$ admitting a finite decomposition in oscillating terms as above.

*Terms in $O(1/\varepsilon^2)$:* The equations are

$$ip\omega(E_{0,p}, v_{e0,p}) = (v_{e0,p}, -E_{0,p}), \tag{3.1}$$

while the oscillating components (i. e. corresponding to $p \neq 0$) of all the other profiles of order 0 vanish. From (3.1), one obtains the dispersion relation: $\omega(\omega^2 - 1) = 0$. We choose $\omega = 1$. Thus $\mathcal{R}_0 \subset \{-1, 0, 1\}$. The leading electric field and electronic velocity are

$$E_0 = E_{0,1} e^{it/\varepsilon^2} + E_{0,-1} e^{-it/\varepsilon^2}, \quad v_{e0} = \frac{i}{\omega} E_{0,1} e^{it/\varepsilon^2} - \frac{i}{\omega} E_{0,-1} e^{-it/\varepsilon^2}. \tag{3.2}$$

*Terms in $O(1/\varepsilon)$:* The equations contain the compatibility condition

$$ip\omega(n_{e0} v_{e0})_p - \theta_e (v_{e0} \times B_0)_p = 0. \tag{3.3}$$

The nonoscillating terms satisfy in particular

$$-\nabla\times B_{0,0} = v_{e1,0} - \frac{1}{\theta_e} v_{i0,0}, \quad E_{1,0} = \theta_e \nabla n_{e0,0}. \tag{3.4}$$

*Terms in $O(1)$:* These equations imply in particular that $\partial_t B_{0,0} = 0$, because $E_{1,0}$ is a gradient. Hence $B_{0,0} = 0$, and (3.3) implies that $n_{e0,0} = 0$. As usual in geometric optics when the ansatz involves three scales, the compatibility condition for the equations in $(E_2, v_{e2}$ is a Schrödinger equation for $E_{0,p}$, $p \in \{-1, 1\}$. For the variable $\tilde{E}_p := e^{ipt/\theta_e^2} E_{0,p}$ (correction of frequency due to the ions), the equation is

$$-2ip\omega\partial_t \tilde{E}_p + \tilde{\Delta}\tilde{E}_p = -n_{e1,0}\tilde{E}_p + \frac{\theta_e}{ip\omega}\tilde{E}_p \times B_{1,0}, \tag{3.5}$$

where $\tilde{\Delta} v := \theta_e^2 \nabla(\nabla\cdot v) - \nabla\times(\nabla\times v)$. The nonoscillating terms satisfy

$$\theta_e\Big((v_{e0}\cdot\nabla)v_{e0}\Big)_0 = -\theta_e \nabla n_{e1,0} - (E_{2,0} + \theta_e(v_{e0}\times B_1)_0). \tag{3.6}$$

(3.6) is the crucial equation that provides the noninear coupling between (3.5) and the evolution equation for $n_{e1,0}$. The third term of the expansion also contains the relation:

$$\nabla\times B_{1,0} = v_{e2,0} - \frac{1}{\theta_e} v_{i1,0} + (n_{e1}v_{e0})_0. \tag{3.7}$$

$v_{i0,0}$ and $n_{i1,0}$ satisfy a linear wave equation, with null initial data. Hence $n_{i0,0} = 0$ and $v_{i0,0} = 0$. With (3.4) and $B_{0,0} = 0$, one has therefore $v_{e1,0} = 0$.

*Terms in $O(\varepsilon)$:* The nonoscillating terms satisfy in particular

$$\begin{aligned} \partial_t v_{i1,0} + \alpha^2 \nabla n_{i1,0} &= \frac{1}{\theta_e} E_{2,0}, \\ \partial_t n_{i1,0} + \nabla v_{i1,0} &= 0, \\ \partial_t n_{e1,0} + \theta_e \nabla\cdot(v_{e2,0} + (n_{e1}v_{e0})_0) &= 0. \end{aligned}$$

This last equation and (3.7) imply the quasineutrality relation: $n_{e1,0} = n_{i1,0}$. The density $\bar{n} := n_{e1,0} = n_{i0,0}$ satisfies the wave equation:

$$\begin{cases} \partial_t v_{i1,0} + (\alpha^2+1)\nabla\bar{n} = -((v_{e0}\cdot\nabla)v_{e0} - v_{e0}\times B_1))_0 \\ \partial_t \bar{n} + \nabla\cdot v_{i1,0} = 0. \end{cases}$$

The nonlinear term in the wave equation is Thus the equation satisfied by $\bar{n}$ is

$$(\partial_t^2 - (\alpha^2+1)\Delta)\bar{n} = -\Delta|\tilde{E}_p|^2. \tag{3.8}$$

Besides, we have $\partial_t B_{1,0} + \nabla\times E_{2,0} = 0$, and this implies with (3.6) that $E_{2,0}$ is a gradient. Hence $B_{1,0} = 0$, and the system (3.5)-(3.8) is the announced Zakharov system (with an elliptic operator operating in three space dimensions). Note that the crucial coupling term $\Delta|\tilde{E}_p|^2$ comes from the convective term *and* from the Lorentz force term.

*High-order terms:* For $m \geq 2$, the terms $E_{m,p}, n_{i,m+1,0}$, $p = -1, 1$ are seen to satisfy a system that corresponds to the linearization of (3.5)-(3.8) around $E_{0,p}$, which can be solved as $(\mathrm{Z})_0$, with an existence time that is a priori smaller than the existence time of the first profiles. The other components of $u_{m,p}$ are given by polarization and compatibility conditions as in the first terms of the expansion.

## 4. Resonances and transparency

For each fixed $z \in \mathbb{R}^{14}$, the characteristic variety is the set of all $(\omega, k) \in \mathbb{R}\times\mathbb{R}^3$, such that det $(-\omega+\mathcal{A}(\varepsilon, \varepsilon z, k)) = 0$. This polynomial equation factorizes as a transverse, degree 4 equation:

$$\omega'\omega''(\omega^2 - 1 - |k|^2 - \frac{\varepsilon^2}{\theta_e^2}) = \varepsilon z_{[v_e]}\omega'' + \frac{\varepsilon^2}{\theta_e^2}\varepsilon^2 z_{[v_i]}\omega',$$

and a longitudinal, degree 5 equation:

$$\omega(\omega''^2 - \varepsilon^2\alpha^2|k|^2)(\omega'^2 - 1 - \theta_e^2|k|^2) = -\varepsilon z_{[v_e]}(\omega''^2 - \varepsilon^2\alpha^2|k|^2) + \frac{\varepsilon^2}{\theta_e^2}\omega''(\omega'^2 - \theta_e^2|k|^2),$$

where $z_{[v_e]}$ and $z_{[v_i]}$ are the electronic and ionic velocity components of $z$, and where $\omega' := \omega - \varepsilon z_{[v_e]}$, $\omega'' := \omega - \varepsilon^2 z_{[v_i]}$. There are six Klein-Gordon modes: $\{\lambda_j\}_{1\leq j\leq 6}$, and eight acoustic modes: $\{\lambda_k\}_{7\leq k\leq 14}$. The corresponding eigenprojectors are denoted by $\Pi_j, \Pi_k$. The acoustic velocities are all $O(\varepsilon)$, as a result of the "cold ions" hypothesis.

Resonances correspond to constructive interactions of characteristic waves. As in [2], because the crucial interaction term $\mathcal{B}$ is bilinear, writing the system as a perturbation system around the approximate solution allows to consider only the resonance relations

$$\Phi_{j,k,p}(\xi) := \lambda_j(0, 0, \xi) - \lambda_k(0, 0, \xi) + p\omega = 0,$$

for $p \in \{-1, 1\}$. Examples of resonances for the Euler-Maxwell system are pictured on Figures 1 and 2.

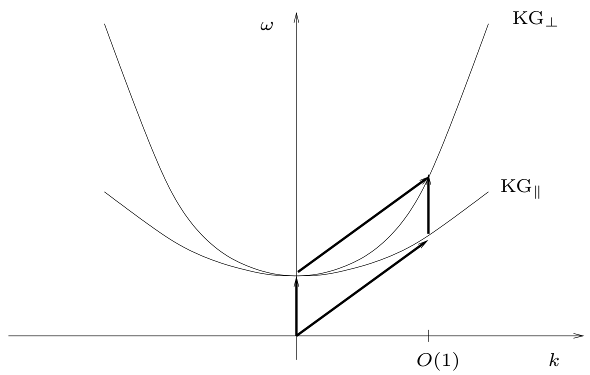


Figure 1: Resonances between the Klein-Gordon branches on the characteristic variety.

Eigenvalues cross at the origin. As a consequence, the eigenvalues and the eigenprojectors may not be infinitely smooth at $k = 0$, and one has to deal with symbols with limited regularity in $k$. This wil not be mentioned in the following (the symbols defined in section 5 are smooth); for more details, see [13].

We let $\Pi_0$ be the total projector on the Klein-Gordon modes and $\Pi_s$ be the total projector on the acoustic modes, so that $\Pi_0 := \sum_{j=1}^{6}\Pi_j$, $\Pi_s := \sum_{k=7}^{14}\Pi_k$.

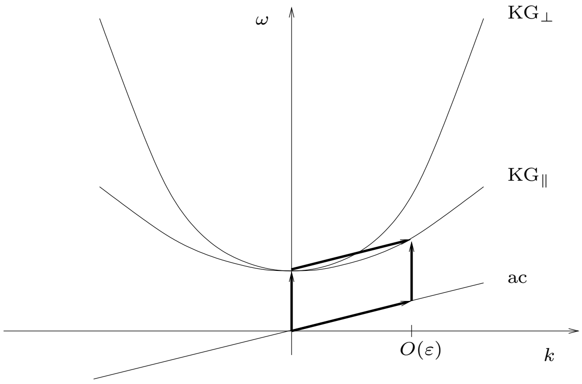


Figure 2: Resonances between the Klein-Gordon and the acoustic branches on the characteristic variety.

To prove existence over time intervals independent of $\varepsilon$ for large-amplitude solutions of the (EM) system, we need to compute how characteristic waves interact through the large source term $\mathcal{B}/\varepsilon$. The eigenvalues can be locally described (with the dispersion relations above) and the eigenvectors can be explicitly computed in terms of the eigenvalues. We can compute

$$\mathcal{B}(u_a,\cdot)+\mathcal{B}(\cdot,u_a)=\left(\begin{array}{cc} * & O(1) \\ * & * \end{array}\right)\begin{array}{l}\Pi_0\\ \Pi_s,\end{array} \tag{4.1}$$

in a basis of eigenvectors. We notice three different behaviours:

(a) transparency: the bottom left entry in (4.1) is small for $k$ close to the Klein-Gordon/acoustic resonance depicted on Figure 2.

(b) absence of transparency: the top right entry in (4.1) is large for $k$ close to the Klein-Gordon/acoustic resonance depicted on Figure 2.

(c) symmetrizability: the top left entry in (4.1) is not small for $k$ close to the Klein-Gordon/Klein-Gordon resonance depicted on Figure 1, but is symmetrizable.

Because of (b), we will rescale the acoustic component of the solution in section 6.2. This procedure makes the equations more singular, but the interaction coefficient mentioned in (a) is sufficiently small to allow such a transformation (it is reduced in section 6.3). Finally, the interaction coefficient mentioned in (c) is symmetrized in section 6.4. Note also that because of the cold ions hypothesis, resonances between acoustic modes are higher-order phenomena, and the interaction coefficients in the bottom right block of (4.1) do not play any role in the analysis.

## 5. Symbols and operators

We consider symbols depending on $(\varepsilon, v, \xi) \in (0,1) \times \mathbb{R}^{14} \times \mathbb{R}^3$, such that for all compact $K \subset \mathbb{R}^{14}$, there exists $0 < \varepsilon_K < 1$, such that for all $\alpha, \beta$, there exists a non-decreasing function $C_{\alpha,\beta,K}$ such that

$$\sup_{\varepsilon\in(0,\varepsilon_K)} \sup_{v\in K} \sup_{\xi} \langle\xi\rangle^{|\beta|-m} |\partial_{\varepsilon,v}^{\alpha} \partial_{\xi}^{\beta} p(\varepsilon, v, \xi)| \leq C_{\alpha,\beta,K}.$$

This class of symbols is denoted $C^\infty\mathcal{M}^m$. We consider semi-classical pseudo- and para-differential operators defined by

$$\mathrm{op}_\varepsilon(q)z := \int e^{ix\cdot\xi} q(\varepsilon, t, x, \varepsilon\xi)\hat{z}(t,\xi)d\xi, \quad \mathrm{op}_\varepsilon^{\psi}(q) := \mathrm{op}_\varepsilon(q^{\psi}), \tag{5.1}$$

where $q^\psi$ is the standard para-differential smooting defined by $\widehat{q^{\psi}}(\eta,\xi) := \psi(\eta,\xi)\hat{q}(\eta,\xi)$, where $\psi$ is an admissible cut-off, defined by $\psi(\eta,\xi) = 1$ if $|\eta| \leq \delta_1(1+\xi|^2)^{1/2}$, and $\psi(\eta,\xi) = 0$ if $|\eta| \geq \delta_2(1+|\xi|^2)^{1/2}$, for some $0 < \delta_1 < \delta_2 < 1$. In (5.1), one will take for instance $q = p(v^\varepsilon)$, where $p \in C^\infty\mathcal{M}^m$ and $v^\varepsilon$ is a bounded family of profiles in $L^\infty([0,t_*(\varepsilon)] \times \mathbb{R}^3)$. Then the following bounds hold [8, 6]:

$$\begin{aligned} \|(\mathrm{op}_\varepsilon(p(v^\varepsilon)) - \mathrm{op}_\varepsilon^{\psi}(p(v^\varepsilon)))u\|_{\varepsilon,s} &\leq \varepsilon C(|v^\varepsilon|_{W^{1,\infty}})(1 + \|v^\varepsilon\|_{\varepsilon,s+1})\|u\|_{\varepsilon,m+d_0}, \\ \|\mathrm{op}_\varepsilon^{\psi}(p(v^\varepsilon))u\|_{\varepsilon,s} &\leq C(|v^\varepsilon|_{L^\infty})\|u\|_{\varepsilon,s+m}, \\ \|\mathrm{op}_\varepsilon(p(v^\varepsilon))u\|_{\varepsilon,s} &\leq C(|v^\varepsilon|_{L^\infty})(\|v^\varepsilon\|_{\varepsilon,s}\|u\|_{\varepsilon,m+d_0} + \|u\|_{\varepsilon,s+m}), \end{aligned}$$

and

$$\|(\mathrm{op}_\varepsilon^{\psi}(q_1)\mathrm{op}_\varepsilon^{\psi}(q_2) - \mathrm{op}_\varepsilon^{\psi}(q_1q_2) - \varepsilon\mathrm{op}_\varepsilon^{\psi}(q_1\sharp q_2))u\|_{\varepsilon,s} \leq \varepsilon^2 C(|v^\varepsilon|_{W^{2,\infty}})|u\|_{\varepsilon,s'}.$$

Above, $C$ represent non-decreasing functions, $d_0$ is greater than $3/2$, and in the last inequality $q_j = p_j(v^\varepsilon)$, $p_j \in C^\infty\mathcal{M}^{m_j}$, $s' = s + 2 - m_1 - m_2$, and $q_1\sharp q_2 := \sum_{|\alpha|=1} \frac{(-i)^\alpha}{\alpha!}\partial_\xi^\alpha q_1 \partial_x^\alpha q_2$.

We use all the above bounds in the proof below, without explicitly referring to them.

## 6. Sketch of proof

We sketch the proof of Theorem 2 in this section. We often drop the epsilons, as we write $u_a$ for $u_a^\varepsilon$, $u$ for $u^\varepsilon$, etc. Standard hyperbolic theory provides the existence of a unique solution $u$ to (2.2) with the initial datum $u(0) = a$, over a small time interval $[0,t_*(\varepsilon)]$, such that $t_*(\varepsilon) = O(\varepsilon)$ (see the discussion following the statement of Theorem 2), with the uniform estimate

$$\sup_{0<\varepsilon<\varepsilon_0} \sup_{0\leq t\leq t_*(\varepsilon)} \|u(t)\|_{\varepsilon,s} \leq \delta. \tag{6.1}$$

We work in the following in $[0,t_*(\varepsilon)]$. Our strategy is to transform the equations to allow uniform energy estimates over $[0,t_*(\varepsilon)]$, then use a standard continuation argument.

## 6.1. The perturbation equations

The exact solution $u$ is sought in the form $u = u_a + \varepsilon^{l_0}\dot{u}$ (where $l_0$ is to be chosen), that is, as a perturbation of the approximate solution. The initial value problem for the variable $\dot{u}$ is

$$\begin{cases} \partial_t \dot{u} + \dfrac{1}{\varepsilon^2}\mathrm{op}^{\psi}_{\varepsilon}(\mathcal{A}(\varepsilon, \varepsilon(u_a + \varepsilon^{l_0}\dot{u})))\dot{u} = \dfrac{1}{\varepsilon}\mathrm{op}_{\varepsilon}(\mathcal{B})\dot{u} + \mathrm{op}_{\varepsilon}(\mathcal{D})\dot{u} + \varepsilon R^{\varepsilon}, \\ \dot{u}(0,x) = \varepsilon^{k-l_0}\phi^{\varepsilon}(x). \end{cases} \tag{6.2}$$

where

$$\begin{aligned} \mathcal{B} &:= \mathcal{B}(u_a, \cdot) + \mathcal{B}(\cdot, u_a) - \mathcal{A}^{(1)}(\varepsilon, \cdot, \xi))u_a, \\ \mathcal{D} &:= (\mathcal{G}^{\varepsilon})'(u_a) - (\mathcal{A}^{(1)}(\varepsilon, u_a + \varepsilon^l \dot{u}, \varepsilon\partial_x) - \mathrm{op}^{\psi}_{\varepsilon}(\mathcal{A}^{(1)}(u_a + \varepsilon^l \dot{u})), \\ \tilde{R}^{\varepsilon} &:= \varepsilon^{-(l_0+1)}(\mathcal{G}^{\varepsilon}(u_a + \varepsilon^l_0 \dot{u}) - \mathcal{G}^{\varepsilon}(u_a) - (\mathcal{G}^{\varepsilon})'(u_a)\dot{u}) + \varepsilon^{l_0-2}\mathcal{B}(\dot{u}, \dot{u}), \end{aligned}$$

and $R^{\varepsilon} := \tilde{R}^{\varepsilon} - \varepsilon^{l-l_0-1}R^{\varepsilon}_a$. With (6.1) and the estimates for $u_a$ following from the construction of section 3: for $\alpha \leq 1$ and $|\beta| \leq 2$,

$$|(\varepsilon^2\partial_t)^{\alpha}\partial_x^{\beta} u|_{L^{\infty}} \leq c_a + \varepsilon^{l_0-2-3/2}\delta,$$

where $c_a$ does not depend on $\varepsilon$, and where the $\varepsilon^{-2-3/2}$ factor comes from the embedding $H^s_{\varepsilon} \hookrightarrow W^{2,\infty}$. We choose now $l_0 > 2 + 3/2$ and let $C_0 := c_a + \delta$. We generically denote by $R_{(0)}$ any pseudo- or para-differential operator such that, for all $z \in H^s_{\varepsilon}(\mathbb{R}^d)$, for all $t \in [0, t_*(\varepsilon)]$ :

$$\|R_{(0)}z\|_{\varepsilon,s} \leq C_0(\|u\|_{\varepsilon,s}\|z\|_{\varepsilon,1+d_0} + \|z\|_{\varepsilon,s}).$$

## 6.2. Projection and rescaling

Let

$$v_0 := \mathrm{op}^{\psi}_{\varepsilon}(\Pi_0)\dot{u}, \qquad v_s := \frac{1}{\varepsilon}\mathrm{op}^{\psi}_{\varepsilon}(\Pi_s)\dot{u}. \tag{6.3}$$

Then $\dot{u} = v_0 + \varepsilon v_s$. Note that in (6.3), the spectral projectors $\Pi_0$ and $\Pi_s$ depend on $\dot{u}$. We multiply (6.2) by $\mathrm{op}^{\psi}_{\varepsilon}(\Pi_0)$ and $\mathrm{op}^{\psi}_{\varepsilon}(\Pi_s)$ to the left to find the equation satisfied by $v := (v_0, v_s)$ :

$$\partial_t v + \frac{1}{\varepsilon^2}\mathrm{op}^{\psi}_{\varepsilon}(iA)v = \frac{1}{\varepsilon^2}(\mathrm{op}^{\psi}_{\varepsilon}(B) + \varepsilon\mathrm{op}_{\varepsilon}(D))v + \frac{1}{\varepsilon}\mathrm{op}^{\psi}_{\varepsilon}(B')v + R_{(0)}v, \tag{6.4}$$

where

- $iA := \begin{pmatrix} \mathcal{A}\Pi_0 & 0 \\ 0 & \mathcal{A}\Pi_s \end{pmatrix} \in C^{\infty}\mathcal{M}^1$;
- $B := \begin{pmatrix} 0 & 0 \\ B_{s0} & 0 \end{pmatrix} \in C^{\infty}\mathcal{M}^0$, $D := \begin{pmatrix} 0 & 0 \\ D_{s0} & 0 \end{pmatrix} \in C^{\infty}\mathcal{M}^0$, with

$$\begin{aligned} B_{s0} &:= \Pi_s\mathcal{B}\Pi_0 + (\varepsilon\partial_t\Pi_s)\Pi_0; \\ D_{s0} &:= \Pi_s\mathcal{D}\Pi_0 - (\Pi_s\sharp\mathcal{A})\Pi_0 - (\Pi_s\mathcal{A})\sharp\Pi_0 + (\Pi_s\sharp\mathcal{B})\Pi_0; \end{aligned}$$

- $B' := \begin{pmatrix} B_0 & 0 \\ 0 & B_s \end{pmatrix} \in C^\infty\mathcal{M}^0$, with

$$B_0 := \Pi_0\mathcal{B}\Pi_0 + (\varepsilon\partial_t\Pi_0)\Pi_0, \qquad B_s := \Pi_s\mathcal{B}\Pi_s + (\varepsilon\partial_t\Pi_s)\Pi_s$$

In (6.4), the variables $v_0$ and $v_s$ are coupled only by order-zero terms, and the leading singular term has a nilpotent structure. The system is prepared.

## 6.3. Reductions

Figure 1 shows that resonances between Klein-Gordon modes happen. Because of the cold ions hypothesis, resonances between acoustic modes are higher-order phenomena. We split the source term $B'$ according to resonant and non-resonant terms as follows: $B' = B^r + B^{nr}$ where

$$B^r := \begin{pmatrix} B_0^r & 0 \\ 0 & 0 \end{pmatrix}, \quad B^{nr} := \begin{pmatrix} B_0^{nr} & 0 \\ 0 & B_s^{nr} \end{pmatrix}.$$

Thus the matrix $B_0^r$ corresponds to the components $\Pi_j B' \Pi_{j'}$ of $\Pi_0 B' \Pi_0$ such that for some $p \in \{-1, 1\}$, $\Phi_{j,j',p}(\xi) = 0$ for some $\xi$; whereas for all $j, k$ such that $\Pi_j B^{nr} \Pi_k$ does not vanish, the phase $\Phi_{j,k,p}$ is bounded away from zero, for all $p$.

**Lemma 3** (reduction of the non-resonant terms)**.** *There exists $M \in C^\infty\mathcal{M}^{-1}$ such that*

$$[\varepsilon^2\partial_t + \mathrm{op}_\varepsilon^\psi(iA), \mathrm{op}_\varepsilon^\psi(M)] = \mathrm{op}_\varepsilon^\psi(B^{nr}) + \varepsilon R_{(0)}.$$

*Sketch of proof.* Up to symbols of the form $\varepsilon R_{(0)}$, the homological equation reduces to $\varepsilon^2\partial_t M + [iA^{(0)}, M] = \tilde{B}^{nr}$, where $\tilde{B}^{nr}$ is the leading term in $B^{nr}$, which in particular is linear in $u_a$, and where $A^{(0)}$, the leading term in $A$, depends only on $\xi$. A solution is given by

$$M = \sum_{p\in\{-1,1\}} e^{ipt/\varepsilon^2}\Big(\sum_{1\le j,j'\le 6} \Phi_{j,j',p}^{-1}\Pi_j B_p^{nr}\Pi_{j'} + \sum_{7\le k,k'\le 14} \Phi_{k,k',p}^{-1}\Pi_k B_p^{nr}\Pi_{k'}\Big).$$

The above actually defines a symbol in $C^\infty\mathcal{M}^{-1}$ because $B^{nr}$ is non-resonant (see above). □

With the above lemma, the change of variables

$$\check{v} := (\mathrm{Id} + \varepsilon\mathrm{op}_\varepsilon^\psi(M))^{-1}v,$$

leads to the reduced system

$$\partial_t\check{v} + \frac{1}{\varepsilon^2}\mathrm{op}_\varepsilon^\psi(iA)\check{v} = \frac{1}{\varepsilon^2}(\mathrm{op}_\varepsilon^\psi(\check{B}) + \varepsilon\mathrm{op}_\varepsilon(D))\check{v} + \frac{1}{\varepsilon}\mathrm{op}_\varepsilon(B^r)\check{v} + R_{(0)}\check{v},$$

where $\check{B} := B + \varepsilon[B, M]$.

**Lemma 4** (reduction of resonant terms)**.** *There exists $N_0, N_1 \in C^\infty\mathcal{M}^{-1}$, such that, up to a term of the form $\varepsilon^2 R_{(0)}$,*

$$\Big[\varepsilon^2\partial_t + \mathrm{op}_\varepsilon^\psi(iA) - \varepsilon\mathrm{op}_\varepsilon^\psi(B^r), \mathrm{op}_\varepsilon^\psi(N_0) + \varepsilon\mathrm{op}_\varepsilon(N_1)\Big] = \mathrm{op}_\varepsilon^\psi(\check{B}) + \varepsilon\mathrm{op}_\varepsilon(D).$$

*Sketch of proof.* The homological equation reduces to the system

$$\varepsilon^2\partial_t N_0 + [iA - \varepsilon B^r, N_0] + i\varepsilon A\sharp N_0 = \check{B}, \tag{6.5}$$
$$\varepsilon^2\partial_t N_1 + [iA, N_1] = D, \tag{6.6}$$

where (6.5) is modulo $\varepsilon^2 R_{(0)}$ and (6.6) is modulo $\varepsilon R_{(0)}$. Equation (6.6) is solved by

$$N_1 = \sum_{p\in\{-1,1\}} e^{ipt/\varepsilon^2} \sum_{k\leq 6<j} \Phi^{-1}_{j,k,p}\Pi_j D\Pi_k.$$

The above actually defines a symbol in $C^\infty\mathcal{M}^{-1}$, as direct computations show that the interactions coefficients $\Pi_j D\Pi_k$ are small at the resonances $\Phi_{j,k,p} = 0$ pictured on Figure 2 (transparency). In equation (6.5), up to symbols of the form $\varepsilon^2 R_{(0)}$, the source term has the form $\check{B}$ by $\check{B}_0 + \varepsilon\check{B}_1$, where $\check{B}_0 = \sum_p e^{ipt/\varepsilon^2}\check{B}_{0,p}$, and $\check{B}_1 = \sum_p e^{ipt/\varepsilon^2}\check{B}_{1,p} + \sum_{p,p'} e^{i(p+p')t/\varepsilon^2}\check{B}_{p+p'}$. Similarly, up to symbols of the form $\varepsilon^2 R_{(0)}$, $A$ has the form $A^{(0)} + \varepsilon A^{(1)}$, where $A^{(0)}$ depends only on $\xi$ and $A^{(1)}$ is linear in $u_a$. Then, to solve (6.5), it suffices to solve the system

$$\begin{aligned} ip\omega N^{(0)}_{0,p} + [iA^{(0)}, N^{(0)}_{0,p}] &= \check{B}_{0,p}, \\ ip\omega N^{(1)}_{0,p} + [iA^{(0)}, N^{(1)}_{0,p}] &= \check{B}_{1,p} - A^{(0)}\sharp N^{(0)}_{0,p}, \\ i(p+p')\omega N_{p+p'} + [iA^{(0)}, N_{p+p'}] &= \check{B}_{p+p'} - [iA^{(1)} - B^{nr}, N^{(0)}]_{p+p'}, \end{aligned}$$

then to let

$$N_0 = \sum_p e^{ipt/\varepsilon^2}(N^{(0)}_{0,p} + \varepsilon N^{(1)}_{0,p}) + \varepsilon\sum_{p,p'} e^{i(p+p')t/\varepsilon^2} N_{0,p,p'}.$$

The last equation in the above system is solved as in Lemma 3, as it corresponds to an homological equation with no resonances: the phases $\lambda_j(0,0,\xi) - \lambda_k(0,0,\xi) + (p+p')\omega$ are indeed all uniformly bounded away from 0. Finally, direct computations show that the interactions coefficients (that is, the source terms in the homological equations) in the last two equations are small at the resonances (that is, when a small divisor appears in the left-hand side of the equation). Thus these equations can be solved in $C^\infty\mathcal{M}^{-1}$. □

With the above lemma, the change of variables

$$w := (\mathrm{Id} - (\mathrm{op}^\psi_\varepsilon(N_0) + \varepsilon\,\mathrm{op}_\varepsilon(N_1)))\check{v},$$

leads to the reduced system

$$\partial_t w + \frac{1}{\varepsilon^2}\mathrm{op}^\psi_\varepsilon(iA)w = \frac{1}{\varepsilon}\mathrm{op}^\psi_\varepsilon(B^r)w + R_{(0)}w.$$

## 6.4. Uniform Sobolev estimates

The system still contains a singular source term in the right-hand side, namely $B^r$. This term cannot be removed using normal form reductions as above, as it corresponds to resonances between Klein-Gordon modes and is not transparent (see section 4). However, $A$ and $B^r$ can be simultaneously symmetrizable: an explicit computation of $B_0^r$ shows that there exists a Fourier multiplier $S$ such that

$$\frac{1}{\varepsilon^2}\Re(S(D_x)\mathrm{op}_\varepsilon^\psi(iA)w, w)_{\varepsilon,s} \leq C_0\|w\|_{\varepsilon,s}^2,$$

and

$$\frac{1}{\varepsilon}\Re(S(D_x)\mathrm{op}_\varepsilon^\psi(B^r)w, , w)_{\varepsilon,s} \leq C_0\|w\|_{\varepsilon,s}^2.$$

The initial datum for $w$ is $O(\varepsilon^{k-l_0-1})$. Because $l_0$ was chosen to be greater than $2+3/2$, a sufficient condition to have a uniform $H_\varepsilon^s$ estimate for $w$ is $k > 3+3/2$. Under this condition, the uniform estimate for $w$ yields a uniform estimate for $\dot{u}$. A classical continuation argument finishes the proof of the existence of a solution over $[0, t^*)$. The error estimate (2.4) then follows from the definition of $\dot{u}$, with the choice $l_0 = k-1$.

## References


[1] C. Cheverry, O. Guès, G. Métivier, *Oscillations fortes sur un champ linéairement dégénéré*, Ann. E.N.S., vol. 36, no. 3 (2003.), 691–745

[2] T. Colin, G. Ebrard, G. Gallice, B. Texier, *Justification of the Zakharov model from Klein-Gordon-waves systems*, Comm. Partial Diff. Eq. 29 (2004), no. 9-10, 1365-1401.

[3] T. Colin, G. Métivier, in preparation.

[4] E. Grenier, *Pseudo-differential estimates of singular perturbations*, Comm. Pure and Applied Math., vol. 50 (1997), 821-865.

[5] J.-L. Joly, G. Métivier, J. Rauch, *Transparent nonlinear geometric optics and Maxwell-Bloch equations*, J. Diff. Eq., vol. 166 (2000), 175-250.

[6] D. Lannes, *Sharp estimates for pseudo-differential operators with limited regularity and commutators*, submitted.

[7] F. Linares, A. Ponce, J. C. Saut, *On a degenerate Zakharov system*, Bull. Braz. Math. Soc. 36 (2005), no. 1, 1-23.

[8] G. Métivier, K. Zumbrun, *Large viscous boundary layers for noncharacteristic nonlinear hyperbolic problems*, Mem. Amer. Math. Soc., vol. 175, no. 826 (2005).

[9] T. Ozawa, Y. Tsutsumi, *Existence and smoothing effect of solution for the Zakharov equation*, Publ. Res. Inst. Math. Sci., vol. 28, no. 3 (1992), 329-361.

[10] S. Schochet, M. Weinstein, *The nonlinear Schrödinger limit of the Zakharov equations governing Langmuir turbulence*, Comm. Math. Physics, vol. 106 (1986), 569-580.

[11] M. Taylor, *Pseudodifferential operators and nonlinear PDE*, Progress in Mathematics vol. 100, Birkhaüser Boston, 1991.

[12] B. Texier, *WKB asymptotics for the Euler-Maxwell equations*, Asymptotic Analysis 42 (2005), no. 3-4, 211-250.

[13] B. Texier, *Derivation of the Zakharov equations*, submitted.

[14] S. Musher, A. Rubenchik, V. Zakharov , *Hamiltonian approach to the description of nonlinear plasma phenomena*, Phys. Reports, vol. 129 (1985), 285-366.

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