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Abstract

This paper is devoted to the proof of almost global existence results for Klein-Gordon equations on compact revolution hypersurfaces with non-Hamiltonian nonlinearities, when the data are smooth, small and radial. The method combines normal forms with the fact that the eigenvalues associated to radial eigenfunctions of the Laplacian on such manifolds are simple and satisfy convenient asymptotic expansions.

1. Introduction

Let (M, g) be a compact Riemannian manifold without boundary, V a nonnegative potential on M, $m \in]0, +\infty[$, and consider a nonlinear Klein-Gordon equation on M

$$\begin{aligned}(\partial_t^2 - \Delta_g + V + m^2)u &= f(x, u, \partial_t u)\\ u|_{t=0} &= \varepsilon u_0\\ \partial_t u|_{t=0} &= \varepsilon u_1\end{aligned} \tag{1}$$

where f is a polynomial in $(u, \partial_t u)$ with smooth dependence in x. We are interested in questions of almost global existence for (1) when the Cauchy data are smooth and small. This problem has been studied in the case of *Hamiltonian nonlinearities* (i.e. nonlinearities $f(x, u)$ independent of $\partial_t u$) by Bourgain [3], Bambusi [1], Bambusi and Grébert [2] on a bounded interval with boundary conditions, or on the circle. In this case, conservation of H^1 norm implies immediately global existence in H^1, and these authors show for almost all values of m boundedness of H^s norms of the solution (for any s) over intervals of time of length $c_N \varepsilon^{-N}$ (for any N), where ε is the size of the Cauchy data. Their method relies on construction of approximate action-angle variables for the Hamiltonian formulation of the equation.

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On the other hand, a considerable amount of work has been done since the 80's on the problem of long time existence for solutions to wave or Klein-Gordon nonlinear equations on $\mathbb{R}^d$ with data which are smooth, small, and rapidly decaying at infinity. We refer to the introduction of [4] for bibliographical references on that problem. Let us just recall that in this framework, global existence has been proved independently by Klainerman [8] and Shatah [10] when $d \geq 3$. The proof relies on the use of dispersive properties of the linear Klein-Gordon equation. In these dimensions, the nonlinearity of the equation can be considered as a short range perturbation of the linear problem, since linear solutions decay when $t \to \pm\infty$ at an integrable rate. On the other hand, in $d = 2$ space dimensions, and for quadratic nonlinearities, the nonlinear equation is a long range perturbation of the linear one. Nevertheless, global existence has been proved by Ozawa, Tsutaya and Tsutsumi [9] combining the use of dispersive properties of the equation together with a method of normal forms. The latter consists in modifying the solution by a quadratic perturbation chosen in such a way that it cancels out the quadratic part of the nonlinearity. One reduces thus the problem to an equation with a cubic nonlinearity, which is a *short range* perturbation of the linear problem. These results do not depend on a possible Hamiltonian structure of the equation. This brings the natural question whether problems of type (1) have almost global H^s-bounded solutions for more general nonlinearities than the Hamiltonian ones considered by Bourgain, Bambusi, Bambusi-Grébert, and for more general manifolds than the circle or the interval. We prove in this paper that such a result holds true for essentially one dimensional problems – i.e. cases when $M = \mathbb{S}^1$ or M is a revolution hypersurface with radial potential and data. As the linear Klein-Gordon equation no longer displays any dispersive effect on a compact manifold, the proof relies only on the use of a normal form method.

Remark. In this paper, we outline the main steps leading to the result of almost global existence. The reader is referred to [7] for the details.

2. Main results

2.1. Statement of the main theorem

Consider (M, g) a compact Riemannian manifold without boundary, of dimension $d \geq 1$. Denote by Δ_g its Laplace-Beltrami. Let W be a closed subspace of $L^2(M)$ such that Δ_g restricted to W is a self-adjoint operator. Let $V : M \to \mathbb{R}$ be a smooth nonnegative potential such that $x \to V(x)w(x)$ belongs to W whenever $w \in W$ and set

$$P = \sqrt{-\Delta_g + V}. \tag{2}$$

We shall assume that

$$\text{the spectrum of } P|_W \text{ consists of simple eigenvalues } (\lambda_n)_{n\geq 1} \tag{3}$$

having the following asymptotic expansion as $n \to +\infty$

$$\lambda_n = \alpha n + \beta + \mathcal{O}\left(\frac{1}{n}\right) \tag{4}$$

where $\alpha > 0, \beta \in \mathbb{R}$.

Remark. Examples of M, V, W satisfying the spectral assumptions (3) and (4) are given in section 2.2.

Let f a polynomial in $(u, \partial_t u, Pu)$ vanishing at least at order 2 at 0 with C^∞ coefficients in x such that:

$$f(x, u, \partial_t u, Pu) \text{ is even in } \partial_t u. \tag{5}$$

We assume furthermore that

$$f(x, u, v, w) \in W \text{ for all } (x, u, v, w) \in M \times (W \cap C^\infty(M))^3. \tag{6}$$

We shall look for a solution u defined on $]-T, T[\times M$ of the following problem

$$\begin{aligned} (\partial_t^2 - \Delta_g + V + m^2)u &= f(x, u, \partial_t u, Pu) \\ u|_{t=0} &= \varepsilon u_0 \\ \partial_t u|_{t=0} &= \varepsilon u_1 \end{aligned} \tag{7}$$

where $m > 0, \varepsilon > 0$ is a small parameter, $u_0 \in H^{s+1}(M) \cap W, u_1 \in H^s(M) \cap W$ are given real valued functions. Our main result is the following:

Theorem 1. *Assume that conditions* (3) *and* (4) *hold true. There is a zero measure subset* $\mathcal{N}$ *of* $]0, +\infty[$ *satisfying the following: for any function* f *satisfying* (5) *and* (6)*, for any* $m \in]0, +\infty[- \mathcal{N}$*, for any* $\bar{p} \in \mathbb{N}$*, there are* $\varepsilon_0 > 0, c > 0, s_0 \in \mathbb{N}$ *such that for any* $s \geq s_0$*, any pair* (u_0, u_1) *of real valued functions belonging to the unit ball of* $H^{s+1}(M) \times H^s(M)$ *and to* $W \times W$*, any* $\varepsilon \in]0, \varepsilon_0[$*, problem* (7) *has a unique solution* u *continuous and bounded on* $]-T_\varepsilon, T_\varepsilon[$ *with values in* $H^{s+1}(M)$ *such that* $\partial_t u$ *is continuous and bounded on* $]-T_\varepsilon, T_\varepsilon[$ *with values in* $H^s(M)$*, and such that* $T_\varepsilon \geq c_{\bar{p}} \varepsilon^{-\bar{p}}$*.*

Remark. For a nonlinearity f vanishing at order 2 at 0, the local existence theory gives the lower bound $T_\varepsilon \geq c\varepsilon^{-1}$. Under the assumptions of Theorem 1, we obtain a much sharper result, namely almost global existence (i.e. $T_\varepsilon \geq c_{\bar{p}} \varepsilon^{-\bar{p}}$ for any integer $\bar{p}$).

Remark. We will outline the main steps of the proof of Theorem 1 in section 3. In particular, we will emphasize the role of the spectral assumptions (3) and (4) and of the assumption (5) on the nonlinearity f. Furthermore, we will explain why the result is obtained for almost every mass m.

2.2. Application to almost global existence on revolution hypersurfaces

In order to apply Theorem 1 to concrete situations, we look for M, V and W such that $P|_W$ satisfies the spectral assumptions (3) and (4). In practice, we consider a manifold M having symmetries and we choose W as the subspace of $L^2(M)$ invariant under these symmetries. Roughly speaking, the assumption on the simplicity of the eigenvalues of $P|_W$ amounts to choose M with enough symmetries such that the problem becomes essentially one dimensional.

We are able to apply Theorem 1 in four situations which are stated in Corollaries 1 to 4.

Corollary 1. *Let $M = \mathbb{S}^1$, let V be a smooth nonnegative odd function, let g be the canonical metric on $\mathbb{S}^1$ and let W be the set of all the odd functions in $L^2(\mathbb{S}^1)$. Assume f satisfies $f(-x,-u,-v,-w) = -f(x,u,v,w)$ for all $(x,u,v,w) \in \mathbb{S}^1 \times \mathbb{R}^3$. Then Theorem 1 holds true.*

Assume now that M and V satisfy the following assumptions (see figure 1):

M is a hypersurface of $\mathbb{R}^d, d \geq 3,$ which is invariant under the action of the rotations with axis x_d. Furthermore, M does not intersect the x_d axis and is symmetric with respect to $x_d = 0$. (8)

V is a smooth nonnegative function which is invariant under the action of the rotations with axis x_d, and even with respect to x_d. (9)

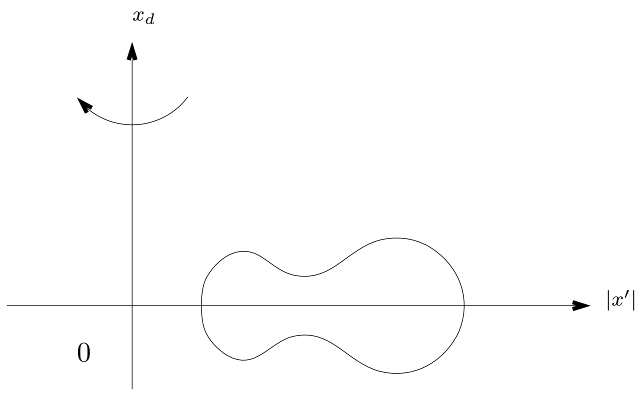

Figure 1: example of a hypersurface of revolution M satisfying (8)

Corollary 2. *Let M and V be chosen as in (8) and (9). Let W consist of all functions in $L^2(M)$ which are invariant under the action of the rotations with axis x_d and even with respect to x_d. Let f be such that $f(Rx,u,v,w) = f(x,u,v,w)$ for all $(x,u,v,w) \in M \times \mathbb{R}^3$ and all rotations R with axis x_d, and $f((x',-x_d),u,v,w) = f((x',x_d),u,v,w)$ for all $(x,u,v,w) \in M \times \mathbb{R}^3$ where $x = (x',x_d)$. Then Theorem 1 holds true.*

Corollary 3. *Let M and V be chosen as in (8) and (9). Let W consist of all functions in $L^2(M)$ which are invariant under the action of the rotations with axis x_d and odd with respect to x_d. Let f be such that $f(Rx,u,v,w) = f(x,u,v,w)$ for all $(x,u,v,w) \in M \times \mathbb{R}^3$ and all rotations R with axis x_d, and*

$$f((x',-x_d),-u,-v,-w) = -f((x',x_d),u,v,w) \text{ for all } (x,u,v,w) \in M \times \mathbb{R}^3$$

where $x = (x',x_d)$. Then Theorem 1 holds true.

Finally, assume that M and V satisfy the following assumptions (see figure 2):

M is a hypersurface of $\mathbb{R}^d, d \geq 3,$ which is invariant under the action of the rotations with axis x_d. Furthermore, M intersects the x_d axis at two points. (10)

V is a smooth nonnegative function which is invariant under the action of the rotations with axis x_d. (11)

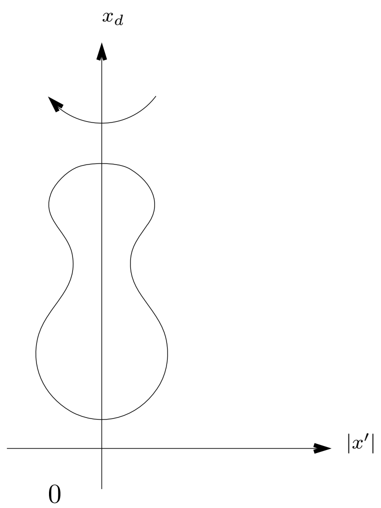

Figure 2: example of a hypersurface of revolution M satisfying (10)

Corollary 4. *Let M and V be chosen as in* (10) *and* (11)*. Let W consist of all functions in $L^2(M)$ which are invariant under the action of the rotations with axis x_d. Let f such that $f(Rx, u, v, w) = f(x, u, v, w)$ for all $(x, u, v, w) \in M \times \mathbb{R}^3$ and all rotations R. Then Theorem 1 holds true.*

In order to prove Corollaries 1 to 4, we have to show that M, V and W satisfy the spectral assumptions (3) and (4). For Corollaries 1 to 3, these properties follow readily from the well known spectral theory of the Hill operator. In the case of Corollary 4, the spectral problem may be reduced to the study of the eigenvalues for an elliptic second order operator on $[0, 1]$, degenerated at the boundary. Since we have been unable to find in the literature references to the spectral results we need, we provide a proof of them in [7]. This proof relies on the combining of WKB expansions for solutions of the corresponding singular ODE with a quantization condition.

3. Main ideas of the proof of Theorem 1

This section consists of four parts:

- In the first part, we explain how to reduce the nonlinear Klein-Gordon equation (7) to the study of a first order equation.

- In the second part, we iterate a first time the normal form method and we obtain an improvement with respect to the estimate given by the local existence theory. The results in this part do not require the spectral assumption (3).

- In the third part, we consider the reiteration of the normal form method. In fact, the improvement given by the second part is not enough for our purposes and we would like to reiterate the procedure as much as needed to gain arbitrary powers of ε^{-1}. At this stage, we explain why we need the extra assumption (3).

- In the forth part, we rely on the first three parts to prove Theorem 1.

3.1. Reduction to a first order equation

Let $u_\pm = (D_t \pm \Lambda_m)u$ where $\Lambda_m = \sqrt{m^2 - \Delta_g + V}$ and $D_t = -i\partial_t$. Instead of looking for a solution u of the nonlinear Klein-Gordon equation (7), we will look for a solution (u_+, u_-) of the corresponding system.

Using the equalities

$$u = \frac{1}{2}\Lambda_m^{-1}(u_+ - u_-) \text{ and } \partial_t u = \frac{i}{2}(u_+ + u_-) \tag{12}$$

we may rewrite (7) in the following form:

$$(D_t \mp \Lambda_m)u_\pm = -f\left(x, \frac{1}{2}\Lambda_m^{-1}(u_+ - u_-), \frac{i}{2}(u_+ + u_-), \frac{1}{2}P\Lambda_m^{-1}(u_+ - u_-)\right). \tag{13}$$

As $(u, \partial_t u) \to (u_+, u_-)$ is bounded from $L^\infty(H^{s+1}) \times L^\infty(H^s)$ to $L^\infty(H^s) \times L^\infty(H^s)$, we will look for (u_+, u_-) in $L^\infty(H^s) \times L^\infty(H^s)$ which are solutions of (13). Finally, as $u_- = -\overline{u_+}$, we further reduce the problem to a scalar first order partial differential equation:

$$(D_t - \Lambda_m)u_+ = g(x, u_+, \overline{u_+}, \Lambda_m^{-1}u_+, \Lambda_m^{-1}\overline{u_+}, P\Lambda_m^{-1}u_+, P\Lambda_m^{-1}\bar{u}_+). \tag{14}$$

In the sequel, we will first consider the following model problem

$$(D_t - \Lambda_m)u_+ = u_+^\ell \overline{u_+}^{p-\ell},\ p \geq 2,\ 0 \leq \ell \leq p. \tag{15}$$

3.2. First improvements using the normal form method

Dropping the index + in (15), we consider the model problem

$$(D_t - \Lambda_m)u = u^\ell \bar{u}^{p-\ell},\ p \geq 2,\ 0 \leq \ell \leq p. \tag{16}$$

The local existence theory applied to (16) yields the lower bound $T_\varepsilon \geq c\varepsilon^{1-p}$. In this section, we use the normal form method to improve this estimate.

For simplicity, we take $M = \mathbb{T}^1$ and $V = 0$ so that we may use Fourier series. We introduce multilinear forms. Let $a : \mathbb{Z}^p \to \mathbb{R}$ be a convenient symbol (see [7] for a definition of the algebra of symbols). We define the p-linear form associated to a by:

$$\mathcal{L}(a)(u_1, \dots, u_p) = \sum_{\xi_1,\dots,\xi_p \in \mathbb{Z}} a(\xi_1, \dots, \xi_p)\widehat{u_1}(\xi_1) \cdots \widehat{u_p}(\xi_p).$$

We introduce the energy $E_s(u)$ defined by

$$E_s(u)(t) = \frac{1}{2}\|u(t,\cdot)\|_{H^s}^2 + \operatorname{Re}\mathcal{L}(b_{p+1}^\ell)(u, \dots, \bar{u}) \tag{17}$$

where $\mathcal{L}(b_{p+1}^\ell)(u, \dots, \bar{u})$ is ℓ linear in u and $p+1-\ell$ linear in $\bar{u}$. b_{p+1}^ℓ is a convenient symbol so that $\mathcal{L}(b_{p+1}^\ell)(u, \dots, \bar{u})$ is bounded on H^s which yields

$$E_s(u)(t) \geq \frac{1}{2}\|u(t,\cdot)\|_{H^s}^2 - C\|u(t,\cdot)\|_{H^s}^{p+1}. \tag{18}$$

As we consider the small data case and as $p \geq 2$, (18) implies that $\|u\|_{H^s}^2$ is bounded by $E_s(u)$ so that it is sufficient to obtain bounds on $E_s(u)$.

We will try to find a good choice of b_{p+1}^ℓ such that for sufficiently large s

$$\frac{d}{dt}E_s(u)(t) = \mathcal{O}(\|u(t,\cdot)\|_{H^s}^{2p}). \tag{19}$$

This implies by integration for sufficiently small ε that $E_s(u)$ (and thus $\|u\|_{H^s}^2$) is bounded by $C\varepsilon^2$ over a time interval of length at least $c\varepsilon^{2-2p}$. This yields immediately $T_\varepsilon \geq c\varepsilon^{2-2p}$ which represents a gain of ε^{1-p} with respect to the local existence theory.

Remark. We will see that there is a unique choice of b_{p+1}^ℓ leading to (19). Other choices would lead to the weaker equality

$$\frac{d}{dt}E_s(u)(t) = \mathcal{O}(\|u(t,\cdot)\|_{H^s}^{p+1})$$

which gives no improvement with respect to the local existence theory.

3.2.1. Computation of $\frac{d}{dt}E_s(u)(t)$

In order to choose conveniently b_{p+1}^ℓ, we compute $\frac{d}{dt}E_s(u)(t)$. According to (17), we must compute the time derivative of $\frac{1}{2}\|u(t,\cdot)\|_{H^s}^2$ and $\mathcal{L}(b_{p+1}^\ell)(u, \dots, \bar{u})$. We start with $\frac{1}{2}\|u(t,\cdot)\|_{H^s}^2$ and using (16), we obtain

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|u(t,\cdot)\|_{H^s}^2 &= \operatorname{Im}\langle (D_t - \Lambda_m)u, u\rangle_{H^s} \\ &= \operatorname{Im}\langle u^\ell \bar{u}^{p-\ell}, u\rangle_{H^s} \\ &= \mathcal{L}(c_{p+1}^\ell)(u, \dots, \bar{u}) \end{aligned} \tag{20}$$

where $c_{p+1}^\ell = (1+\xi_{p+1}^2)^s \mathbf{1}_{\{\xi_{p+1}=\xi_1+\dots+\xi_p\}}$.

Using again (16), we obtain

$$
\begin{aligned}
D_t[\mathcal{L}(b^\ell_{p+1})(u,\dots,\bar u)] = \\
&\sum_{j=1}^{\ell}\mathcal{L}(b^\ell_{p+1})(u,\dots,\underbrace{(D_t-\Lambda_m)u}_{u^\ell\bar u^{p-\ell}},\dots,\bar u)\\
&+\sum_{j=\ell+1}^{p+1}\mathcal{L}(b^\ell_{p+1})(u,\dots,\underbrace{(D_t+\Lambda_m)\bar u}_{-\bar u^\ell u^{p-\ell}},\dots,\bar u)\\
&+\sum_{j=1}^{\ell}\mathcal{L}(b^\ell_{p+1})(u,\dots,\Lambda_m u,\dots,\bar u)\\
&-\sum_{j=\ell+1}^{p+1}\mathcal{L}(b^\ell_{p+1})(u,\dots,\Lambda_m \bar u,\dots,\bar u).
\end{aligned}
\tag{21}
$$

The first two terms are 2p linear so that

$$
\sum_{j=1}^{\ell}\mathcal{L}(b^\ell_{p+1})(u,\dots,u^\ell\bar u^{p-\ell},\dots,\bar u)+\sum_{j=\ell+1}^{p+1}\mathcal{L}(b^\ell_{p+1})(u,\dots,-\bar u^\ell u^{p-\ell},\dots,\bar u) = \mathcal{O}(\|u(t,\cdot)\|^{2p}_{H^s}).
\tag{22}
$$

Using the fact that Λ_m is diagonal on the Fourier basis with eigenvalues $\sqrt{m^2+\xi^2}$, we may rewrite the other terms of (21) as

$$
\sum_{j=1}^{\ell}\mathcal{L}(b^\ell_{p+1})(u,\dots,\Lambda_m u,\dots,\bar u)-\sum_{j=\ell+1}^{p+1}\mathcal{L}(b^\ell_{p+1})(u,\dots,\Lambda_m \bar u,\dots,\bar u) = \mathcal{L}(F_m^{p+1,\ell}b^\ell_{p+1})(u,\dots,\bar u)
\tag{23}
$$

where

$$
F_m^{p+1,\ell}=\sqrt{m^2+\xi_1^2}+\dots+\sqrt{m^2+\xi_\ell^2}-\sqrt{m^2+\xi_{\ell+1}^2}-\dots-\sqrt{m^2+\xi_{p+1}^2}.
\tag{24}
$$

Finally, (17), (20), (21), (22) and (23) yield

$$
\frac{d}{dt}E_s(u)(t)=\mathrm{Im}\left[-\mathcal{L}(F_m^{p+1,\ell}b^\ell_{p+1})(u,\dots,\bar u)+\mathcal{L}(c^\ell_{p+1})(u,\dots,\bar u)\right]+\mathcal{O}(\|u(t,\cdot)\|^{2p}_{H^s}).
\tag{25}
$$

In order to obtain (19), we would like to choose $b^\ell_{p+1}=c^\ell_{p+1}(F_m^{p+1,\ell})^{-1}$. This requires to obtain a lower bound on $F_m^{p+1,\ell}$.

3.2.2. A lower bound on $F_m^{p+1,\ell}$

We would like to choose b^ℓ_{p+1} such that $\mathcal{L}(b^\ell_{p+1})(u,\dots,\bar u)$ is bounded on H^s. As $c^\ell_{p+1}=(1+\xi_{p+1}^2)^s\mathbf{1}_{\{\xi_{p+1}=\xi_1+\dots+\xi_p\}}$, the choice $b^\ell_{p+1}=c^\ell_{p+1}(F_m^{p+1,\ell})^{-1}$ leads to

$$
b^\ell_{p+1}=(1+\xi_{p+1}^2)^s\mathbf{1}_{\{\xi_{p+1}=\xi_1+\dots+\xi_p\}}(F_m^{p+1,\ell})^{-1}.
\tag{26}
$$

Thus, we must prove that there is $C > 0$ and $N \in \mathbb{N}$ such that

$$|F_m^{p+1,\ell}(\xi_1,\dots,\xi_{p+1})| \geq C\mu(|\xi_1|,\dots,|\xi_{p+1}|)^{-N} \tag{27}$$

where $\mu(|\xi_1|,\dots,|\xi_{p+1}|)$ is the third largest frequency. In fact, (26) allows us to put s derivatives on the term corresponding to the largest frequency and s derivatives on the term corresponding to the second largest frequency. To avoid a loss of derivatives, the additional derivatives coming from $F_m^{p+1,\ell}$ must fall on the term corresponding to the third largest frequency, whence (27). Taking the terms corresponding to the two largest frequencies in L^2 and the other ones in L^∞, and using (27) and the Sobolev embedding implies

$$|\mathcal{L}(b_{p+1}^\ell)(u,\dots,\bar{u})| \leq C\|u\|_{H^s}^2\|u\|_{H^{N+1/2+\delta}}\|u\|_{H^{1/2+\delta}}^{p-2} \tag{28}$$

which yields the continuity of $\mathcal{L}(b_{p+1}^\ell)(u,\dots,\bar{u})$ on H^s for $s > N+1/2$.

We are left with proving (27). Unfortunately, such a bound does not hold in general as $F_m^{p+1,\ell}$ may vanish. If $p+1$ is even and $\ell = (p+1)/2$, then there are as many square roots with a plus sign as square roots with a minus sign in (24). Thus, $F_m^{p+1,\ell}$ vanishes on the set $\Sigma(p+1)$ such that there is a bijection between the squares of the frequencies corresponding to a plus sign, and the squares of the frequencies corresponding to a minus sign:

$$\Sigma(p+1) = \Big\{(\xi_1,\dots,\xi_{p+1}) \in \mathbb{Z}^{p+1}; \{\xi_1^2,\dots,\xi_{(p+1)/2}^2\} = \{\xi_{(p+3)/2}^2,\dots,\xi_{p+1}^2\}\Big\}.$$

If we restrict ourselves to frequencies outside the set $\Sigma(p+1)$, we are able to obtain the lower bound (27):

Proposition 1. *There is a zero measure subset $\mathcal{N}$ of $]0,+\infty[$ such that for $m \in]0,+\infty[-\mathcal{N}$:*

$$|F_m^{p+1,\ell}(\xi_1,\dots,\xi_{p+1})| \geq C\mu(|\xi_1|,\dots,|\xi_{p+1}|)^{-N}$$

where $(\xi_1,\dots,\xi_{p+1}) \in \mathbb{Z}^{p+1}$ if $\ell \neq \frac{p+1}{2}$ (resp. $(\xi_1,\dots,\xi_{p+1}) \in \mathbb{Z}^{p+1} - \Sigma(p+1)$ if p is odd and $\ell = \frac{p+1}{2}$).

Remark. The proof of Proposition 1 requires the spectral assumption (4) and relies in particular on the Łojaciewiecz inequalities.

Remark. Proposition 1 does not hold for all mass m. This explains why Theorem 1 is obtained for almost every mass m.

3.2.3. First improvements using the normal form method

As $F_m^{p+1,\ell}$ vanishes on $\Sigma(p+1)$, we may not choose $b_{p+1}^\ell = c_{p+1}^\ell(F_m^{p+1,\ell})^{-1}$. We first decompose c_{p+1}^ℓ in $(c_{p+1}^\ell)' + (c_{p+1}^\ell)''$ where $(c_{p+1}^\ell)'' = 0$ if $\ell \neq (p+1)/2$, and for $\ell = (p+1)/2$:

$$(c_{p+1}^\ell)''(\xi_1,\dots,\xi_{p+1}) = c_{p+1}^\ell(\xi_1,\dots,\xi_{p+1})\mathbf{1}_{\{(\xi_1,\dots,\xi_{p+1})\in\Sigma(p+1)\}}.$$

By Proposition 1, we may now choose $b_{p+1}^\ell = (c_{p+1}^\ell)'(F_m^{p+1,\ell})^{-1}$ as $(c_{p+1}^\ell)'$ vanishes on $\Sigma(p+1)$. Together with (25), this yields

$$\frac{d}{dt}E_s(u)(t) = \operatorname{Im}\Big[\mathcal{L}((c_{p+1}^\ell)'')(u,\dots,\bar{u})\Big] + \mathcal{O}(\|u(t,\cdot)\|_{H^s}^{2p}).$$

In order to obtain (19), we are left with proving the 'miraculous' cancellation

$$\operatorname{Im}\left[\mathcal{L}((c^{\ell}_{p+1})'')(u,\dots,\bar{u})\right]=0. \tag{29}$$

A straightforward computation yields

$$\mathcal{L}((c^{\ell}_{p+1})'')(u,\dots,\bar{u})=\sum_{(\xi_1,\dots,\xi_{\frac{p+1}{2}},\eta_1\dots,\eta_{\frac{p+1}{2}})\in\Sigma(p+1)}K(\xi,\eta)\prod_{j=1}^{\frac{p+1}{2}}\hat{u}(\xi_j)\prod_{j=1}^{\frac{p+1}{2}}\overline{\hat{u}(\eta_j)}$$

where

$$K(\xi,\eta)=(1+\eta^2_{\frac{p+1}{2}})^s\mathbf{1}_{\{\xi_1+\dots+\xi_{\frac{p+1}{2}}=\eta_1+\dots+\eta_{\frac{p+1}{2}}\}}. \tag{30}$$

Since

$$2i\operatorname{Im}\left[\mathcal{L}((c^{\ell}_{p+1})'')(u,\dots,\bar{u})\right]=$$
$$\sum_{(\xi_1,\dots,\xi_{\frac{p+1}{2}},\eta_1\dots,\eta_{\frac{p+1}{2}})\in\Sigma(p+1)}(K(\xi,\eta)-\overline{K(\eta,\xi)})\prod_{j=1}^{\frac{p+1}{2}}\hat{u}(\xi_j)\prod_{j=1}^{\frac{p+1}{2}}\overline{\hat{u}(\eta_j)},$$

checking (29) amounts to prove that

$$\overline{K(\eta,\xi)}=K(\xi,\eta)\text{ on }\Sigma(p+1)\text{ up to permutations among the }\xi_j{}'s\text{ and }\eta_j{}'s. \tag{31}$$

We see from (30) that $K(\xi,\eta)$ is real so that we only have to check that we can exchange ξ and η without changing $K(\xi,\eta)$ on $\Sigma(p+1)$ up to permutations. It is clearly true for $\mathbf{1}_{\{\xi_1+\dots+\xi_{\frac{p+1}{2}}=\eta_1+\dots+\eta_{\frac{p+1}{2}}\}}$. As $\xi^2_{\frac{p+1}{2}}=\eta^2_{\frac{p+1}{2}}$ on $\Sigma(p+1)$ up to permutations, it is also true for $(1+\eta^2_{\frac{p+1}{2}})^s$. This yields (31) which in turn yields (29). Thus, our choice for b^{ℓ}_{p+1} implies (19), whence the lower bound $T_\varepsilon\geq c\varepsilon^{2-2p}$.

Remark. In the case of our model problem (16), we have used the fact that $K(\xi,\eta)$ is real. In order to maintain this property in the general case (14), we must assume that the nonlinearity $f(x,u,\partial_t u,Pu)$ is even in $\partial_t u$. In fact, each power of $\partial_t u$ brings a power of i as shown by (12).

Remark. This first improvement with respect to the local existence theory has been obtained without making the spectral assumption (3). In fact, we have explained the proof in the case of $M=\mathbb{T}^1$ and $V=0$ for which the eigenvalues are double. We have also obtained this improvement in the case of spheres [5] and Zoll manifolds [6]. In this framework, one must replace the Fourier coefficients by the spectral projectors associated to the eigenvalues (resp. to the clusters of eigenvalues in the case of Zoll manifolds). One of the difficulties comes from the generalization to the spheres and Zoll manifolds of inequality (28). Here, we have used formula (26) and in particular the fact that $\xi_{p+1}=\xi_1+\dots+\xi_p$ on the support of b^{ℓ}_{p+1}. This comes from the fact that the product of eigenvectors on $\mathbb{T}^1$ is an eigenvector. This does not hold on spheres and Zoll manifolds, but we are able to prove a property of almost orthogonality of products of eigenfunctions which is sufficient for our purpose (see [6]).

3.3. Reiteration of the procedure

In the previous section, we have obtained the lower bound $T_\varepsilon \geq c\varepsilon^{2-2p}$. This is not sufficient to prove a result of almost global existence. Therefore, we would like to reiterate the normal form method as much as needed to gain any power of ε^{-1}. In this section, we explain the additional difficulties coming from the reiteration of this procedure.

We would like to improve the estimate (19). This requires to replace $\mathcal{O}(\|u\|_{H^s}^{2p})$ by a higher nonlinearity using the normal form method. $\mathcal{O}(\|u\|_{H^s}^{2p})$ comes from (21) and is given by

$$\sum_{j=1}^{\ell} \mathcal{L}(b_{p+1}^{\ell})(u,\ldots,u^\ell \bar{u}^{p-\ell},\ldots,\bar{u}) + \sum_{j=\ell+1}^{p+1} \mathcal{L}(b_{p+1}^{\ell})(u,\ldots,-\bar{u}^\ell u^{p-\ell},\ldots,\bar{u}).$$

We focus on the second term which may be rewritten as

$$\sum_{j=\ell+1}^{p+1} \mathcal{L}(b_{p+1}^{\ell})(u,\ldots,-\bar{u}^\ell u^{p-\ell},\ldots,\bar{u}) = \mathcal{L}(c_{2p}^{p})(u,\ldots,\bar{u})$$

for a convenient symbol c_{2p}^p. As $F_m^{2p,p}$ vanishes on $\Sigma(2p)$, we proceed as in the previous section, namely we decompose c_{2p}^p in $(c_{2p}^p)' + (c_{2p}^p)''$. The normal form method allows us to get rid of the contributions of $(c_{2p}^p)'$ and we are left with proving

$$\operatorname{Im}\left[\mathcal{L}((c_{2p}^p)'')(u,\ldots,\bar{u})\right] = 0.$$

This amounts to check that $\overline{K(\eta,\xi)} = K(\xi,\eta)$ on $\Sigma(2p)$ up to permutations, where $K(\xi,\eta)$ is given this time by:

$$\begin{aligned} K(\xi,\eta) = c_{p+1}^{\ell}&(\xi_1,\ldots,\xi_\ell,\xi_{\ell+1}+\ldots+\xi_p-\eta_1-\ldots-\eta_\ell,\eta_{l+1},\ldots,\eta_p) \\ &\times\Big(\sqrt{m^2+\xi_1^2}+\cdots+\sqrt{m^2+\xi_\ell^2} \\ &\quad -\sqrt{m^2+(\xi_{\ell+1}+\ldots+\xi_p-\eta_1-\ldots-\eta_\ell)^2} \\ &\quad\quad -\sqrt{m^2+\eta_{\ell+1}^2}-\cdots-\sqrt{m^2+\eta_p^2}\Big)^{-1} + \cdots . \end{aligned} \tag{32}$$

For $K(\xi,\eta)$ given by (30), we concluded using the fact that the expressions were symmetric in (ξ,η) or involved only squares of frequencies. For $K(\xi,\eta)$ given by (32), we do not know how to deal with expressions of the form $\xi_{\ell+1}+\ldots+\xi_p-\eta_1-\ldots-\eta_\ell$ which do not satisfy these properties.

However, there is one particular case which is easy to deal with. Remember that $\xi_j = \pm\eta_j$, $j=1,\ldots,p$, up to permutations on $\Sigma(2p)$. If we make the further restriction that $\xi_j = \eta_j$, $j=1,\ldots,p$, up to permutations on $\Sigma(2p)$, then $\xi=\eta$ up to permutations, and as K is real, we have clearly $\overline{K(\eta,\xi)} = K(\xi,\xi) = K(\xi,\eta)$ up to permutations. In other words, the difficulty comes from the choice of sign $\xi_j = \pm\eta_j$ which is a consequence of the fact that the eigenvalues on $\mathbb{T}^1$ are double and are associated to the eigenvectors $e^{\pm i\xi}$. Therefore, we expect to overcome this difficulty by assuming that the eigenvalues are simple which corresponds to the spectral assumption (3).

3.4. Proof of Theorem 1

For simplicity, we still take $M = \mathbb{T}^1$ and $V = 0$. Furthermore, we choose W as the subspace of all odd functions in $L^2(M)$. Thus, these M, V, and W satisfy the spectral assumptions (3) and (4), and we may use the sinusoidal Fourier series. Instead of dealing with the model problem (16), we now consider the general case (14).

To a convenient symbol $a : \mathbb{N}^p \to \mathbb{R}$, we associate a p-linear form

$$\mathcal{L}(a)(u_1, \ldots, u_p) = \sum_{n_1,\ldots,n_p \in \mathbb{N}} a(n_1, \ldots, n_p)\widehat{u_1}(n_1) \cdots \widehat{u_p}(n_p)$$

where $\hat{u}(n)$ are now the sinusoidal Fourier coefficients of u. We fix an integer $\bar{p}$. In order to get rid of all terms of order less than $\bar{p}$, we introduce the energy

$$E_s(u)(t) = \frac{1}{2}\|u(t,\cdot)\|^2_{H^s} + \mathrm{Re} \sum_{3 \leq p \leq \bar{p}} \sum_{\ell=0}^{p} \mathcal{L}(b_p^\ell)(u, \ldots, \bar{u}).$$

A computation similar to the one carried on in section 3.2.1 yields

$$\begin{aligned} \frac{d}{dt}E_s(u)(t) = \mathrm{Im}\Bigg[\sum_{3\leq p \leq \bar{p}} \sum_{\ell=0}^{p} \Bigg(& - \mathcal{L}(F_m^{p,\ell} b_p^\ell)(u, \ldots, \bar{u}) \\ & + \mathcal{L}(c_p^\ell)(u, \ldots, \bar{u})\Bigg)\Bigg] + \mathcal{O}(\|u(t,\cdot)\|^{\bar{p}+1}_{H^s}) \end{aligned} \tag{33}$$

where c_p^ℓ depends on b_q^k for $3 \leq q < p$, $0 \leq k \leq q$. For $3 \leq p \leq \bar{p}$, $0 \leq \ell \leq p$, we decompose c_p^ℓ in $(c_p^\ell)' + (c_p^\ell)''$ and we choose

$$b_p^\ell(n_1, \ldots, n_p) = \frac{(c_p^\ell)'(n_1, \ldots, n_p)}{F_m^{p,\ell}(n_1, \ldots, n_p)}, 3 \leq p \leq \bar{p}, 0 \leq \ell \leq p. \tag{34}$$

Remark. In order to define b_p^ℓ, we need to know c_p^ℓ. This is indeed the case as c_p^ℓ depends on b_q^k for $3 \leq q < p$, $0 \leq k \leq q$ which have already been constructed.

Finally, the spectral assumption (3) implies immediately

$$\mathrm{Im}\left[\mathcal{L}((c_p^\ell)'')(u, \ldots, \bar{u})\right] = 0,\ 3 \leq p \leq \bar{p}, 0 \leq \ell \leq p, \tag{35}$$

as noticed at the end of the previous section. (33), (34) and (35) yield

$$\frac{d}{dt}E_s(u)(t) = \mathcal{O}(\|u(t,\cdot)\|^{\bar{p}+1}_{H^s}).$$

Thus, $T_\varepsilon \geq c\varepsilon^{-\bar{p}}$ which finishes the proof of Theorem 1.

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