JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (1982), p. 1-3

<http://www.numdam.org/item?id=JEDP_1982____A15_0>

AN EXAMPLE OF THE HEISENBERG GROUP

by E. M. STEIN

Let us consider the Kohn-Laplacian $\square_b^{(q)}$ which acts on appropriate q-forms on the Heisenberg group H_n. Then as Kohn showed (more generally), $\square_b^{(q)}$ is C^∞ hypoelliptic and locally solvable, when $0 < q < n$; however this is not the case when $q = 0$, or $q = n$, and the latter fact goes back to the fundamental example of Lewy.

If we introduce the coordinates (z,t) $z \in \mathbb{C}^n$, $t \in \mathbb{R}^n$, and the complex vectorfields

$$Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t} \quad , \quad j = 1,\dots,n$$

then one can write

$$\square_b^{(q)} = \mathcal{L}_\alpha \otimes I, \qquad \text{with} \qquad \alpha = n - 2q$$

$$\mathcal{L}_\alpha = -\frac{1}{2} \sum_j (Z_j \bar{Z}_j + \bar{Z}_j Z_j) + i\alpha \frac{\partial}{\partial t},$$

and there is an explicit fundamental solution, given by

$$\psi_\alpha \, c_\alpha^{-1} \quad , \quad c_\alpha = \frac{c}{\Gamma(\frac{n-\alpha}{2}) \; \Gamma(\frac{n+\alpha}{2})}$$

with $\psi_\alpha = (|z|^2 - it)^{-(n+\alpha)/2} \, (|z|^2 + it)^{-(n-\alpha)/2}$.

All of this holds for $\alpha \neq \pm n, n+2, \dots$ and incidentally shows the analytic hypoellipticity of $\square_b^{(q)}$, $0 < q < n$, on the Heisenberg group (see [1]).

Our first question, is what happens when $q = 0$ (i.e. $\alpha = n$) ? These is then ([2]) a relative fundamental solution $\underset{\sim}{K}$ so that

$$\square_b \underset{\sim}{K} = I - \underset{\sim}{C}$$

where $\underset{\sim}{C}$ is the projection operator on the null space of $\Box_b$ (i.e. the Cauchy-Szego projector) and $\underset{\sim}{K}$ has a description similar to the above fundamental solutions.

Finally what happens to $\Box_b^o + \mu I$? (From now on, for simplicity of notation $\Box_b = \Box_b^o$, and n = 1).

Theorem : Suppose $\mu \neq 0$, the operator $\Box_b + \mu I$ is locally solvable, C^∞ and analytic hypoelliptic.

The C^∞ hypoellipticity was already observed by Melin

The idea will be to construct a parametrix (analytic away from the diagonal).

The formal solution to our problem is

$$(\Box_b + \mu I)^{-1} = \underset{\sim}{C}/\mu + \sum_{n=1}^{\infty} (-\mu)^{n-1} \underset{\sim}{K}^n .$$

The main difficulty is to give a meaning to this infinite series, and prove the appropriate properties of the kernel it represents. This requires some definitions.

Let $E(u) = \sum_{n\geqslant 3}^{\infty} \frac{u^n}{n!(n-3)!}$.

This is a Bessel function. What is important for us is that $|E(u)| \leqslant e^{C|u|^{1/2}}$, u complex. Next let $w = |z|^2 - it$. **Write**

$$P_\mu = C/\mu + K^1 - \mu K^2 + \frac{\bar{w}}{2\mu\pi^2} \int_0^\infty E\{2\mu(w+\bar{w}s)\frac{\log s}{s-1)}\} \log(w+\bar{w}s) \times \frac{s-1}{(w+\bar{w}s)^3} ds$$

with C the Cauchy-Szego kernel $= \frac{1}{\pi^2 w^2}$

$$K = \frac{1}{2\pi^2}(\log w - \log \bar{w}) w^{-1} ,$$

and $K^2 = \frac{1}{\pi^2} \bar{\omega} \int_0^\infty (w + \bar{w}s)^{-1} \frac{\log s}{s-1} ds$.

Proposition : (1) P_μ is real-analytic away from the origin

(2) $(\Box_b + \mu I)P_\mu = S + R_\mu$, where R_μ is everywhere real-analytic, with δ = Dirac function of the origin.

The proof of the proposition is somewhat complicated; its details will appear elsewhere [3].

Bibliography

[1] Folland and Stein : Comm. Pure and Applied Math. 27 (1974), 429-522.

[2] Greiner, Kohn and Stein : Proc. Nat. Acad. Sci. 72 (1975), 3287-3289.

[3] To appear.

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