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LOWER BOUNDS FOR SCHRODINGER EQUATIONS

by C. FEFFERMAN and D. H. PHONG

The main purpose of this article is to establish the following theorem on the lowest eigenvalue of Schrödinger equations with no regularity assumptions on the potentials :

Theorem : Let $V(x)$ be a negative function on $\mathbb{R}^n$, and set

$$E_{small} = \sup_Q \left[\left(\frac{1}{|Q|}\int_Q |V(x)|\,dx\right) - C(\operatorname{diam} Q)^{-2}\right]$$

$$E_{big} = \sup_Q \left[\left(\frac{1}{|Q|}\int_Q |V(x)|^p dx\right)^{1/p} - c(\operatorname{diam} Q)^{-2}\right] \qquad p > 1$$

where Q ranges over cubes in $\mathbb{R}^n$ with sides parallel to the axes, $|Q|$ and diam Q are respectively the measure and the diameter of Q, and C,c denote throughout constants depending only on the dimension of $\mathbb{R}^n$.

Then the lowest bound $-E$ of the operator $-\Delta + V(x)$ satisfies the inequalities

$$cE_{small} \leqslant E \leqslant CE_{big} \tag{1}$$

Remarks (a) It may happen that $E_{big} = 0$, in which case $-\Delta + V(x)$ has been proved to be positive. When $n \geqslant 3$ we may thus apply the theorem with $p < n/2$ and obtain as a corollary the Sobolev inequality, i.e., $\nabla f \in L^2$ implies $f \in L^{2n/(n-2)}$. In $\mathbb{R}^2$ the sharp form of the Sobolev inequality states that $\nabla f \in L^2$ implies $e^{\lambda|f|^2} \in L^1_{loc}$ for small positive λ . This is sharper than our result.

(b) In the definition of E_{big}, one cannot replace $p > 1$ simply by 1. In fact in $\mathbb{R}^2$ this would imply $\nabla f \in L^2 \Rightarrow f \in L^\infty$, which is false. To get counterexamples in $\mathbb{R}^n$ $(n > 2)$ just consider potentials of the form $V(x',x'') = V(x')$ with $x' \in \mathbb{R}^2$, $x'' \in \mathbb{R}^{n-2}$.

(c) It is a natural guess that the sharp lower bound should involve $L \operatorname{Log}^+ L$ norms of $|V|$ rather than L^p norms. This would agree with the Sobolev inequality on $\mathbb{R}^2$. If this were the case, however, a study of the function $f^\#$ (see below) alone would not suffice, since $f^\# \in L^\infty$ would merely imply $f \in \exp(L)$.

(d) Similar information can be obtained for higher eigenvalues, this development will be reported on in detail elsewhere.

The proof of the theorem is based on three lemmas, the first providing the key estimate in this work, while the other two are variants of results from harmonic analysis established in the last decade (see e. g. [1] [2] and the references therein). We begin by recalling some definitions :

(1) A positive measure μ is said to satisfy condition (A_∞) if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$E \subset Q \quad \text{and} \quad |E| < \delta |Q| \quad \text{implies} \quad \mu(E) < \varepsilon\, \mu(Q).$$

(2)

$$f_Q = \frac{1}{|Q|} \int_Q f(x)\,dx$$

$$f^*(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\,dy$$

$$f^\#(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\,dy$$

Fix a cube Q^o.

<u>Lemma 1</u> : Let μ be a positive measure on Q^o such that

$$\mu(Q) \leqslant C|Q| \,(\text{diam } Q)^{-2} \qquad \text{for all cubes } Q \subseteq Q^o. \tag{2}$$

Then

$$\int_{Q^o} |f^\#|^2 d\mu \leqslant C' \int_{Q^o} |\nabla f|^2 dx \tag{3}$$

<u>Lemma 2</u> : If μ satisfies (A_∞) then

$$\int_{Q^o} |f(x) - f_{Q^o}|^2 d\mu(x) \leqslant C \int_{Q^o} |f^\#(x)|^2 d\mu(x) \tag{4}$$

<u>Lemma 3</u> : Let $d\mu = |V(x)|dx$, and assume that $(p > 1)$

$$\left(\frac{1}{|Q|}\int_Q |V(x)|^p dx\right)^{1/p} \leqslant C(\text{diam } Q)^{-2} \qquad \text{for all cubes } Q \subset Q^o .$$

Then the measure $d\mu^{+}$ defined by $d\mu^{+} = [(|V(x)|^{p})^{*}]^{1/p}dx$ will satisfy both (A_∞) and the condition (2).

The constants in the conclusions can evidently be taken to be small if the constants in the hypotheses are small.

Proof of the theorem : Applying $-\Delta + V(x)$ to the dilates and translates of a fixed C_0^∞ function, we get the estimate involving E_{small} at once. To establish the estimate involving E_{big}, cut $\mathbb{R}^n$ into a grid of cubes Q^o of equal size diam $Q^o = (cE_{big})^{-1/2}$, and define $d\mu, d\mu^{+}$ as in Lemma 3. Observe then that if the constants c appearing in the definition of E_{big} and Q^o are small, we have

$$\begin{aligned}\int_{Q^o}|f_{Q^o}|^2 d\mu &\leqslant |f_{Q^o}|^2\ \mu(Q^o)\\ &\leqslant \left(\frac{1}{|Q^o|}\int_{Q^o}|f|^2dx\right)[C|Q^o|\ (\text{diam } Q^o)^{-2}]\\ &\leqslant \frac{E_{big}}{2}\int_{Q^o}|f|^2dx \qquad (5)\end{aligned}$$

while

$$\begin{aligned}\int_{Q^o}|f(x) - f_{Q^o}|^2 d\mu(x) &\leqslant \int_{Q^o}|f(x) - f_{Q^o}|^2\, d\mu^{+}(x)\\ &\leqslant C\int_{Q^o}|f^{\#}(x)|^2 d\mu^{+}(x)\\ &\leqslant \frac{1}{2}\int_{Q^o}|\nabla f(x)|^2dx \qquad (6)\end{aligned}$$

in view of Lemmas 1,2, and 3. The inequalities (5) and (6) in turn yield

$$\int_{Q^o}|f(x)|^2 d\mu(x) \leqslant \int_{Q^o}|\nabla f(x)|^2dx + E_{big}\int_{Q^o}|f(x)|^2dx$$

and the desired bound follows by summing over all cubes Q^o.

Proof of Lemma 1 : Let $\alpha_o = \frac{1}{|Q^o|}\int_{Q^o}|f(x) - f_{Q^o}|dx$. For any $\alpha > \alpha_o$, $K >> 1$ we may apply the Calderón-Zygmund stopping process to obtain a decomposition of the sets $\{x \in Q^o,\ f^{\#}(x) > \alpha\}$ and $\{x \in Q^o\ ;\ f^{\#}(x) > K\alpha\}$ into cubes with the following properties.

$$\{f^{\#} > \alpha\} = \bigcup_k Q_k \qquad \{f^{\#} > K\alpha\} = \bigcup_{k,\nu} Q_{k\nu}$$

(i) $Q_{k\nu} \subseteq Q_k$

(ii) $\dfrac{1}{|Q_k|}\displaystyle\int_{Q_k} |f(x) - f_{Q_k}|dx \sim \alpha$

(iii) $\dfrac{1}{|Q_{k\nu}|}\displaystyle\int_{Q_{k\nu}} |f(x) - f_{Q_{k\nu}}|dx \sim K\alpha$

Next observe that

$$(\nabla f)^* \geqslant \alpha(\text{diam } Q_k)^{-1} \text{ throughout } Q_k \tag{7}$$

$$|Q_k| \geqslant 2|\bigcup_\nu Q_{k\nu}| \tag{8}$$

In fact (7) follows from

$$\frac{1}{|Q_k|}\int_{Q_k} |f(x) - f_{Q_k}|dx \leqslant (\text{diam } Q_k)\frac{1}{|Q_k|}\int_{Q_k} |\nabla f(x)|dx$$

while (8) is a consequence of

$$\alpha|Q_k| \sim \int_{Q_k} |f(x) - f_{Q_k}|dx \geqslant \sum_\nu \int_{Q_{k\nu}} |f(x) - f_{Q_k}|dx$$

$$\geqslant \frac{1}{2}\sum_\nu \int_{Q_{k\nu}} |f(x) - f_{Q_{k\nu}}|dx \sim \frac{1}{2}K\alpha \sum_\nu |Q_{k\nu}|$$

and the fact that K is large. We may thus write

$$\int_{Q_k\setminus\bigcup_\nu Q_{k\nu}} |(\nabla f)^*(x)|^2dx \geqslant \alpha^2(\text{diam } Q_k)^{-2}|Q_k\setminus\bigcup_\nu Q_{k\nu}|$$

$$\geqslant \frac{1}{2}\alpha^2(\text{diam } Q_k)^{-2}|Q_k|$$

$$\geqslant \alpha^2\mu(Q_k)/(2C).$$

which implies in turn

$$\int_{\{\alpha < f^{\#} \leqslant K\alpha\}} |(\nabla f)^{*}(x)|^{2}dx \geqslant \alpha^{2}\mu(f^{\#} > \alpha)/C .$$

Let now α range over $2K^{m}\alpha_{o}$, $m = 0,1,2,\ldots$, and sum over m to obtain

$$\int_{Q^{o}} |(\nabla f)^{*}(x)|^{2}dx \geqslant \Big(\int_{\{f^{\#} > 2\alpha_{o}\}} |f^{\#}(x)|^{2}d\mu(x)\Big)/C \tag{9}$$

On the other hand note that

$$\begin{aligned}\int_{\{f^{\#} \leqslant 2\alpha_{o}\}} |f^{\#}|^{2}d\mu &\leqslant 4\alpha_{o}^{2}\, \mu(Q^{o}) \\ &\leqslant 4[(\text{diam } Q^{o})^{2} \frac{1}{|Q^{o}|} \int_{Q^{o}} |\nabla f(x)|^{2}dx]\ [C|Q^{o}|(\text{diam } Q^{o})^{-2}] \\ &\leqslant 4\ C\int_{Q_{o}} |\nabla f(x)|^{2}dx \end{aligned}\tag{10}$$

Combining (9), (10) and applying the maximal theorem establihes (3).

<u>Proof of Lemma 2</u> : We may assume that $f_{Q^{o}} = 0$, so it suffices to show that

$$\int_{Q^{o}}|f^{*}(x)|^{2}d\mu(x) \leqslant C\int_{Q^{o}}|f^{\#}(x)|^{2}d\mu(x) \tag{11}$$

Let $\alpha_{o} = \frac{1}{|Q^{o}|}\int_{Q^{o}}|f(x)|dx$, and let K be a large positive number. A Calderon-Zygmund decomposition yields for each $\alpha > \alpha_{o}$:

$$\{f^{*} > \alpha\} = \bigcup_{k} Q_{k} \qquad \{f^{*} > K\alpha\} = \bigcup_{k,\nu} Q_{k\nu}$$

(i) $Q_{k\nu} \subseteq Q_{k}$

(ii) $\frac{1}{|Q_{k}|}\int_{Q_{k}} |f(x)|dx \sim \alpha$

(iii) $\frac{1}{|Q_{k\nu}|}\int_{Q_{k\nu}} |f(x)|dx \sim K\alpha$

Obviously $f^{\#} > \delta\alpha$ on cubes Q_k satisfying $\frac{1}{|Q_k|}\int_{Q_k} |f(x) - f_{Q_k}|dx > \delta\alpha$; on the remaining cubes, for any $\varepsilon > 0$ a choice of δ small enough will guarantee that

$$\mu\Big(\bigcup_{\nu} Q_{k\nu}\Big) \leqslant \varepsilon\,\mu(Q_k) \tag{12}$$

Indeed we have in that case

$$\delta\alpha|Q_k| \geqslant \int_{Q_k} |f(x) - f_{Q_k}|dx \geqslant \sum_{\nu}\int_{Q_{k\nu}} |f(x) - f_{Q_k}|\,dx$$

$$\geqslant \frac{1}{2}\sum_{\nu}\int_{Q_{k\nu}} |f(x) - f_{Q_{k\nu}}|\,dx \sim \frac{1}{2}K\alpha\sum_{\nu}|Q_{k\nu}|$$

and (12) follows from (A_∞). Thus

$$\mu(f^* > K\alpha) \leqslant \varepsilon\;\mu(f^* > \alpha) + \mu(f^{\#} > \delta\alpha) \quad \text{for} \quad \alpha > \alpha_o .$$

For $\alpha \leqslant \alpha_o$ this estimate is trivial, since $f^{\#} > \alpha_o)$ throughout Q^o. To get (11), integrate with respect to $\alpha d\alpha$ and choose $\varepsilon = K^{-2}/2$. Q.E.D.

Proof of Lemma 3 : We need only prove a reverse Hölder inequality for some $r > 1$

$$\Big(\frac{1}{|Q|}\int_Q (V^+(x))^r dx\Big)^{1/r} \leqslant C\,\frac{1}{|Q|}\int_Q V^+(x)dx \qquad \text{all} \quad Q \subseteq Q^o \tag{13}$$

since then

$$\mu(E) = \int_Q \chi_E V^+(x)dx \leqslant |E|^{1/r'}\Big(\int_Q (V^+(x))^r dx\Big)^{1/r} \leqslant \Big(\frac{|E|}{|Q|}\Big)^{1/r'}\mu(Q)$$

for $E \subset Q$. Now introduce

$$K(Q) = \sup_{Q' \supseteq Q} \Big[\frac{1}{|Q'|}\int_{Q'} |V(x)|^p dx\Big]^{1/p}$$

$$V_Q(x) = \sup_{x \in Q' \subseteq Q} \Big[\frac{1}{|Q'|}\int_{Q'} |V(y)|^p dy\Big]^{1/p}$$

and observe that

$$V^+(x) = \max(K(Q),\, V_Q(x))$$

$$K(Q) \leqslant \min\{C(\mathrm{diam}\ Q)^{-2}, \quad \frac{1}{|Q|}\int_Q V^+(x)\,dx\}$$

Thus (13) and condition (2) for $d\mu^+$ are simple consequences of the following inequality

$$\frac{1}{|Q|}\int_Q (V_Q(x))^r dx \leqslant C\, K(Q)^r \tag{14}$$

In view of the maximal theorem we may write

$$\frac{|\{x \in Q;\ V_Q(x) > \alpha\}|}{|Q|} \leqslant \frac{C}{\alpha^p}\ \frac{1}{|Q|}\int_Q |V_Q(x)|^p dx$$

$$\leqslant \frac{C}{\alpha^p}\ [K(Q)]^p$$

Integrating with respect to α for $\alpha >$ $K(Q)$, $1 < r < p$ yields

$$\frac{1}{|Q|}\int_{V_Q(x) > K(Q)} (V_Q(x))^r dx \leqslant C[K(Q)]^r$$

The inequality (14) now follows since $\frac{1}{|Q|}\int_{V_Q(x) \leqslant K(Q)} (V_Q(x))^r dx$ is also bounded by $K(Q)^r$. The proof of Lemma 3 and hence of the theorem is complete.

REFERENCES

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[2] C. Fefferman, E. M. Stein : H^p spaces in several variables, Acta Math. 129 (1972), pp. 137-193.

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