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VECTOR FIELDS

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§0. The purpose of this paper is to present a classification of smooth, globally hipoelliptic complex vector fields on orientable compact surfaces. It has been known for some time that the existence of some globally hypoelliptic but (locally) non-hypoelliptic differential first-order operators imposes restrictions on the topology of the manifold where the operator is defined. For instance, if the Lie derivative L_X associated to a real vector field X is globally hypoelliptic, the Euler characteristic $\chi(M)$ of the compact manifold M is zero ([2]) (L_X acts on sections of $\Lambda^n T^*(M)$, n= dim M, and is the transpose of X relative to the pairing $(\alpha,\beta) \longrightarrow \int_M \alpha_\Lambda \beta$,

$\alpha \in C^\infty(M)$, $\beta \in C^\infty \Lambda^n T^*(M)$). As far as I know all examples of globally hypoelliptic vector fields take a torus for M. In the case of a real vector field X, this is achieved if all orbits of X are dense and they recur with an appropriate speed. Thus an analysis of global behavior of orbits, limit points and other notions related to the theory of dynamical systems seems natural. This is particularly adequate when dealing with compact surfaces, where the Poincare-Bendixon theory is valid for single or several vector fields, [4], [8], (regard a complex vector field as a pair of real vector fields).

A real vector field cannot be hypoelliptic in dimension higher than one but surfaces may admit elliptic complex vector fields yielding

a source of global hypoellipticity of a different kind from that one caused by recurrence of orbits. The study of global hypoellipticity for vector fields on a surface coordinate patch is well understood [6]; to pass from local to global one uses the aforementioned Poincaré-Bendixon techniques.

The theorems we state below imply that the only orientable compact surface that carries globally hypoelliptic vector fields is the torus T^2. There are two types of globally hypoelliptic vector fields. A real vector field with dense orbits and appropriate recurrence is of type I. An example of of a vector of type II is $L = X+iY$ with X and Y linearly independent everywhere. In general, L may be globally hypoelliptic even if X and Y are linearly dependent at certain points but the set where this happens must be "small" in some sense if L is of type II. Vector fields of type I are highly unstable and relatively few: they form a nowhere dense set of the space g of all globally hypoelliptic vector fields with the C° topology. On the other hand the vector fields of type II constitute an open dense subset of g.

Another consequence of the classification is that every null solution of a globally hypoelliptic vector field is constant.

Complete proofs of results presented here will appear in [5].

§1. Let M^2 be a compact connected, orientable, two-dimensional smooth manifold and consider a complex vector field L on M^2.

Def. 1.1: L _is said to be globally hypoelliptic if for every dis_
tribution u ε $\mathcal{D}'(M^2)$ _such that_ Lu ε $C^\infty(M^2)$, _it follows_ that
u ε $C^\infty(M^2)$.

The principal symbol ℓ of L is defined on the cotangent
bundle $T^*(M^2)$ by the identity

$$\ell(d\phi) = L(\phi), \quad u \in C^\infty(M^2;R)$$

Def. 1.2: L _is said to satisfy condition_ (P) _in_ M^2 _if there_
is no complex valued function g _in_ M^2 _such that_ Im(gℓ) _takes_
both positive and negative values on a null bicharacteristic of
Re(gℓ) _where_ g≠0.

We recall that a bicharacteristic of a real function f de
fined on $T^*(M)$ is an integral curve of the Hamilton field H_f.
Since $H_f f = 0$, f is constant along its bicharacteristics; when
the constant is zero the bicharacteristic is said to be null.

We shall assume that

(1.1) L _does not vanish on_ M^2.

Let X and Y be respectively the real and imaginary parts
of L and consider the group of diffeomorphisms G generated by
the one-parameter groups whose infinitesimal generators are X and
Y. A set Ω M^2 _will be said to be L-invariant if_ g Ω ⊆ Ω _for all_
g ε G. _A set will be called L-minimal if it is closed, L-invariant,_
non-empty and contains no such proper subset. The orbits of G

shall be called the orbits of L or L-orbits. The L-orbits are connected submanifolds of M^2 with a natural differentiable structure ([10]). It is known ([4]) that L-minimal sets Ω exist and must be one of the following:

i) a point which is a common zero of X and Y

ii) an L-orbit homeomorphic to S^1

iii) all of M^2.

Case i) cannot occur because of (1.1). If ii) occurs, $\Omega \subset M^2$ is an imbedding and X and Y are tangent to Ω. Since M^2 is orientable, Ω has a tubular neighborhood V homeomorphic to $S^1 \times (0,1)$ which is disconnected by Ω into two components V^+ and V^-. The function f that takes the value 1 at v^+ and 0 at V^- verifies Lf=0 in V and sing supp f = Ω, so if $\phi \in C^\infty(V)$ and is equal to one in a neighborhood of Ω it follows that $u = \phi f \in \mathcal{D}'(M^2) \setminus C^\infty(M^2)$ and Lu $\in C^\infty(M^2)$. This proves that if L is globally hypoelliptic and nonsingular there is a unique L-minimal set equal to M^2; in particular M^2 is the closure of an orbit of L.

Def. 1.3: We say that a globally hypoelliptic vector field L is of type I if M is the closure of a one-dimensional L-orbit. Otherwise we say that L is of type II.

We are ready to state our main results.

Theorem A. Assume that L verifies (1.1). The following statements are equivalent:

i) *L is a globally hypoelliptic vector field of type I*

ii) *There exists a diffeomorphism of* M^2 *onto* R^2/Z^2 *that takes* L *into a non-vanishing multiple of*

(1.2)
$$\frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y}$$

where γ *is an irrational non-Liouville number.*

We recall that an irrational number γ is a non-Liouville number if there exists a positive constant k such that $k\left|\gamma - \frac{n}{m}\right| > \left|m\right|^{-k}$ for all integers n,m. For instance, all algebraic irrational numbers are non-Liouville.

Theorem B. Assume that L *verifies (1.1) and* (P). *The following statements are equivalent:*

   i) L *is a globally hypoelliptic vector field of type II*

   ii) G *acts transitively*

   iii) M^2 *is the only L-minimal set and* $X \wedge Y \in \Lambda^2 T(M^2)$ *is not identically zero.*

Furthermore, if the equivalent conditions i), ii) and iii) are fulfilled, M^2 *must be homeomorphic to* $R^2/Z^2 = T^2$.

If L is globally hypoelliptic in M^2, it follows from general arguments ([1], p. 206) that its transpose L^t is solvable at M^2 in the sense of definition 2.1 of [3]. Then, Theorem 3.2 of [3] shows that L^t must verify (P) which implies that L itself verifies (P). Thus, we have

Corollary 1.1. *Assume that* L *verifies (1.1). Then,* L *is globally hypoelliptic of type II if and only if* L *satisfies* (P) *and* M^2 *is an orbit of* L.

Corollary 1.2. *If* M^2 *admits a nonsingular globally hypoelliptic complex vector field, it must be* homeomorphic to a *torus* T^2.

§2.   One of the tools used in the proof of theorem B above is a geometric characterization of property (P) that   enhances   its two-dimensional character (this two-dimensional behavior is   exploited for instance, in the study of uniqueness in   the   Cauchy problem in [9]). It seems interesting enough on its own right   to state it separately. A local version of a theorem equivalent   to Theorem C below was known in the case of a vector field L   with analytic coefficients ([11], ch. 1). Its proof relied on a theorem of Nagano [7] on the integrability of the distribution associated to the Lie algebra generated by X= ReL and Y= ImL.

When X and Y are just smooth, this distribution is not   integrable, in general. In Theorem C no assumptions are made concerning the existence of foliations. The basic idea is to replace distributions which may not be integrable by the orbits of L.   Then   we observe that if L satisfies (P), the orbits have dimension one or two and the two-dimensional orbits Σ^2 are orientable. Hence, $\Lambda^2 T(\Sigma^2)$ has a global generator θ and we may write $X \wedge Y = \rho\theta$, where ρ is a real valued function. The property that ρ does not change sign in Σ^2 is independent of the choice of θ and we may state:

Theorem C. Let L = X+iY *be a complex vector field without singular points on a paracompact manifold* M. *The following conditions are equivalent:*

i) L satisfies *condition* (P) *in* M.

ii) The *orbits of* L *are orientable, of dimension less than or equal to* 2 *and* X∧Y *does not change sign on the two-dimensional orbits.*

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