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COMPOSITIONS OF PSEUDO-DIFFERENTIAL OPERATORS

by

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The purpose of this report is to describe a certain class of operators arising typically in boundary value problems and in particular in the $\bar{\partial}$-Neumann problem for strongly pseudo-convex domains which represents work done in collaboration with D. H. Phong [5] and [6] . Besides the fact that these operators lead to a better understanding of the "sharp" estimates that can be made for these boundary problems, the operators may also have an interest of their own.

For the clarity of exposition we shall consider four classes of operators, the first three we review briefly and the fourth is the real object of our attention.

First class

The simplest example arises here when we consider a distribution K on $\mathbb{R}^n$, which is homogeneous of degree $-n$, smooth away from the origin, and has vanishing mean value on the unit sphere. Then the principal value convolution operator $f \to Tf = f * K$ is a basic Mihlin-Calderón-Zygmund operator. It can also be realized in the pseudo-differential form

$$(Tf)(x) = \int a(\xi)\hat{f}(\xi)e^{i\xi.x}d\xi \quad ,$$

where a is homogeneous of degree 0 and smooth away from the origin. (A basic example of this is $K(x) = c \dfrac{\partial^2}{\partial x_i \partial x_j}\left(\dfrac{1}{|x|^{n-2}}\right)$, $n \geq 3$, and then $a(\xi) = \dfrac{\xi_i \, \xi_j}{|\xi|^2}$ which is the operator arising when one makes the estimates of maximal gain for the Laplacian). There are many generalizations of those operators in the same spirit. In effect : $a(\xi)$ may be replaced by $a(x,\xi)$ with smooth dependence on x, and the functions $\xi \to a(x,\xi)$ having asymptotic expansions into functions which of infinity behave like homogeneous functions. As is well known operators of this kind satisfy a host of basic estimates : L^p estimates, of which L^2, and "weak type 1,1" are the most fundamental, and also Hölder estimates, etc...

Second class

This class arises when we replace the homogeneity $(x_1,\dots,x_n) \to (\delta x_1,\dots,\delta x_n)$, $\delta > 0$, which dominates the first class, by more general homogeneities, but the operators are still convolution operators with respect to the usual group of $\mathbb{R}^n$.

We shall briefly mention the example of the heat equation $\frac{\partial u}{\partial t} = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$. Using the fundamental solution $h(x,t) = ct^{-n/2} e^{-x^2/2t}$, $t > 0$, $\equiv 0$, $t \leqslant 0$, the basic kernels become $K(x,t) = \frac{\partial^2}{\partial x_i \partial x_j} h(x,t)$, or $K(x,t) = \frac{\partial}{\partial t} h(x,t)$. The operator is then $Tf = f * K$ with the convolution in the full space $\mathbb{R}^n \times \mathbb{R}$ of the x and t variables. The theory for this class of operators (starting with the book of F. Kohn Jr.) can be carried out in wide ranging analogy with that of the previous class.

We now come to the $\bar{\partial}$-Neumann problem. This problem (arising from the $\bar{\partial}$-complex) and its $\bar{\partial}_b$ analogue was studied by Kohn (see e.g. [0]), where a variety of existence and regularity results were obtained. the natural question is what kind of operators -analogous to the first class or second class described above - can be used to represent solutions, make estimates, etc. Two ideas are needed : first : it is best to attack the $\bar{\partial}_b$ problem first before $\bar{\partial}$, because it is "pure". second : we look for exact solutions in the case of the Heisenberg group, because of its many symmetries and the fact that it is equivalent with the boundary of the complex ball :

Third class

Here we consider the Heisenberg group $H^n = \{(z,t) \mid z \in \mathbb{C}^n, t \in \mathbb{R}\}$. The basic vector fields are $Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}$, $\bar{Z}_j$, $1 \leqslant j \leqslant n$, and $T = \frac{\partial}{\partial t}$.

The fundamental operator is $\square_b^{(q)} = -\frac{1}{2} \Sigma(Z_j\bar{Z}_j + \bar{Z}_jZ_j) + i\,\alpha T$ with $\alpha = n - 2q$, which is the Kohn-Laplacian on q-forms. Now in [2] a fundamental solution

$$F = F(z,t) = \frac{C_\alpha}{(|z|^2-it)^{\frac{n-\alpha}{2}} \cdot (|z|^2+it)^{\frac{n+\alpha}{2}}}$$

was found for $\square_b^{(q)}$ (of course, $0 < q < n$),

and the fundamental operators become $Tf = f * K$, with $K = Z_j \bar{Z}_j(F)$, or $T(F)$, and the convolution being with respect the Heisenberg group multiplication. Notice the same kind of homogeneity $(z,t) \to (\partial z, \partial^2 t)$ as the heat equation, but a different convolution. These operators can also be written with pseudo-differential form. More precisely

$$(Tf)(x) = \int a(x,\xi)\hat{f}(\xi)e^{i\xi.x}d\xi \;,$$

with $a(x,\xi) = \hat{K}(L_x^{t-1}(\xi))$, since $y^{-1}.x = L_x(x-y)$.

See [3] where explicit formulas for the symbols are discussed. Notice that $\hat{K}$ is homogeneous of degree 0.

The theory of L^p estimates, Hölder space estimates etc... for these operators can be carried out either in the singular integral form as in [3], or in the pseudo-differential form as in [4] , and these apply to the general class of strictly pseudo-convex boundaries.

Fourth class

The fourth class arises by composition of operators belonging to the first and third class. We describe this more precisely, we let σ,τ be the dual variable to z and t and write $\xi = (\sigma,\tau)$. We consider operators U of the form U_1U_2 where U_1 is a classical operator (of the first class) of order 0 and U_2 is of the third class of order 0) i.e. the symbol of U_2 is $a_2(x,\xi) = a_2(L_x^{t-1}(\xi))$ with a_2 homogeneous in the Heisenberg sense of degree 0 for large ξ and everywhere smooth. It is fundamental that we make the assumption that $a_2(\sigma,0) \equiv 0$ $(\xi = (\sigma,\tau))$.

Theorem 1

(a) U is bounded on L^p , $1 < p < \infty$

(b) U is of weak-type (1,1)

(c) U preserves Λ_α (the usual Hölder spaces)

(d) U preserves Γ_α (the non-isotropic Hölder spaces appropriate for the Heisenberg group).

Remarks : (a) is of course true without the assumption on $a_2(\sigma,0)$, but not the conclusions (b) (c) and (d) for that assumption is essentially necessary.

The proof of this theorem can be found in [5].

$\bar\partial$-Neumann problem

We now discuss the relevance of Theorem 1 to the $\bar\partial$-Neumann problem. The solution of this problem can be reduced to the problem of inversion of a certain first-order pseudo-differential operator on the boundary. This operator, $\Box^+$, is studied in [3] ; on the Heisenberg group it can be exactly defined as follows :

$$\Box^+ = (2\,\Box_b^{(1)} - T^2)^{1/2} + iT$$

Notice $\Box_b^{(1)}$ is non negative Hermitian and commutes with T. Consider $\Box^- = (2\,\Box_b^{(1)} - T^2)^{1/2} - iT$. Then

$$\Box^+\Box^- = 2\Box_b \quad , \text{ so } (\Box^+)^{-1} = \frac{1}{2}\,\Box^-(\,\Box_b^{(1)})^{-1} \; .$$

Using the fundamental solution operator F for $\Box_b^{(1)}$ described above, and the fact that $\frac{1}{2}\square^{-} = U_1 T + \Sigma\, U^{(k)} Z_k + \Sigma\, U'^{(k)} \bar{Z}_k$, we can see that :

$$(\square^{+})^{-1} = U_1 U_2 + \text{smoothing operator},$$

where U_1 = operator of class 1 of order 0, and U_2 = TF, and so has symbol vanishing for $\tau = 0$. So the composition $U_1 U_2$ is the heart of the $\bar{\partial}$-Neumann problem.

Fourth class : second version

It is interesting to find the kernel of this operator, i.e. $(\square^{+})^{-1}$. Again it is a convolution operator with kernel K (on the Heisenberg group). K may be taken to vanish for large x = (z,t), but its main term for small x (in the case of $H^2 = \mathbb{C}^2 \times \mathbb{R} \subset \mathbb{C}^3$) is given by

$$K(x) = \frac{|z|^2}{(2|z|^2+t^2)^2} \qquad \frac{1}{(|z|^2 - it)^2}$$

Thus we are lead to consider kernels of the form :

$$K(x) = E_k(z,t) H_\ell(z,t)$$

where E_k is homogeneous in the usual sense (in the variables z and t) of degree $-k$, and H_ℓ is homogeneous in the Heisenberg group sense of degree $-\ell$.

There are two critical ranges of (k,ℓ). The first range is when $k + \ell = 2n+2$, but $k < 2n$. In that case :

Theorem 2 : Suppose K is as above one $E_k(z,0)H_\ell(z,t)$ has mean value zero on the unit sphere. Then the operator $f \to f * K$, defined as a principal value convolution satisfies the conclusions of Theorem 1.

For a proof see [5].

Remarks : (1) The second critical range is $k + \ell/2 = 2n+1$, $\ell < 2$. Then if $E_k(z,t)H_\ell(0,t)$ has mean value zero the same conclusions holds.

(2) In the "super critical" case, i.e. $K = 2n$, $\ell = 2$, it would seem necessary that $E_k(z,0)$, and $H_\ell(0,t)$ separately have vanishing mean values. Whether this is sufficient for e.g. L^p boundedness is not known.

The Neumann operator

There are operators akin to those that appear in Theorem 2, but in the setting of $H^n \times \mathbb{R}^+$. These operators arise when one takes the asymptotic series for the Neumann operator (the operator which solves the $\bar{\partial}$-Neumann problem) and subjects it to appropriate differentiation.

One is thus lead to operators of the form

$$(Tf)(x,\rho) = \int_{H^n} \int_0^\infty K(y^{-1}.x,\ \rho + \mu) f(y,\mu) dy\, d\mu$$

where K is a function on $\mathbb{H}^n \times \mathbb{R}^+$ of compact support which near the origin is of the form

$$K(x,\rho) = E_k(x,\rho) H_\ell(x,\rho).$$

with $E_k(x,\rho) = E_k(z,t,\rho)$ homogeneous (with respect to $(z,t,\rho) \to (\delta z,\delta t,\delta\rho)$) of degree $-k$, while $H_\ell(x,\rho) = H_\ell(z,t,\rho)$ is homogeneous (with respect to $(z,t,\rho) \to (\delta z,\delta^2 t,\delta^2\rho)$) of degree $-\ell$. Both are assumed to be smooth away from the origin.

There are two critical ranges, the first is $k + \ell = 2n+4$, $k < 2n$; the second is $k + \ell/2 = 2n+2$, $\ell < 4$. The "super critical" point is $k = 2n$, $\ell = 4$.

Theorem 3 : (a) In the two critical ranges, the operator T is bounded on $L^p(\mathbb{H}^n \times \mathbb{R}^+)$, even it K is replaced by $|K|$, $(1 < p < \infty)$.

(b) In the case $k = 2n$, $\ell = 4$, the operator is bounded on L^p if $E_k(z,0,0)$ has vanishing mean value on the sphere.

The proof will appear in [6]. It is based in turn on the following result obtained by D. Geller and the author [1].

Consider the following singular homogeneous distribution on $\mathbb{H}^n$. $K_0(z)$ is a homogeneous distribution on $\mathbb{C}^n$ of degree $-2n$, which is smooth away from the origin. $\delta(t)$ is the delta dunction (at the origin) in the t-variable $K(z,t) = K_0(z)\delta(t)$.

Theorem 4 : The operator $f \to f * K$ is bounded on $L^p(\mathbb{H}^n)$ to itself, $1 < p < \infty$.

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