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# **ÉQUATIONS AUX DÉRIVÉES PARTIELLES**

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# Canonical commutation relations and interacting Fock spaces

Zied Ammari

## Abstract

We introduce by means of reproducing kernel theory and decomposition in orthogonal polynomials canonical correspondences between an interacting Fock space a reproducing kernel Hilbert space and a square integrable functions space w.r.t. a cylindrical measure. Using this correspondences we investigate the structure of the infinite dimensional canonical commutation relations. In particular we construct test functions spaces, distributions spaces and a quantization map which generalized the work of Krée-Rączka [KR] and Janas-Rudol [JR1]-[JR3].

## 1. Canonical commutation relations

Canonical commutation relations have their roots in the basic concepts of quantum mechanics and quantum field theory. The abstract formulation of those theories consists of considering in analogy with the classical mechanics formulation the moments coordinates  $p_1, \dots, p_n, \dots$  and the positions coordinates  $q_1, \dots, q_n, \dots$  of a quantum system as a self-adjoint operators on a Hilbert space  $\mathcal{H}$ , satisfying the *Heisenberg commutation relations*:

$$(1.1) \quad [q_k, q_l] = [p_k, p_l] = 0,$$

$$(1.2) \quad [q_k, p_l] = i\delta_{k,l}I.$$

It can be elementary noticed that Equ. (1.2) implies that both  $q_k$  and  $p_l$  are unbounded self-adjoint operators. In order to avoid several complications arising from domain problem one considers the following operators:

$$U(a) := \prod_k e^{ia_k p_k}, \text{ and } V(b) := \prod_l e^{ib_l q_l},$$

where  $a = (a_1, \dots, a_n, 0, \dots)$  and  $b = (b_1, \dots, b_m, 0, \dots)$  in  $\mathbb{R}^{\mathbb{N}}$ . Clearly  $U(a)$  and  $V(b)$  are unitary bounded operators and using a formal calculus one obtain from Equ. (1.1)-(1.2) that  $U$  and  $V$  satisfy the relation:

$$(1.3) \quad U(a)V(b) = e^{i\sum_k a_k b_k} V(b)U(a).$$

The above identity is called the *restricted Weyl commutation relation*.

It is von Neumann who suggested to combine  $U$  and  $V$  in one operator namely the so-called Weyl operator:

$$(1.4) \quad W(a, b) = e^{-\frac{i}{2} \sum_k a_k b_k} U(a) V(b)$$

We get from Equ. (1.3):

$$\begin{aligned} (i) \quad & W(a, b)W(c, d) = e^{\frac{i}{2} \sigma[(a,b),(c,d)]} W(a + c, b + d). \\ (ii) \quad & W(a, b)^* = W(-a, -b). \end{aligned}$$

where  $\sigma[(a, b), (c, d)] = \sum_k a_k d_k - b_k c_k$  is the canonical symplectic form over the space  $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ .

We say that a family of unitary operators satisfies the *Weyl commutation relations* if (i) – (ii) hold. If in addition

$$(iii) \quad \mathbb{R} \ni t \mapsto W(te_k, te_l) \text{ is continuous, with } e_k = (0, \dots, 1, 0, \dots).$$

holds then we call it a *regular Weyl commutation relations*. Some care is needed when we pass from a form of commutation relations to another. For example the regular Weyl form of commutation relations leads to the Heisenberg form however the opposite is in general not valid.

In conclusion we have different forms of the canonical commutation relations. The Weyl form is widely considered in the literature for the simplification that carries and we will do the same. An elegant way to formulate Weyl canonical commutation relation is briefly described in the following subsection.

## 1.1. The CCR algebras

Let  $(H, \sigma)$  be a symplectic space (i.e: a linear space endowed with an anti-symmetric non-degenerate form). A *CCR algebra*  $\mathfrak{U}$  is a  $\mathcal{C}^*$ -algebra generated by a family of elements  $\{W(z), z \in H\}$  satisfying:

$$\begin{aligned} (1) \quad & W(z_1)W(z_2) = e^{\frac{i}{2} \sigma(z_1, z_2)} W(z_1 + z_2). \\ (2) \quad & W(z)^* = W(-z). \end{aligned}$$

One can remark that the element  $W(0)$  is the unit of the algebra  $\mathfrak{U}$ . The following fact is due to Slawny [Sl].

**Theorem 1.1.** *For any symplectic space  $(H, \sigma)$  there exists a CCR algebra over  $(H, \sigma)$  unique modulo  $*$ -isomorphisms.*

In the sequel  $\mathcal{A}(H, \sigma)$  will denote the unique class of equivalence of CCR algebras over  $(H, \sigma)$  relative to the equivalence relation induced by  $*$ -isomorphisms. Let us recall some jargon used later.

- A representation  $(\pi, \mathcal{H})$  of  $\mathcal{A}(H, \sigma)$  is a  $*$ -morphism  $\pi$  and a Hilbert space  $\mathcal{H}$  such that  $\pi$  maps  $\mathcal{A}(H, \sigma)$  into  $B(\mathcal{H})$ .

- A representation  $(\pi, \mathcal{H})$  of  $\mathcal{A}(H, \sigma)$  is said regular if the map  $\mathbb{R} \ni t \mapsto \pi(W(tz))$  is continuous for the weak topology in  $\mathcal{B}(\mathcal{H})$ .

- A representation  $(\pi, \mathcal{H})$  is said irreducible if the only invariant subspaces w.r.t.  $\pi(\mathcal{A}(H, \sigma))$  are the trivial ones.

- A representation  $(\pi, \mathcal{H})$  of  $\mathfrak{U}$  is called cyclic if and addition it admits a vector  $\Omega$  such that  $\pi(\mathfrak{U})\Omega$  generates the space  $\mathcal{H}$ .

- Two representations  $(\pi_1, \mathcal{H}_1)$  and  $(\pi_2, \mathcal{H}_2)$  are called unitary equivalent if there exists an unitary transformation  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  such that

$$U\pi_1(A)U^{-1} = \pi_2(A).$$

## 1.2. Realization of CCR's in finite dimension

We consider as a symplectic space  $\mathbb{R}^{2d}$  equipped with its canonical symplectic form.

A- The Schrödinger representation:

This is the main arena for the study of Schrödinger equation. The Hilbert space  $\mathcal{H}$  in this representation is  $L^2(\mathbb{R}^d, dx)$ , and  $q_j = x_j, p_j = -i\partial_{x_j}$ . Starting from  $q_j, p_j$  a large class of observables can be constructed using different procedures for instance using Weyl, Kohn-Nirenberg, or Anti-Wick quantization. However two operators are of particular interest in QFT. Namely the creation and annihilation operators:

$$(1.5) \quad z_j^* = \frac{1}{\sqrt{2}}(q_j - ip_j); \quad z_j = \frac{1}{\sqrt{2}}(q_j + ip_j).$$

Hence if we introduce the field operator  $\phi(a) = \sum_k a_k z_k^* + \bar{a}_k z_k$  then the Weyl operator is given by:

$$W(a) = e^{i\phi(a)},$$

leading to a representation of the CCR algebra over  $\mathbb{R}^{2d}$ .

There exists an algebraic structure related to the creation and annihilation operators. In fact  $z_j$  for all  $j$  has eigenvector associated to the 0 eigenvalue given by the normalized gaussian  $H_0(x) = \frac{1}{\pi^{d/4}} e^{-|x|^2/2}$  and called usually the *vacuum*. Moreover using Hermite functions

$$H_\alpha(x) := \frac{(-1)^{|\alpha|}}{\pi^{d/4}} \frac{1}{\sqrt{2^{|\alpha|} \alpha!}} e^{|x|^2/2} \partial^\alpha e^{-|x|^2}, \quad \alpha \in \mathbb{N}^d,$$

we get

$$z_j^* H_\alpha = \sqrt{\alpha_j + 1} H_{\alpha+1_j}, \quad z_j H_\alpha = \sqrt{\alpha_j} H_{\alpha-1_j}.$$

It follows from that fact that Hermite polynomials are the orthogonal polynomials associated to the gaussian measure that one can decompose the  $L^2(\mathbb{R}^d, dx)$  as a direct sum of the orthogonal subspaces  $\text{Vect}\{z_\alpha^* H_0, |\alpha|_1 = n\}$ , where  $z_\alpha^* = z_1^{*\alpha_1} \cdots z_d^{*\alpha_d}$ . We can establish a correspondence between  $\text{Vect}\{z_\alpha^* H_0, |\alpha|_1 = n\}$  and  $\otimes_s^{|\alpha|_1} \mathbb{C}^d$  where the subscript "s" stands for the symmetric tensor. Consider  $\{e_j\}$  a basis of  $\mathbb{C}^d$  then the following map

$$z_\alpha^* H_0 \leftrightarrow e_1^{\otimes \alpha_1} \otimes_s \cdots \otimes_s e_d^{\otimes \alpha_d} \in \otimes_s^{|\alpha|_1} \mathbb{C}^d.$$

extends to an unitary transform between  $L^2(\mathbb{R}^d, dx)$  and  $\bigoplus_{n=0}^{\infty} \bigotimes_s^n \mathbb{C}^d$  which is the symmetric Fock space over  $\mathbb{C}^d$ .

B- The Fock representation:

The symmetric Fock space is denoted by  $\Gamma_s(\mathbb{C}^d) = \bigoplus_{n=0}^{\infty} \bigotimes_s^n \mathbb{C}^d$ . In this representation the annihilation operator is given by:

$$A(f)f_1 \otimes_s \cdots \otimes_s f_n = \sqrt{n} \frac{1}{n!} \sum_{\sigma \in S_n} (f, f_{\sigma_1}) f_{\sigma_2} \otimes \cdots \otimes f_{\sigma_n},$$

and the creation operator by:

$$A^*(f)f_1 \otimes_s \cdots \otimes_s f_n = \sqrt{n+1} f \otimes_s f_1 \cdots \otimes_s f_n.$$

Furthermore  $A^*(f)$  and  $A(g)$  satisfy the relation

$$(1.6) \quad [A(f), A^*(g)] = (f, g)I.$$

We have the field operator  $\Phi(f) = A^*(f) + A(f)$  and the Weyl operator  $W(f) = e^{i\Phi(f)}$  satisfying the Weyl commutation relations. This representation is particularly preferred by physicists and one can implement quantization procedures in the Fock space. For instance the quantized Weyl operator is defined by:

$$(1.7) \quad B_{\varphi}^w := \int_{\mathbb{R}^{2d}} \tilde{\varphi}(x, y) W(x + iy) \frac{dx dy}{(2\pi)^{2d}}.$$

where  $\tilde{\varphi}(x, y) = \int_{\mathbb{R}^{2d}} e^{-i(x \oplus y, \xi \oplus \eta)} \varphi(\eta, \xi) d\xi d\eta$ . In a similar way one can define also the left/right quantization however we mention that the mainly used observables in QFT are the so-called Wick polynomials for a reason which will be clear later.

C- The Segal-Bargmann representation:

It is realized in the Segal-Bargmann space (see for instance [B1]):

$$\mathcal{HL}^2(\mathbb{C}^d, d\mu_g) = \{F : \mathbb{C}^d \rightarrow \mathbb{C}, \text{ analytic} : \int_{\mathbb{C}^d} |F(z)|^2 d\mu_g(z) < \infty\},$$

where  $d\mu_g(z) = \pi^{-d} e^{-|z|^2} dz$ . The creation and annihilation operators are respectively  $A_j^* = z_j, A_j = \partial_{z_j}$ . The vacuum is the vector  $\Omega_0 = 1$  annihilated by  $A_j$ . This representation is the most rich in structures. The Segal-Bargmann space possess the properties of a reproducing kernel Hilbert space and moreover quantization procedures can be naturally constructed. For a symbol  $\varphi$  one can corresponds:

- The Toeplitz operators:

$$(1.8) \quad T_{\varphi}^{AW} F(z) = \int_{\mathbb{C}^d} \varphi(\bar{w}, w) e^{z\bar{w}} F(w) d\mu_g(w).$$

- The Wick operators:

$$(1.9) \quad T_{\varphi}^W F(z) = \int_{\mathbb{C}^d} \varphi(\bar{w}, z) e^{z\bar{w}} F(w) d\mu_g(w).$$

- The Quantized Weyl operators:

$$(1.10) \quad T_\varphi^w F(z) = \int_{\mathbb{C}^d} \varphi(\bar{w}, \frac{w+z}{2}) e^{z\bar{w}} F(w) d\mu_g(w).$$

The Segal-Bargmann space in finite dimension was carefully studied by Bargmann in [B1] and [B2] for test functions and distributions spaces.

All of the above three CCR's representations are regular. In fact we are only concerned by regular CCR representations although there exists non regular representation of physical interest [Re].

In the finite dimension case we have the following fundamental fact due to Stone and von Neumann [St], [vN].

**Theorem 1.2.** *Let  $(H, \sigma)$  be a finite dimension symplectic space. Then any regular irreducible representation of  $\mathcal{A}(H, \sigma)$  is unitary equivalent to the Schrödinger representation.*

The situation is quite different in the infinite dimensional case since there exists an infinite number of non equivalent irreducible regular representations of the CCR algebra  $\mathcal{A}(H, \sigma)$ .

### 1.3. Extension to the infinite dimension

Before discussing how to extend the above representations to the infinite dimensional case one need to introduce a complex structure on the infinite dimensional symplectic space  $(H, \sigma)$ . A complex structure is anti-involution  $J : H \rightarrow H$ ,  $J^2 = -I$  compatible with the symplectic form  $\sigma$ , i.e:

$$(1.11) \quad \sigma(Ju, Jv) = \sigma(u, v)$$

$$(1.12) \quad \sigma(u, Ju) > 0 \text{ for all } u \neq 0.$$

$H$  becomes a complex pre-Hilbert space when equipped with the inner product  $(u, v) := i\sigma(u, v) + \sigma(u, Jv)$  and we will denote by  $H_{\mathbb{C}}$  its completion.

The extension of the Schrödinger representation fails since we lack an extension of the Lebesgue measure to infinite dimension spaces. However the Fock representation is well adapted to the infinite dimension and it is extended easily by replacing in (B) the space  $\mathbb{C}^d$  by the Hilbert space  $H_{\mathbb{C}}$ .

The Segal-Bargmann representation extends also to the context of infinite number of degrees of freedom. This can be achieved using two approaches. Namely the inductive approach and the theoretical measure approach. For simplicity here we only consider the inductive approach formulated in a non invariant way by considering the space  $\ell_{\mathbb{C}}^2(\mathbb{N})$ . This means that we restrict our selves to separable Hilbert spaces and we fix a basis.

Let  $\ell_{\mathbb{C}}^2(\mathbb{N})$  be the space of square summable complex sequences. We denote by  $T_L : \mathbb{C}^d \rightarrow \ell_{\mathbb{C}}^2(\mathbb{N})$  the injection associated to a subspace  $L$  of finite dimension  $d \in \mathbb{N}$  of  $\ell_{\mathbb{C}}^2(\mathbb{N})$ . The Segal-Bargmann space can be generalized as below:

$$\mathcal{HL}^2(\ell_{\mathbb{C}}^2(\mathbb{N})) := \{F : \ell_{\mathbb{C}}^2(\mathbb{N}) \mapsto \mathbb{C} \text{ analytic} : \sup_L \int_L |F \circ T_L(z)|^2 d\mu_g(z) < \infty\}.$$

The creation and annihilation operators are defined as:

$$A^*(u)F(z) = (u, z)_{\ell^2(\mathbb{N})} F(z);$$

$$A(u)F(z) = \lim_{\lambda \rightarrow 0} \frac{F(z + \lambda u) - F(z)}{\lambda}.$$

The Fock representation is unitary equivalent to the Segal-Bargmann representation. The correspondence is given by the Segal isomorphism:

$$I_S f_1 \otimes_s \cdots \otimes_s f_n = \frac{2^{n/2}}{\sqrt{n!}} : \prod_{j=1}^n (f_j, z) :$$

where  $: :$  stands for the Wick polynomials (i.e: the orthogonal projection on the subspace generated by Hermite polynomials of degree  $n$  but orthogonal to Hermite polynomials of degree  $n - 1$ ).

## 2. Classification of CCR's representations

The classification of regular representations of the CCR algebra which have induced sub-representations leading to a restricted Weyl commutation relations can be considered using two different approaches as mentioned before, the inductive approach due to Segal and the theoretical measure approach due to Araki and Gelfand.

In this talk we will only consider the inductive approach. Let  $H$  be a Hilbert space and consider an exhaustion  $\mathcal{K}$ :

- (i) Every  $K \in \mathcal{K}$  is finite dimension subspace of  $H$ .
- (ii)  $\forall K, L \in \mathcal{K}, \exists M \in \mathcal{K} : K \subset M, L \subset M$ .
- (iii)  $\overline{\cup_{K \in \mathcal{K}} K} = H$ .

Let  $\tau_\sigma(K, H)$  be the Borel  $\sigma$ -algebra over  $H$  generated by the family of sets  $C(K, A) = \{x \in H : P_K x \in A, A \in \mathcal{B}(K)\}$  where  $\mathcal{B}(K)$  is the Borel  $\sigma$ -algebra of  $K$ . We denote by  $\tau(\mathcal{K}, H)$  the union of all  $\tau_\sigma(K, H)$ ,  $K \in \mathcal{K}$  which is a boolean algebra.

A cylindrical measure is a positive map on  $\tau(\mathcal{K}, H)$  such that the restriction  $\mu_K$  on  $\tau_\sigma(K, H)$  is a  $\sigma$ -additive Radon probability measure (i.e:  $\mu(H) = 1$ ) and satisfying a compatibility condition:

$$\mu_K|_{\tau_\sigma(L, H)} = \mu_L \text{ when } L \subset K.$$

Cylindrical measures can also be formulated using the theory of martingales.

A tame function is a measurable function  $F : H \rightarrow \mathbb{C}$  such that there exists  $K \in \mathcal{K}$ ,  $F(P_K w) = F(w)$ . In such case we call  $F$  a  $K$ -tame function. The family of spaces  $L^p(H, \tau_\sigma(K), \mu_K)$ ,  $K \in \mathcal{K}$  forms an inductive system and we will denote by  $L_\infty^p(H, \mu)$  the completion of  $\cup_{K \in \mathcal{K}} L^p(H, \tau_\sigma(K), \mu_K)$  with respect to the natural norm carried by those of  $L^p(H, \tau_\sigma(K), \mu_K)$ .

We recall that a measure in finite dimension vector space is said quasi-invariant if the translation preserve null sets. This is equivalent to the fact that the measure is absolutely continuous w.r.t. to the Lebesgue measure. The notion of quasi-equivalence need to be modified in the case of infinite dimension topological spaces

by the almost quasi-invariance notion. We say that a cylindrical measure  $\mu$  is almost quasi-invariant if there exists a function  $D^b \in L^1_\infty(H, \mu), b \in H$  such that for every  $K \in \mathcal{K}$ ,

$$\int_H F(x) d\mu_K^b(x) = \int_H D^b(x) F(x) d\mu_K(x), \text{ for all } F \text{ } K\text{-tame function.}$$

where  $\mu_K^b$  is the translation by  $b$  of the measure  $\mu_K$ .

The following theorem gives a classification of CCR representations and it is due to Segal [BSZ].

**Theorem 2.1.** *For every cyclic regular representation  $(\pi, \mathcal{H}, \Omega)$  of the CCR algebra  $\mathcal{A}(H, \text{Im}(\cdot, \cdot))$  over a Hilbert space  $H$  there exists an almost quasi-invariant cylindrical measure  $\mu$  on  $H$  such that  $(\pi, \mathcal{H})$  is unitary equivalent to the representation over the space  $L^2_\infty(H, \mu)$  defined by:*

$$\begin{aligned} U(a)F(x) &= e^{i(a,x)} F(x); \\ V(b)F(x) &= \frac{d\mu^{b^{1/2}}}{d\mu}(x) F(x+b); \end{aligned}$$

where  $\frac{d\mu^b}{d\mu}(x)$  stands for the inductive limit of the Radon-Nikodym derivatives of the translated measure  $\mu^b$  w.r.t.  $\mu$ .

We briefly describe the questions considered in the work [Am].

**A)- Realization of CCR's:**

Using an extension of the correspondence in Accardi-Bożejko [AB], Accardi-Nahni [AN], Asai [As], Asai-Kubo-Kuo [AKK], we provide provide a realization of a large class of CCR's representation on a reproducing kernel Hilbert space and an interacting Fock space.

**B)- Construction of test functions/distributions spaces:**

Using the work of Martens [M] in inductive/projective limits of spaces of analytic functions (an alternative to the work in non-gaussian analysis of Albeverio-al[ADKS] and Kondratiev-al [KSWY]).

**C)- Quantization procedures:**

Generalization of the work of Krée-Rączka [KR] and Janas-Rudol [JR2].

Our starting point for all the following is an analytic cyclic representation  $(\pi, \mathcal{H}, \Omega)$  of the CCR algebra over the space  $\ell^2_{\mathbb{C}}(\mathbb{N}) \oplus \ell^2_{\mathbb{C}}(\mathbb{N})$  carried by the induced sub-representations  $(U, V)$  satisfying the restricted Weyl commutation relations and such that  $\Omega$  is cyclic vector for the family of operators  $U(a), a \in \ell^2_{\mathbb{C}}(\mathbb{N})$ . Therefore by Thm. 2.1 the representation  $(\pi, \mathcal{H}, \Omega)$  is unitary equivalent to the realization of CCR on the Hilbert space  $L^2_\infty(\ell^2_{\mathbb{C}}(\mathbb{N}), \mu)$  with a given almost quasi-invariant cylindrical measure  $\mu$ .

### 3. Polynomials of infinitely many variables

In a concern of simplicity we consider polynomials on the space  $\ell^2(\mathbb{N})$ . The general case can be treated using tensor analysis.



Let us introduce some notations. Let  $\mathbb{D}$  be a countable set (here  $\mathbb{D} = \mathbb{N}$  although we can consider countable sets without order using Köthe sequences spaces). We denote by  $\omega(\mathbb{D})$  the space of real sequences and by  $\varphi(\mathbb{D})$  the space of finite sequences,

$$\ell^p(\mathbb{D}) = \{a \in \omega(\mathbb{D}) : \sum_{j \in \mathbb{D}} |a(j)|^p < \infty\}$$

Let  $M(\mathbb{D}) = \{a \in \varphi(\mathbb{D}) : a(j) \in \mathbb{N}\}$ , and  $M_n(\mathbb{D}) = \{a \in M(\mathbb{D}) : |a|_1 \leq n\}$ . We set  $x^\alpha = \prod_{j \in \mathbb{D}} x(j)^{\alpha(j)}$ .

Let  $T = \{\tau \in \omega(\mathbb{D}) : \tau(j) \geq 1\}$  be a weight set. We define the following scaled Hilbert spaces:

$$(3.1) \quad H_\pm[\tau] := \{a \in \ell(\mathbb{D}) : \sum_{j \in \mathbb{D}} \tau(j)^{\pm 1} a(j)^2 < \infty\}.$$

The family  $(H_\pm[\tau])_{\tau \in T}$  is respectively a projective/inductive system. Moreover we have the following nuclear triplet:

$$(3.2) \quad \varphi(\mathbb{D}) = \lim_{\tau \in T} \text{pr} H_+[\tau] \hookrightarrow \ell_2(\mathbb{D}) \hookrightarrow \lim_{\tau \in T} \text{ind} H_-[\tau] = \omega(\mathbb{D})$$

In the infinite dimension case we can distinguish three type of polynomials of infinitely many variables in the space  $L_\infty^2(\ell^2(\mathbb{D}), \mu)$ .

- Cylindrical polynomials:

$$\mathcal{P}_{\text{cyl}}^n := \{P : \ell^2(\mathbb{D}) \rightarrow \mathbb{R}; P(x) = \sum_{\beta \in M_n(\mathbb{D})} a_\beta x^\beta, a_\beta \in \varphi(\mathbb{D})\}.$$

- Continuous polynomials:

$$\mathcal{P}^n := \{P : \ell^2(\mathbb{D}) \rightarrow \mathbb{R}; P(x) = \sum_{\beta \in M_n(\mathbb{D})} a_\beta x^\beta; a_\beta \in \ell^2(\mathbb{D})\}.$$

- Measurable polynomials: Consider

$$\mathcal{V}_\mu^n := \overline{\mathcal{P}^n} \ominus \mathcal{P}^{n-1},$$

where  $\overline{\mathcal{P}^n}$  is the closure of  $\mathcal{P}^n$  in  $L_\infty^2(\mathbb{D}, \mu)$ . The space of measurable polynomials is defined by

$$\mathcal{P}_\mu^n = \bigoplus_{k=0}^n \mathcal{V}_\mu^k.$$

Therefore we obtain

$$\mathcal{P}_{\text{cyl}}^n \subset \mathcal{P}^n \subset \mathcal{P}_\mu^n.$$

Let  $\mathbb{P}_n$  be the orthogonal projection on  $\mathcal{V}_\mu^n$ . We denote by  $:x^\alpha := \mathbb{P}x^\alpha$ .

**Lemma 3.1.** (i) The family  $(:x^\alpha :)_{\alpha \in M_n(\mathbb{D})}$  spans  $\mathcal{V}_\mu^n$ .

(ii) The family  $(:x^\alpha :)_{\alpha \in M_n(\mathbb{D})}$  is linearly independent.

The classical three terms relations hold in this context.

**Theorem 3.2.** *There exists three operators  $A_{n,i} : \mathcal{V}_\mu^n \rightarrow \mathcal{V}_\mu^{n+1}$ ,  $B_{n,i} : \mathcal{V}_\mu^n \rightarrow \mathcal{V}_\mu^n$  and  $C_{n,i} : \mathcal{V}_\mu^n \rightarrow \mathcal{V}_\mu^{n-1}$  satisfying:*

$$(3.3) \quad (e_i, x)\mathbb{P}_n = \mathbb{P}_{n+1}A_{n,i} + \mathbb{P}_nB_{n,i} + \mathbb{P}_{n-1}C_{n,i}$$

where  $\mathbb{P}_{-1} = 0$  and  $C_{-1,i} = 0$ .

The proof is elementary since the relation follows from remarking that  $(e_i, x)\mathbb{P}_n\psi$  has vanishing components w.r.t.  $\mathcal{V}_\mu^k$  if  $k \neq n-1, n, n+1$ . Moreover  $A_{n,i} = \mathbb{P}_{n+1}(e_i, x)\mathbb{P}_n$ ,  $B_{n,i} = \mathbb{P}_n(e_i, x)\mathbb{P}_n = B_{n,i}^*$ , and  $C_{n,i} = \mathbb{P}_{n-1}(e_i, x)\mathbb{P}_n = A_{n-1,i}^*$ .

The operators  $A_{n,i}$ ,  $B_{n,i}$ , and  $C_{n,i}$  are not arbitrary. They satisfy some additional commutation relations specified in the following theorem.

**Theorem 3.3.** *The families of operators  $A_{n,i}, B_{n,i}, C_{n,i}$ ,  $n, i \in \mathbb{N}, i \in \mathbb{N}$  introduced in the above theorem satisfy the following relations:*

$$(3.4) \quad A_{k,i}A_{k+1,j} = A_{k,j}A_{k+1,i};$$

$$(3.5) \quad A_{k,i}B_{k,j} + B_{k+1,i}A_{k,j} = B_{k+1,j}A_{k,i} + A_{k,j}B_{k,i},$$

$$(3.6) \quad C_{k,i}A_{k-1,j} + B_{k,j}B_{k,i} + C_{k+1,i}A_{k,j} = \\ A_{k-1,j}C_{k,i} + B_{k,i}B_{k,j} + C_{k+1,j}A_{k,i},$$

for  $i \neq j$  and where  $A_{-1,i} = 0$ .

**Lemma 3.4.** *We have:*

- (i)  $L_\infty^2(\ell^2(\mathbb{D}), \mu) = \bigoplus_{n=0}^\infty \mathcal{V}_\mu^n$ ,
- (ii)  $\overline{\sum_{j=1}^n \text{Ran}(A_{n,j})} = \mathcal{V}_\mu^{n+1}$ ,
- (iii)  $A_{n,j} : \mathcal{V}_\mu^n \rightarrow \mathcal{V}_\mu^{n+1}$  is injective.

Results and proofs in this section are quite similar to those of the well known situation of polynomials of several variables [DX].

## 4. Interacting Fock spaces

We recall the definition of interacting Fock space as in [AN].

**Definition 4.1.** Let  $\tilde{\mathcal{H}}_0$  be a pre-Hilbert space. An interacting Fock space is a Hilbert space

$$\mathcal{H} = \bigoplus_{n=0}^\infty \mathcal{H}_n, \quad \text{where } \mathcal{H}_0 := \mathbb{C}\Phi,$$

such that there exists a family of densely defined operators  $a^+(v)$  on  $\mathcal{H}$  for all  $v \in \tilde{\mathcal{H}}_0$  and furthermore:

- (i) The map  $\tilde{\mathcal{H}}_0 \ni v \mapsto a^+(v)$  is linear.
- (ii) The set

$$\mathcal{N}_n := \{a^+(v_1) \cdots a^+(v_n)\Phi, v_i \in \tilde{\mathcal{H}}_0\Phi\} \subset \mathcal{D}(a^+(v)), \text{ for all } n \in \mathbb{N}; \\ \text{and } \mathcal{N} := \text{Vect}\{\mathcal{N}_n, n \in \mathbb{N}\} \text{ is dense in } \mathcal{H}.$$

- (iii)  $a^+(v)$  has densely defined adjoint operator  $a^-(v)$  for all  $v \in \tilde{\mathcal{H}}_0$ .

**Proposition 4.2.** *The space  $L_\infty^2(\ell^2(\mathbb{N}), \mu)$  has an interacting Fock space structure given by:*

$$L_\infty^2(\ell^2(\mathbb{N}), \mu) = \bigoplus_{n=0}^{\infty} \mathcal{V}_\mu^n, \quad \tilde{\mathcal{H}}_0 = \varphi(\mathbb{N}), \quad \mathcal{N}_n = \mathcal{P}_n^{\text{cyl}}.$$

and the operators

$$a^+(v) := \bigoplus_{n=0}^{\infty} A_{n,v} \text{ where } A_{n,v} = \sum_j v_j A_{n,j}, \text{ and } v = \sum_j v_j e_j \in \varphi(\mathbb{D});$$

$$a^-(v) := \bigoplus_{n=0}^{\infty} C_{n,v} \text{ where } C_{n,v} = \sum_j v_j C_{n,j}.$$

Moreover we have a family of commuting operators given by:

$$X_j = a^+(e_j) + a^0(e_j) + a^-(e_j),$$

where  $a^0(v) := \bigoplus_{n=0}^{\infty} B_{n,v}$  and  $B_{n,v} = \sum_j v_j B_{n,j}$ .

**Theorem 4.3.** *Let  $\mathcal{H}$  be a Hilbert space and  $\mathcal{H}_0$  a dense subspace in  $\mathcal{H}$ . Consider*

$$\Gamma(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{W}_n,$$

where  $\mathcal{W}_n$  is the completion of the symmetric algebraic tensor product  $\bigotimes_{\text{Alg}}^{n,s} \mathcal{H}_0$  w.r.t a given family of norms  $(\cdot, \cdot)_n$ . Assume that  $\mathcal{H}$  is endowed with a family of bounded operators

$$a_n^+(j) : \mathcal{W}_n \rightarrow \mathcal{W}_{n+1}, \quad a_n^-(j) : \mathcal{W}_n \rightarrow \mathcal{W}_{n-1}, \quad a_n^0(j) : \mathcal{W}_n \rightarrow \mathcal{W}_n,$$

satisfying the commutation relations (3.4)-(3.6) and defining a commuting family of operators  $a_n^+(j) + a_n^0(j) + a_n^-(j)$ .

(i) *The space  $\Gamma(\mathcal{H})$  has an interacting Fock space structure.*

(ii) *We have a canonical isomorphism*

$$T : \quad L_\infty^2(\ell^2(\mathbb{N}), \mu) \rightarrow \Gamma_\mu(\ell_{\mathbb{C}}^2(\mathbb{N}))$$

$$\mathbb{P} \prod_{i=1}^n (v_i, x) 1 \mapsto \bigotimes_{i=1}^n v_i,$$

where  $\Gamma_\mu(\ell_{\mathbb{C}}^2(\mathbb{N}))$  is constructed w.r.t. to the norms on  $\mathcal{V}_\mu^n$ . Hence  $T$  transforms  $A_{n,i}, B_{n,i}, C_{n,i}$  respectively into  $a_n^+(i), a_n^0(i), a_n^-(i)$  satisfying (3.4)-(3.6).

In gaussian analysis the Segal-Bargmann space can be defined using reproducing kernel Hilbert spaces (RKHS). We use a similar construction to generalize the Segal-Bargmann space and establish a canonical correspondence with the interacting Fock space.

**Theorem 4.4.** *We have the canonical isomorphism:*

$$\mathcal{L} : \Gamma_\mu(\ell_{\mathbb{C}}^2(\mathbb{N})) \rightarrow F(\ell_{\mathbb{C}}^2(\mathbb{N}), K_\mu)$$

$$\Psi \mapsto \mathcal{L}(\Psi)(z) = \left( \Psi, \frac{z^{\otimes n}}{\sqrt{n!}} \right),$$

where  $F(\ell_{\mathbb{C}}^2(\mathbb{N}), K_\mu)$  is the RKHS defined by the kernel  $K_\mu(x, y) := \sum_{n=0}^{\infty} \frac{1}{n!} (x^{\otimes n}, y^{\otimes n})$  and having an interacting Fock space structure.

To summarize we have the diagram of isomorphisms:

$$\begin{array}{ccc} \Gamma_\mu(\ell_{\mathbb{C}}^2(\mathbb{N})) & \rightarrow & L_\infty^2(\ell^2(\mathbb{N}), \mu) \\ \downarrow & & \downarrow \\ F(\ell_{\mathbb{C}}^2(\mathbb{N}), K_\mu) & \rightarrow & \mathcal{H}L^2(\ell_{\mathbb{C}}^2(\mathbb{N}), \tilde{\mu}) \end{array}$$

where  $F(\ell_{\mathbb{C}}^2(\mathbb{N}), K_\mu)$  is the functional RKHS and the space  $\mathcal{H}L^2(\ell_{\mathbb{C}}^2(\mathbb{N}), \tilde{\mu})$  is the a generalized Segal-Bargmann space defined by

$$\{F : \ell_{\mathbb{C}}^2(\mathbb{N}) \mapsto \mathbb{C}; \text{ analytic} : \sup_L \int_L |F \circ T_L(z)|^2 d\tilde{\mu} < \infty\},$$

where the cylindrical measure  $\tilde{\mu}$  obtained form the complex (Hamburger) moment problem.

## 5. Test functions/Distributions and quantization

The space  $F(\ell_{\mathbb{C}}^2(\mathbb{N}), K_\mu)$  is rich in structure and we will take advantage form this fact to construct the spaces of test functions and distributions using technics elaborated in the work of Martens [M].

For  $a \in \omega^+(\mathbb{N})$  positive sequences we define

$$K_\mu^a(x, y) := K_\mu(ax, ay).$$

Let  $T := \{a \in \omega(\mathbb{N}) : a(j) \geq 1\}$ . We introduce the family of spaces  $F_{ind}[a] := F(\varphi(\mathbb{N}), K_\mu^a)$ . It is clear that  $(F_{ind}[a])_{a \in \omega^+(\mathbb{N})}$  is an inductive system i.e:

$$a \leq b \implies F_{ind}[a] \hookrightarrow F_{ind}[b].$$

For  $a \in T$  we introduce the space  $F_{proj}[a] := F(\ell_{\mathbb{C}}^2(\mathbb{N}), K_\mu^{a^{-1}})$ . The family  $(F_{proj}[a])_{a \in T}$  is a projective system i.e:

$$a \leq b \implies F_{proj}[b] \hookrightarrow F_{proj}[a].$$

We define

$$F_{ind}[T] := \lim_{a \in T} ind F_{ind}[a], \quad F_{proj}[T] := \lim_{a \in T} ind F_{proj}[a].$$

**Proposition 5.1.** *We have the following nuclear Gelfand triplet:*

$$F_{proj}[T] \hookrightarrow F(\ell_{\mathbb{C}}^2(\mathbb{N}), K_\mu) \hookrightarrow F_{ind}[T].$$

**Proposition 5.2.** *For any bounded operator  $Q : F_{proj}[T] \rightarrow F_{ind}[T]$  there exists kernel  $Q_K(z, z') = (K_\mu(z, \cdot), QK_\mu(\cdot, z'))$  in  $F_{ind}[T] \otimes F_{ind}[T]$  such that*

$$Q = \int \int |K_\mu(z, \cdot)\rangle \langle K_\mu(\cdot, z')| Q_K(z, z') d\mu(z) d\mu(z'),$$

*holds in the weak sense.*

Using the reproducing kernel Hilbert space representation of the canonical commutation relations, one can introduce quantization procedure similar to the case with respect to the gaussian measure and in finite dimension case. Hence we define:  
- The Wick operators:

$$T_p^W F(z) := (K_\mu(z, \cdot), p(\cdot, z)F),$$

- The Anti-Wick operators:

$$T_p^{AW} F(z) := (K_\mu(z, \cdot), p(\cdot)F),$$

- The quantized Weyl operators:

$$T_p^w F(z) := (K_\mu(z, \cdot), p(\cdot, \frac{\cdot + z}{2})F).$$

First the above definitions can be applied to cylindrical symbols then it can be extended using the strong topology as in [JR2]. The construction of a test/distribution spaces allows to study unbounded quantized operators. For a detailed exposition of the ideas presented in this talk we refer the reader to the paper [Am].

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