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Abstract

After a short introduction on micromagnetism, we will focus on a scalar micromagnetic model. The problem, which is hyperbolic, can be viewed as a problem of Hamilton-Jacobi, and, similarly to conservation laws, it admits a kinetic formulation. We will use both points of view, together with tools from geometric measure theory, to prove the rectifiability of the singular set of micromagnetic configurations.

1. Micromagnetism: a short introduction

Micromagnetism is the study of the spontaneous magnetization which exists in any ferromagnetic material. A thorough presentation of micromagnetism can be found in [HS]. Here, we consider a thin cylindrical sample of a ferromagnetic material. The magnetization u, defined at every point of the domain D of $\mathbf{R}^3$ delimited by the sample, has a constant norm in D which we choose to be equal to 1 (after renormalization). Because of the small thickness of the sample, we will assume that the magnetization is invariant under translation along the axis of the cylinder D so that u is actually defined on a domain Ω of $\mathbf{R}^2$ which corresponds to a section of D. The magnetization distribution is then given by a vector field u defined in Ω, taking values in S^2 the unit sphere of $\mathbf{R}^3$.
To this magnetization vector field is associated an energy composed of three terms:
- **the exchange energy**, $A\int_\Omega |\nabla u|^2$, which penalizes variations of u,
- **the demagnetizing energy**, $K_d\int_{\mathbf{R}^2} |H(u)|^2$, which comes from the existence of a demagnetizing field $H(u)$, created by the magnetization u, defined by the usual Maxwell equations in the whole $\mathbf{R}^2$:

$$\begin{cases} \operatorname{div}(H(u) + \mathbf{1}_\Omega u) = 0 & \text{in } \mathbf{R}^2, \\ \operatorname{curl}(H(u)) = 0 & \text{in } \mathbf{R}^2, \end{cases}$$

where $\mathbf{1}_\Omega$ denotes the characteristic function of Ω, this energy forces u to be divergence free,
- **the anisotropic energy**, $K\int_\Omega \Psi(u)$, where Ψ is a non-negative function which vanishes only when the component of u along the axis of D vanishes, so that planar magnetizations are preferred.
The free micromagnetic energy is then given by

$$E = A\int_\Omega |\nabla u|^2 + K_d\int_{\mathbf{R}^2} |H(u)|^2 + K\int_\Omega \Psi(u).$$

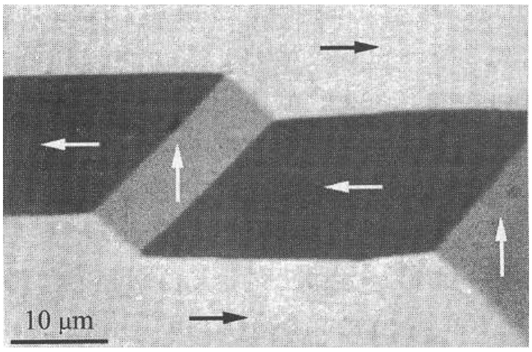

Figure 1: Magnetic domains and walls, image of a silicon-iron crystal ([HS])

It depends on two parameters $d := \sqrt{\frac{A}{K_d}}$, the exchange length, and $Q := \frac{K}{K_d}$, the anisotropy coefficient. In typical experiments, one can observe areas where the magnetization is constant (see Figure 1). These areas, called "magnetic domains", are separated by "walls" whose thickness, proportional to d , is very small ($d \simeq 10\,nm$) compared to the size of the sample ($\simeq 10\,\mu m$). It is then natural to introduce in mathematical models a small parameter ε which takes into account the different scalings of physical coefficients.
The AG model, introduced by P. Aviles and Y. Giga, modelizes the low anisotropy case ($Q << 1$). Since the demagnetizing energy is the strongest term, u is constrained to be divergence free and the energy is given by

$$E_\varepsilon^{AG}(u) = \varepsilon \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_\Omega \left|1-|u|^2\right|^2,$$

where $u : \Omega \to \mathbf{R}^2$ is the projection of the magnetization in the plane of the cylinder D and is divergence free.
The RS model, introduced by T. Rivière and S. Serfaty, modelizes the strong anisotropy case ($Q >> 1$). In this case, the strongest term in the energy is the anisotropic energy and u is constrained to be planar and then to take values in S^1 the unit sphere of the plane. The energy is given by

$$E_\varepsilon^{RS}(u) = \varepsilon \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_{\mathbf{R}^2} |H(u)|^2,$$

where $u : \Omega \to S^1$. In a physical point of view, this model is not relevant, since the vortex configuration ($u(x) = |x|^{-1}x^\perp$) carries in this model some positive energy at the limit $\varepsilon \to 0$ (i.e. $\lim E_\varepsilon(u_\varepsilon) = 0$ for any family u_ε approaching u in L^1). The frequent observation of such a configuration in experiments suggests that its energy should be minimal. That's why F. Alouges, T. Rivière and S. Serfaty have introduced a third model (**the ARS model**) to modelize the strong anisotropy case:

$$E_\varepsilon^{ARS}(u) = \varepsilon \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_{\mathbf{R}^2} |H(u)|^2 + \frac{1}{c_\varepsilon}\int_\Omega |u\cdot k|^2,$$

where $u : \Omega \to S^2$, k is the direction of the axis of D and $c_\varepsilon << \varepsilon$.
In this note, we will focus our analysis on the RS model which is simpler in a math-

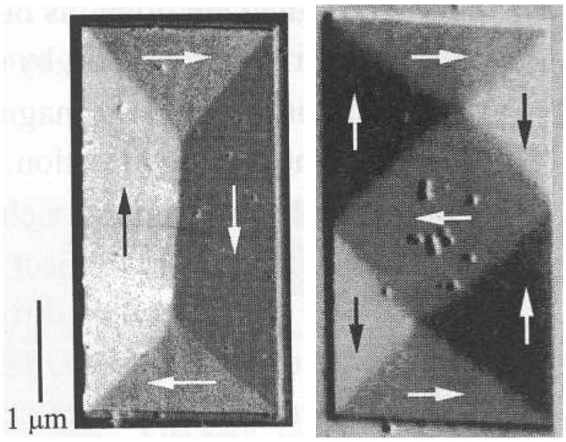

Figure 2: The rectangle 2×1 ([HS])

ematical point of view. Indeed, any $u \in H^1(\Omega, S^1)$ admits a lifting $\phi \in H^1(\Omega, \mathbf{R})$ such that $\int_\Omega |\nabla u|^2 = \int_\Omega |\nabla \phi|^2$. The RS model is then a scalar model, whereas the two other models are truly vectorial.

In order to study the asymptotic problem when ε goes to 0, we need a compactness result which has been given by T. Rivière and S. Serfaty in [RS1] by a compensated compactness argument and in [RS2] by a kinetic averaging argument via a kinetic formulation. Let (ϕ_ε) be a family in $H^1(\Omega, \mathbf{R})$ such that $\|\phi_\varepsilon\|_\infty + E_\varepsilon^{RS}(e^{i\phi_\varepsilon}) \leq C$, then there exist $\varepsilon_n \to 0$ and $\phi \in L^1(\Omega, \mathbf{R})$ such that $\phi_{\varepsilon_n} \to \phi$ strongly in L^1. Moreover, the limit ϕ satisfies the hyperbolic problem:

$$\begin{cases} \text{div } (\mathbf{1}_\Omega u) = 0 \text{ in } \mathcal{D}'(\mathbf{R}^2), \\ |u| = 1 \text{ a.e. in } \Omega, \end{cases} \tag{1.1}$$

where $u = e^{i\phi}$. By "a.e.", we mean almost everywhere with respect to the Lebesgue measure when no measure is precised.
There exist an infinite number of solutions of (1.1): a particular one, called the viscosity solution $u = -\nabla^\perp \text{dist}(\cdot, \partial\Omega)$, where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$, and many configurations built from the distance function as, for instance, the configuration in the right of Figure 2: $u = -\nabla^\perp \text{dist}(\cdot, S \cup \partial\Omega)$ where the segment S separates the rectangle Ω into two squares. The question is: is it possible to characterize, among all solutions of (1.1), those which come from the micromagnetic relaxation?

2. Singular set of limiting configurations

Any configuration $u = e^{i\phi}$ in $L^1(\Omega, S^1)$ such that ϕ is the limit in L^1 of a sequence of $L^\infty \cap H^1$ functions ϕ_{ε_n} satisfying the following uniform bound

$$\|\phi_{\varepsilon_n}\|_\infty + E_{\varepsilon_n}^{RS}(e^{i\phi_{\varepsilon_n}}) \leq C \tag{2.1}$$

is called a limiting configuration. Such a configuration satisfies the hyperbolic problem (1.1) which can be reformulated in the following equivalent way : there exists

$g \in W^{1,\infty}(\Omega, \mathbf{R})$ such that $u = -\nabla^\perp g$ (since u is divergence free) and such that g satisfies the Hamilton-Jacobi problem:

$$\begin{cases} |\nabla g| = 1 \text{ in } \Omega, \\ g = 0 \text{ on } \partial\Omega. \end{cases}$$

One can also understand this problem as a conservation law: indeed the divergence free condition implies the following equation on the lifting ϕ,

$$\frac{\partial}{\partial x_1}(\cos\phi) + \frac{\partial}{\partial x_2}(\sin\phi) = 0.$$

To characterize micromagnetic limiting configurations among all solutions of (1.1), we need to know what further information on ϕ is yielded by the uniform bound (2.1). An answer is given in [RS2]:

Proposition 2.1.

$$\int_\Omega \int_{\mathbf{R}} |div\ e^{i\phi\wedge a}|\, da\, dx \leq \liminf_{n\to+\infty} E^{RS}_{\varepsilon_n}(e^{i\phi_{\varepsilon_n}})$$

where $\phi \wedge a := \inf(\phi, a)$, ϕ_{ε_n} strongly converges to ϕ in L^1 and ϕ_{ε_n} satisfies (2.1).

Ideas of proof (we refer to [RS2] *for a complete proof)*
The main trick is the following inequality

$$E^{RS}_\varepsilon(e^{i\phi_\varepsilon}) = \varepsilon \int_\Omega |\nabla\phi_\varepsilon|^2 + \frac{1}{\varepsilon}\int_{\mathbf{R}^2} |H_\varepsilon|^2 \geq \int_\Omega |\nabla\phi_\varepsilon \cdot H_\varepsilon|$$

where $H_\varepsilon = H(u_\varepsilon)$. Then, the conclusion is given by the following equality which holds in the space of distributions in Ω:

$$\nabla\phi_\varepsilon \cdot H_\varepsilon = -\int_{\mathbf{R}} \operatorname{div}\left(e^{i\phi_\varepsilon\wedge a} + \mathbf{1}_{\{\phi_\varepsilon \leq a\}} H_\varepsilon\right)\, da. \tag{2.2}$$

Indeed, since when $\varepsilon \to 0$, $\mathbf{1}_{\{\phi_\varepsilon\leq a\}}H_\varepsilon \to 0$ in L^2, Proposition 2.1 follows. (2.2) is obtained applying co-area formula (CF) and integration by parts (IP): let $\xi \in C^\infty_c(\Omega)$,

$$\begin{aligned}
\int_\Omega \xi\, \nabla\phi_\varepsilon \cdot H_\varepsilon &\underset{(CF)}{=} \int_{\mathbf{R}} \int_{\{\phi_\varepsilon = a\}} H_\varepsilon \cdot \frac{\nabla\phi_\varepsilon}{|\nabla\phi_\varepsilon|}\, \xi\, dx\, da \\
&\underset{(IP)}{=} \int_{\mathbf{R}} \int_{\{\phi_\varepsilon \leq a\}} (\xi \text{ div } H_\varepsilon + H_\varepsilon \cdot \nabla\xi)\, dx\, da \\
&= -\int_{\mathbf{R}} \int_\Omega \operatorname{div}\left(e^{i\phi_\varepsilon\wedge a} + \mathbf{1}_{\{\phi_\varepsilon\leq a\}} H_\varepsilon\right) \xi\, dx\, da.
\end{aligned}$$

■

By Proposition 2.1, for any limiting configuration $u = e^{i\phi}$, the distribution div $e^{i\phi\wedge a}$ is a finite Radon measure on $\Omega \times \mathbf{R}$. We then introduce the following limiting set of configurations as the best candidate for the Γ-limit problem:

$$\mathcal{M}_{\text{div}}(\Omega) := \left\{u = e^{i\phi},\ \phi \in L^\infty(\Omega, \mathbf{R}),\ \text{ div } u = 0 \text{ and } \operatorname{div}(e^{i\,\phi\wedge a}) \in \mathcal{M}(\Omega\times\mathbf{R})\right\}$$

$\mathcal{M}(\Omega \times \mathbf{R})$ denotes the space of finite Radon measures on $\Omega \times \mathbf{R}$. The Γ-limit problem is still open. In this note, we present the result we obtained with L. Ambrosio, B. Kirchheim and T. Rivière in [AKLR] on the geometric structure of the singular set of any configuration u in $\mathcal{M}_{\rm div}(\Omega)$. Precisely, let us consider the jump set J_ϕ of the lifting ϕ of u which is defined as soon as ϕ is in L^1 by

$$x \in J_\phi \iff \exists\, \phi^+(x) \neq \phi^-(x) \in \mathbf{R}, \exists\, \nu_x \in S^1 \,\Big|\, \lim_{r\to 0} \frac{1}{r^2} \int_{B_r^\pm(x)} |\phi(y) - \phi^\pm(x)| dy = 0$$

where $B_r^\pm(x) = \{y \in B_r(x) \mid \pm (y-x)\cdot \nu_x > 0\}$.
First, let us assume that ϕ has locally bounded variations in Ω ($\phi \in BV_{loc}(\Omega)$). Then, J_ϕ is 1-rectifiable (i.e. is a countable union of C^1 curves). The derivative of ϕ, which is a Radon measure, can be written as follows:

$$D\phi = \nabla\phi\, \mathcal{L}^2 + (\phi^+ - \phi^-)\nu_\phi \mathcal{H}^1 \llcorner J_\phi + D^c\phi,$$

where $\mathcal{H}^1 \llcorner J_\phi$ is the one-dimensional Hausdorff measure restricted to J_ϕ, ϕ^+, ϕ^- and ν_ϕ come from the definition of J_ϕ, $\nabla\phi\, \mathcal{L}^2$ is the absolutely continuous part of $D\phi$ with respect to $\mathcal{L}^2$ the Lebesgue measure, the remaining term $D^c\phi$ is called the Cantor part of $D\phi$. For any $f : \mathbf{R} \to \mathbf{R}$ Lipschitz, $f(\phi) \in BV_{loc}(\Omega)$ and by the Vol'Pert chain rule (see [AFP]), we have:

$$D\big(f(\phi)\big) = f'(\phi)\nabla\phi\, \mathcal{L}^2 + \big(f(\phi^+) - f(\phi^-)\big)\nu_\phi \mathcal{H}^1 \llcorner J_\phi + f'(\phi) D^c\phi.$$

Applying this equality to $f(\phi) = \cos(\phi \wedge a)$ and $f(\phi) = \sin(\phi \wedge a)$ we get

$$\operatorname{div}\left(e^{i\phi\wedge a}\right) = \chi(\phi^+, \phi^-, a)\left(e^{ia} - e^{i\phi^-}\right) \cdot \omega_\phi\, \mathcal{H}^1 \llcorner J_\phi, \tag{2.3}$$

where $\chi(\phi^+, \phi^-, a) = 1$ if $\phi^- < a < \phi^+$, -1 if $\phi^+ < a < \phi^-$ and 0 otherwise, and $\omega_\phi = e^{i(\phi^+ + \phi^-)/2}$.
It is not hard to see, with formula (2.3), that the control on the mass of the defect measure, $\int_\Omega \int_{\mathbf{R}} |\text{div } e^{i\phi\wedge a}|\, da\, dx$ yields a control on $\int_{J_\phi} |\phi^+ - \phi^-|^3 d\mathcal{H}^1$ which is not a BV type control. Actually, $\mathcal{M}_{\rm div}(\Omega)$ is not included in $BV_{loc}(\Omega)$ (see [ADM] for a counter-example). However, the control given by Proposition 2.1 may imply that ϕ has the same properties as BV functions, namely J_ϕ is 1-rectifiable and the defect measure

$$\mu = \int_{\mathbf{R}} |\text{ div } e^{i\phi\wedge a}| da \quad \in \mathcal{M}_{\rm div}(\Omega)$$

is concentrated on J_ϕ and is given by (2.3).

In [AKLR], the 1-rectifiability of J_ϕ is proved. The strategy of the proof is based on the study of the defect measure μ at the one-dimensional level.
The first observation is that μ is absolutely continuous with respect to $\mathcal{H}^1$. Indeed, μ is nearly defined as the divergence of a L^∞ function. Let us assume for simplicity that $\mu =$ div m where $m \in L^\infty$ and $\mu \geq 0$. Let K be a compact subset of Ω such that $\mathcal{H}^1(K) < +\infty$. Then, K can be approached by a sequence of sets A_n where

each A_n is a union of balls covering K such that $\mathrm{Per}(\partial A_n) \to \pi\mathcal{H}^1(K)$ (where Per denotes the perimeter). For all $n \in \mathbf{N}$,

$$\mu(A_n) = \int_{A_n} \text{div } m = \int_{\partial A_n} m \cdot \nu \leq \|m\|_\infty \ \mathrm{Per}(\partial A_n)$$

(ν is the unit normal of A_n). Passing to the limit, we have $\mu(K) \leq \|m\|_\infty \pi\,\mathcal{H}^1(K)$. Hence, for any set B such that $\mathcal{H}^1(B) = 0$, $\mu(B) = 0$.

We then use a blow-up technic, defining the rescaled measures at $x \in \Omega$, for any $r > 0$, by

$$\mu_{r,x}(B) = \frac{\mu(x + rB)}{r}. \tag{2.4}$$

If $\limsup_{r\to 0} \frac{\mu(B_r(x))}{r} < +\infty$ (which is true for $\mathcal{H}^1$ a.e. x, then for μ a.e. x), then there exists a sequence $r_n \to 0$ and $\nu \in \mathcal{M}(\mathbf{R}^2)$, such that $\mu_{x,r_n} \rightharpoonup \nu$ in $\mathcal{M}'(\mathbf{R}^2)$, i.e. $\forall\, \xi \in C_c^0(\Omega)$, $\langle \mu_{x,r_n}, \xi\rangle \to \langle \nu, \xi\rangle$. The set of such a limit ν is called the set of tangent measures of μ at x and is denoted by $\mathrm{Tan}(\mu, x)$.

In order to have information on the jump set J_ϕ, we focus on points x of Ω such that $\mathrm{Tan}(\mu, x) \neq \{0\}$. Therefore, we introduce the set

$$\Sigma = \left\{ x \in \Omega \;\Big|\; \limsup_{r\to 0} \frac{\mu(B_r(x))}{r} > 0 \right\}.$$

Since Σ is σ-finite with respect to $\mathcal{H}^1$ (see [AFP]), the restriction of μ on Σ can be written as $\mu \llcorner \Sigma = f\mathcal{H}^1 \llcorner \Sigma$. The remaining part $\delta = \mu\llcorner(\Omega \setminus \Sigma)$ is orthogonal to $\mathcal{H}^1$, i.e. for any $B \subset \Omega$ such that $\mathcal{H}^1(B) < +\infty$, $\mu(B) = 0$. The goal is then to show that Σ is rectifiable and that it coincides with J_ϕ up to a $\mathcal{H}^1$-negligible set. This can be done in an indirect way showing in a first step the rectifiability of the subset Σ' of Σ

$$\Sigma' = \left\{ x \in \Omega \;\Big|\; \liminf_{r\to 0} \frac{\mu(B_r(x))}{r} > 0 \right\}$$

using the following rectifiability criteria (see [AKLR]):

Theorem 2.1. *If, for μ almost every x in Σ', there exists $\omega(x) \in S^1$ such that for any ν in Tan(μ, x), supp ν is included in a line oriented by $\omega(x)$, then Σ' is 1-rectifiable.*

The second step, which we won't detail in this note, consists in showing that Σ and Σ' coincide μ almost everywhere (i.e. $\mu(\Sigma \setminus \Sigma') = 0$). We then obtain the main result of [AKLR]:

Theorem 2.2. *Let $u = e^{i\phi} \in \mathcal{M}_{\mathrm{div}}(\Omega)$, then*

1. *J_ϕ is 1-rectifiable and*

$$div\left(e^{i\phi\wedge a}\right) \llcorner J_\phi = \chi(\phi^+, \phi^-, a)\left(e^{ia} - e^{i\phi^-}\right) \cdot \omega_\phi \mathcal{H}^1 \llcorner J_\phi.$$

2. *$\mu\llcorner(\Omega \setminus J_\phi)$ is orthogonal with respect to $\mathcal{H}^1$.*

In the last section, we detail the proof of the rectifiability of Σ'.

3. Some details of the proof of Theorem 2.2

Before going into the proof of Theorem 2.2, let us go back to the analogy with conservation laws. The defect measure div $e^{i\phi\wedge a}$ can be seen as Kruzhkov entropy measures where a is the kinetic parameter. This interpetation is hightlighted by the kinetic formulation which was given in [RS2] for the RS model. For scalar conservation laws, unicity for the Cauchy problem holds as soon as a uniform sign condition is satisfied by Kruzhkov entropy measures (see [Se] for instance). In the micromagnetic framework, we obtain in [ALR] such a unicity result:

Theorem 3.1. *Let $u = -\nabla^\perp g \in \mathcal{M}_{\rm div}(\Omega)$. If $\forall\, a \in \mathbf{R}$, div $e^{i\phi\wedge a} \geq 0$, then g is a viscosity solution of $|\nabla g| = 1$, which is uniquely determined by $g_{|\partial\Omega}$:*

$$g(x) = \inf_{y\in\partial\Omega} \{g(y) + |x-y|\}\,.$$

In particular g is locally semiconcave in Ω (i.e. $D^2 g \leq CId$) and $u = -\nabla^\perp g$ is in $BV_{loc}(\Omega)$.

Remark 1: In the case when $\Omega = \mathbf{R}^2$, g is a viscosity solution of $|\nabla g| = 1$ in $\mathbf{R}^2$ so that g is concave ($D^2 g \leq 0$).

Remark 2: The BV regularity of u implies the BV regularity of the lifting ϕ as soon as the total mass of div $e^{i\phi\wedge a}$ is controlled (see [AKLR]).

Let us go back to the proof of Theorem 2.2. We define the rescaled functions by $\phi_r(y) = \phi(x + ry)$, they are related to the rescaled measures defined by (2.4) as follows:

$$\mu_{x,r} = \int_{\mathbf{R}} |\ \text{div}\ e^{i\phi_r\wedge a}|\, da.$$

The compactness result of [RS1] can be adapted to this situation and we have that, possibly extracted a subsequence still denoted by r_n,

$$\phi_{r_n} \to \phi_\infty \quad \text{in } L^1_{loc}(\mathbf{R}^2),$$

$$\mu_{x,r_n} \rightharpoonup \nu \quad \text{in } \mathcal{M}'(\mathbf{R}^2).$$

The proof of Theorem 2.2 is based on the following observation:

$$\text{div}\ e^{i\phi_\infty\wedge a} = h(a)\nu \quad \text{where } h\,:\,\mathbf{R}\to\mathbf{R} \ \text{ is Lipschitz} \tag{3.1}$$

Proof of (3.1): Let D be a countable dense subset of $C^0_c(\mathbf{R})$. For all $g \in D$, let us define $\nu_g = \int_{\mathbf{R}} g(a)$ div $e^{i\phi\wedge a} da \in \mathcal{M}(\Omega)$. ν_g is absolutely continuous with respect to μ, therefore, by the Radon-Nikodym theorem, there exists h_g in $L^1_\mu(\Omega)$, the space of integrable functions with respect to μ, such that $\nu_g = h_g\mu$. Let us assume that x is a Lebesgue point of h_g for any $g \in D$ (which is true for μ a.e. x in Ω). Then, the rescaled measure $(\nu_g)_{x,r_n} = h_g(x+r_n\cdot)\mu_{x,r_n}$ weakly converges to the measure $h_g(x)\nu$. But, using the rescaled function ϕ_{r_n} we also have $(\nu_g)_{x,r_n} = \int_{\mathbf{R}} g(a)$ div $e^{i\phi_{r_n}\wedge a}da$ and it weakly converges to $\int_{\mathbf{R}} g(a)$ div $e^{i\phi_\infty\wedge a}da$, so that the following equality holds:

$$\int_{\mathbf{R}} g(a)\ \text{div}\ e^{i\phi_\infty\wedge a} = h_g(x)\nu. \tag{3.2}$$

Let us fix $a \in \mathbf{R}$ and choose a sequence (g_k) in D weakly converging to the Dirac mass at a. The sequence $\Big(h_{g_k}(x)\Big)$ is bounded, because of (3.2). Hence, up to an extraction, it converges to some real h satisfying div $e^{i\phi_\infty \wedge a} = h\nu$. This last equality implies that the limit h does not depend on the extraction but only on a. The Lipschitz regularity of h comes from the Lipschitz regularity of the map $a \mapsto$ div $e^{i\phi_\infty \wedge a}$ (see [RS2]).

■

Let us decompose the space $\mathbf{R}$ of the parameter a into three subsets : $\{h = 0\}$, $\{h > 0\}$ and $\{h < 0\}$. For each subset, let us take the connected components of their interior and denote them (b_l, c_l). We then have a countable set of intervals (b_l, c_l) such that $\mathbf{R} \setminus \underset{l\in\mathbf{N}}{\cup}(b_l, c_l)$ has an empty interior.

We define the truncated function $\phi_l := \sup\Big(\inf(\phi_\infty, c_l), b_l\Big)$. On one hand, these new functions satisfy a uniform sign condition on the defect measures:

$$\text{div } e^{i\phi_l \wedge a} \geq 0, \ \forall\, a \in \mathbf{R} \quad \text{or} \quad \text{div } e^{i\phi_l \wedge a} \leq 0, \ \forall\, a \in \mathbf{R}.$$

Applying Theorem 3.1, we have that for any $l \in \mathbf{N}$, the function g_l such that $e^{i\phi_l} = -\nabla^\perp g_l$ is a viscosity solution of $|\nabla g| = 1$ in $\mathbf{R}^2$, therefore g_l is concave by Remark 1. Moreover, $\phi_l \in BV_{loc}(\mathbf{R}^2)$ by Remark 2.

On the other hand, we recover some information on ϕ_∞ using the information we have on all ϕ_l. Indeed, a point $x \in \Omega$ belongs to J_{ϕ_∞} if and only if it belongs to at least one J_{ϕ_l}. Moreover since $\phi_l \in BV_{loc}$ for all $n \in \mathbf{N}$, we have that $\phi_\infty \in BV_{loc}$. By (3.1) and by the BV regularity of ϕ_∞, ν is concentrated on J_{ϕ_∞}. Our goal is to show that J_{ϕ_∞} is included in one line whose direction does not depend on the subsequence r_n.

First, we show that for any $l, m \in \mathbf{N}$, $l \neq m$, $\mathcal{H}^1(J_{\phi_l} \setminus J_{\phi_m}) = 0$ and that for any $l \in \mathbf{N}$, ϕ_l^+ and ϕ_l^- are constant on J_l, using the two formulations for the defect measures:

$$\text{div } e^{i\phi_l \wedge a} = \begin{cases} h(a)\nu & \text{if } a \in (b_l, c_l), \\ 0 & \text{else,} \end{cases}$$

and

$$\text{div}\left(e^{i\phi_l \wedge a}\right) = \chi(\phi_l^+, \phi_l^-, a)\left(e^{ia} - e^{i\phi_l^-}\right) \cdot \omega_l \mathcal{H}^1 \llcorner J_{\phi_l}, \quad \text{where } \omega_l = e^{i(\phi_l^+ + \phi_l^-)/2},$$

the last one holds since $\phi_l \in BV_{loc}(\mathbf{R}^2)$. In particular, ω_l, the unit normal of J_{ϕ_l}, is constant on J_{ϕ_l}, so that J_{ϕ_l} is included in a set of parallel lines.

Secondly, one can show that J_{ϕ_l} is actually included in one line arguing by contradiction using the concavity of g_l on a line passing through two points of J_{ϕ_l} and which is not parallel to $\mathbf{R}\omega_l^\perp$.

Then, $J_{\phi_\infty} = J_{\phi_l}$ for any $l \in \mathbf{N}$ up to a $\mathcal{H}^1$-negligible set and J_{ϕ_∞} is included in one line.

Finally, we have to show that the direction ω of this line does not depend on the sequence r_n. Let us introduce the vector-valued measure $\vec{\lambda} = \int_{\mathbf{R}} e^{ia}$ div $e^{i\phi \wedge a} da$ which is absolutely continuous with respect to μ, so that $\vec{\lambda} = \vec{H}\mu$, where $\vec{H} \in L^1_\mu(\Omega)$.

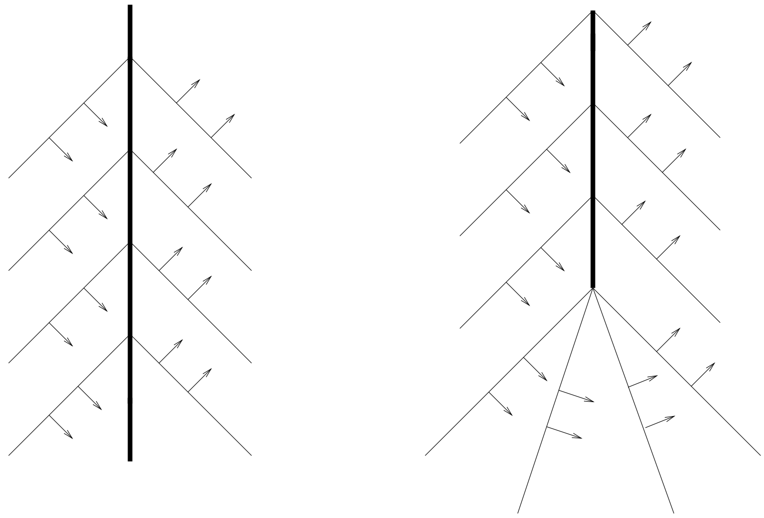

Figure 3: Limits of the blow-up, J_{ϕ_∞} is either a line or a half-line

Let us assume that x is a Lebesgue point of $\vec{H}$, then

$$\int_{\mathbf{R}} e^{ia} \text{ div } e^{i\phi_\infty \wedge a} da = \vec{H}(x)\nu.$$

But, ϕ_∞ is in BV_{loc} so that div $e^{i\phi_\infty \wedge a}$ can be evaluated and we also have

$$\int_{\mathbf{R}} e^{ia} \text{ div } e^{i\phi_\infty \wedge a} da = \frac{1}{2}\Big(\phi_\infty^+ - \phi_\infty^- - \sin(\phi_\infty^+ - \phi_\infty^-)\Big)\omega \mathcal{H}^1 \llcorner J_{\phi_\infty}.$$

Therefore, ω is given by $\vec{H}(x)$, and it does not depend on the sequence r_n.
Using Theorem 2.1, Σ' is 1-rectifiable.

To finish the proof of Theorem 2.2, in particular to show that $\mu(\Sigma \setminus \Sigma') = 0$, one has to go further in the characterization of limits of the blow-up: indeed, only two configurations can be obtained at the limit of the blow-up (see Figure 3).

4. Conclusion

Our goal, in this note, was to provide the main ideas of the proof of the rectifiability of the jump set of any limiting micromagnetic configurations. Many technical details are missing and we refer to [AKLR] for the whole proof.
For the two other micromagnetic models (the AG and ARS models), a similar study was carried through by C. De Lellis and F. Otto. Since the problem can't be reduce to a scalar problem anymore, the proof is more technical: indeed, no unicity result (such as Theorem 3.1) exists in the vectorial situation, so that the BV regularity of the limit of the blow-up is not known a priori. Nevertheless, they manage to obtain the same rectifiability result as Theorem 2.2 (see [DO]).
The method used in this context of micromagnetism can be applied to conservation

laws. A rectifiability result on "shock waves", the singular set of solutions of conservation laws, can be obtained without any sign assumption on entropy measures. We refer to [LR] where the proof presented in this note is adapted to conservation laws when the space dimension is equal to 1. The general case (any space dimension) is studied in [DOW].

The problem to know if the defect measure μ is concentrated on the jump set of configurations (like in the BV case) is left open in all the above papers. It is a crucial problem in order to solve the Γ-limit problem. As far as we know, the only result on the concentration of the defect measure has been obtained by C. De Lellis and T. Rivière in [DR] for a problem of conservation laws in one space dimension, with a sign assumption on Kruzhkov entropy measures but in a general situation where the solution is not necesseraly BV.

Eventually, we would like to mention the review made by T. Rivière at Forges-les-Eaux in 2002 (see [Ri]) on the whole study of micromagnetic problems which have been described in the introduction. We suggest to the reader to refer to it for other references, in particular about compactness issues and kinetic formulations.

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