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1. Introduction

In this talk, we study the existence and regularity properties of weak solutions to the following nonlinear wave equation

$$\begin{cases} \partial_t^2 u - c(u)\partial_x(c(u)\partial_x u) = 0, \\ u|_{t=0} = u_0, \\ \partial_t u|_{t=0} = u_1, \end{cases} \tag{1.1}$$

where $c(\cdot)$ is a given smooth, bounded, and positive function with $c'(\cdot) \geq 0$ and $c'(u_0) > 0$, $u_0(x) \in H^1(\mathbb{R})$, and $u_1(x) \in L^2(\mathbb{R})$.

One motivation for study (1.1) comes from liquid crystals. We give a brief explanation of how the equation arises in that context. The mean orientation of the molecules in a nematic liquid crystal is described by a director field of unit vectors, $\mathbf{n} \in \mathcal{S}^2$. We consider a regime in which inertia effects dominate viscosity. The propagation of orientation waves in the director field is then modeled by a constrained variational principle

$$\delta \int\int \{\mathbf{n}_t \cdot \mathbf{n}_t - W(\mathbf{n}, \nabla\mathbf{n})\}\, dx\, dt = 0, \quad \mathbf{n}\cdot\mathbf{n} = 1,$$

where W is the Oseen-Franck potential energy density,

$$W(\mathbf{n}, \nabla\mathbf{n}) = \alpha|\mathbf{n} \times (\mathbf{n} \times \mathbf{n})|^2 + \beta(\nabla \cdot \mathbf{n})^2 + \gamma(\mathbf{n} \cdot \nabla \times \mathbf{n})^2.$$

This potential energy is determined (up to a null Lagrangian) by the requirement that it is invariant under reflection $\mathbf{n} \to -\mathbf{n}$ and under simultaneous rotations of the spatial variables and the director field. The positive constants α, β, γ are elastic constants of the liquid crystal.

A commonly used special case is the one-constant approximation in which $\alpha = \beta = \gamma$. The potential energy density then reduces to

$$W(\mathbf{n}, \nabla\mathbf{n}) = \alpha|\nabla\mathbf{n}|^2.$$

The associate variational problem is identical to the variational problem for wave maps from $(1+3)$-dimensional Minkowski space into two sphere.

The simplest class of solutions for the orientational waves in a liquid crystal consists of planar deformations depending on a single space variable. The director field then has the special form

$$\mathbf{n} = \cos u(t,x)\mathbf{e}_x + \sin u(t,x)\mathbf{e}_y.$$

Here, the dependent variable u measures the angle of the director field to the x-direction, $\mathbf{e}_x$ and $\mathbf{e}_y$ are coordinate vectors in the x and y directions, respectively. In this case, the variational principle for $\mathbf{n}$ reduces to

$$\delta \int\int \{u_t^2 - c^2(u)u_x^2\}\, dx\, dt = 0,$$

with the wave speed c given by

$$c^2(u) = \alpha \cos^2 u + \beta \sin^2 u. \tag{1.2}$$

the Euler-Lagrange equation for this variational principle is (1.1). In the wave map case, we have $\alpha = \beta$, and equation (1.1) reduces to the standard linear wave equation.

We point out that, early in the study of (1.1), Hunter and Saxton ([5]) derived an asymptotic equation

$$\partial_t v + u\partial_x v = -\frac{1}{2}v^2, \quad v = \partial_x u, \tag{1.3}$$

for (1.1) via weakly nonlinear geometric optics. The global existence and uniqueness of solutions to the Cauchy problem is fairly complete, see Hunter and Zheng [6] and the authors' Zhang and Zheng [11]. The study of (1.3) has been very beneficial for both the blow-up result Glassey, Hunter and Zheng [4] and the current global existence result for the wave equation (1.1) [13].

The difficulty to prove the global existence of weak solutions to (1.1) are that the potential oscillations, in terms of DiPerna and Majda [2], get amplified unboundedly by quadratic growth terms of the equation, and the possible concentrations in the approximate solutions. We use the generalized compensated compactness (Gerard [3] or Tartar [9]), the latest development in the L^p Young measure method of Lions [8] and Joly, Metivier and Rauch [7], the renormalization method in DiPerna and Lions [1], and the techniques used in our paper Zhang and Zheng [12] to treat the oscillations. We obtain high regularity for the space derivative of the wave amplitude $\partial_x u$ away from $c'(u) = 0$, to control the possible concentrations.

Before we present our main result, let us first give the following definition. Our notations are $\mathbb{R}^+ = (0, \infty)$, Lip stands for Lipschitz, and

$$R := \partial_t u + c(u)\partial_x u, \quad S := \partial_t u - c(u)\partial_x u, \quad \widetilde{c}(\cdot) := \frac{1}{4}\ln c(\cdot), \tag{1.4}$$

so that $\tilde{c}'(u) = \frac{c'(u)}{4c(u)}$.

With the above notations, we can also write (1.1) in the following form:

$$\begin{cases} \partial_t R - c(u)\partial_x R = \tilde{c}'(u)(R^2 - S^2), \\ \partial_t S + c(u)\partial_x S = \tilde{c}'(u)(S^2 - R^2), \\ \partial_x u = \frac{R-S}{2c(u)}, \\ R|_{t=0} = R_0, \quad S|_{t=0} = S_0. \end{cases} \tag{1.5}$$

Definition 1.1 *We call $u(t,x)$ an admissible weak solution of (1.1) if*
1) $u(t,x) \in L^\infty(\mathbb{R}^+, H^1(\mathbb{R})) \cap Lip(\mathbb{R}^+, L^2(\mathbb{R}))$, and

$$\int_{\mathbb{R}} (|\partial_t u|^2 + |c(u)\partial_x u|^2)\, dx \le \int_{\mathbb{R}} (|u_1|^2 + |c(u_0)\partial_x u_0|^2)\, dx; \tag{1.6}$$

2) For any test function $\phi(t,x) \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$, there holds

$$\int\int_{\mathbb{R}^+ \times \mathbb{R}} (\partial_t \phi \partial_t u - \partial_x \phi c^2(u)\partial_x u - \phi c'(u)c(u)(\partial_x u)^2)\, dx\, dt = 0; \tag{1.7}$$

3) (The entropy condition) For any (t_0, x_0) with $t_0 > 0$, there always exists a positive constant $M(t_0, x_0)$ such that

$$R(t,x) \ge -M(t_0,x_0), \qquad S(t,x) \ge -M(t_0,x_0), \tag{1.8}$$

hold in a neighborhood $\mathcal{N}(t_0,x_0)$ of (t_0,x_0);
4) $u(t,x) \to u_0(x)$ in $L^2(\mathbb{R})$ and $\partial_t u(t,x) \to u_1(x)$ in the distributional sense as $t \to 0+$.

We shall always assume that there exist two positive constant C_1, C_2 such that

$$0 < C_1 \le c(\cdot) \le C_2, \quad \text{and} \quad |c^{(l)}(\cdot)| \le M_l, l \ge 1 \tag{1.9}$$

for some positive constants M_l.

Theorem 1.1 *Let $c'(\cdot) \ge 0, c'(u_0(\cdot)) > 0, u_0 \in H^1(\mathbb{R})$, and $u_1 \in L^2(\mathbb{R})$. Then (1.1) has a global admissible weak solution u in the sense of Definition 1.1. Moreover,*

$$\int\int_{\Omega} |\partial_x u|^p\, dx\, dt \le C_{\Omega,p}, \quad \forall p < 3, \tag{1.10}$$

where Ω is a small neighborhood of any point $(t,x) \in \overline{\mathbb{R}^+} \times \mathbb{R}$ at which $c'(u(t,x)) \ne 0$, and $C_{\Omega,p}$ is a positive constant which depends only on $\Omega, p, \|u_0\|_{H^1}$, and $\|u_1\|_{L^2}$.

Remark 1.1 *Theorem 1.1 still holds if we replace the assumptions $c'(\cdot) \ge 0$ and $c'(u_0(\cdot)) > 0$ by $c'(\cdot) \le 0$ and $c'(u_0(\cdot)) < 0$. One needs only to replace the entropy condition in (1.8) by*

$$R(t,x) \le M(t_0,x_0), \qquad S(t,x) \le M(t_0,x_0),$$

for $(t,x) \in \mathcal{N}(t_0,x_0)$.

Remark 1.2 *Suppose that c' keeps sign, and $(R_0, S_0) \in L^\infty(\mathbb{R})$, then by Zhang and Zheng [12], we know that the following ordinary differential equations have global solutions $\Phi_t^\pm(x) \in Lip([0,\infty) \times \mathbb{R})$:*

$$\begin{cases} \frac{d\Phi_t^\pm(x)}{dt} = \pm c(u(t, \Phi_t^\pm(x))), \\ \Phi_0^\pm(x) = x. \end{cases} \tag{1.11}$$

But here as the initial data $(R_0, S_0) \in L^2(\mathbb{R})$, the entropy condition (1.8) is not enough to prove this result for (1.11). Actually we do not even know that (1.11) has solutions $\Phi_t^\pm(x) \in C([0,\infty) \times \mathbb{R})$.

Remark 1.3 *Motivated by Hunter and Zheng [6] and Zhang and Zheng [11], we point out that we expect multiple weak solution to problem (1.1). Our weak solutions in Theorem 1.1 are solutions of the dissipative type, because the entropy condition (1.8) guarantees that: On almost all the blow-up points (τ, y), $R(t,x) \to +\infty$ as $(t,x) \to (\tau, y)$, and similarly for S. But in the construction of the conservative weak solutions to (1.3) (see Hunter and Zheng [6]), $v(t,x) \to -\infty$ as $t \to \tau-$ and $v(t,x) \to +\infty$ as $t \to \tau+$, if τ is the blow-up time of the solution.*

2. Approximate solutions and uniform estimates

Define for $\varepsilon > 0$

$$Q_\varepsilon(\xi) := \begin{cases} \frac{1}{\varepsilon}(\xi - \frac{1}{2\varepsilon}), & \xi\varepsilon \geq 1, \\ \frac{1}{2}\xi^2, & \xi\varepsilon < 1. \end{cases} \tag{2.1}$$

Let us also use the notations $\zeta^+ := \max(0, \zeta)$ and $\zeta^- := \min(0, \zeta)$. We now define the approximate solution sequence by the equations

$$\begin{cases} \partial_t R^\varepsilon - c(u^\varepsilon)\partial_x R^\varepsilon = \tilde{c}'(u^\varepsilon)^+(2Q_\varepsilon(R^\varepsilon) - (S^\varepsilon)^2) + \tilde{c}'(u^\varepsilon)^-(2Q_{-\varepsilon}(R^\varepsilon) - (S^\varepsilon)^2), \\ \partial_t S^\varepsilon + c(u^\varepsilon)\partial_x S^\varepsilon = \tilde{c}'(u^\varepsilon)^+(2Q_\varepsilon(S^\varepsilon) - (R^\varepsilon)^2) + \tilde{c}'(u^\varepsilon)^-(2Q_{-\varepsilon}(S^\varepsilon) - (R^\varepsilon)^2), \\ \partial_x u^\varepsilon = \frac{R^\varepsilon - S^\varepsilon}{2c(u^\varepsilon)}, \\ \lim_{x\to-\infty} u^\varepsilon(t,x) = 0, \\ (R^\varepsilon, S^\varepsilon)|_{t=0} = (R_0, S_0)(x). \end{cases} \tag{2.2}$$

Assume that $c(\cdot)$ satisfies (1.9), but $c'(\cdot)$ may change sign.

Lemma 2.1 *(Solution of (2.2) with smooth data). Let $(R_0, S_0)(x) \in C_c^\infty(\mathbb{R})$. Then, problem (2.2) has a global smooth solution $(R, S)(t, x) \in L^\infty(\mathbb{R}^+, W^{1,\infty}(\mathbb{R}))$, $u(t, x) \in L^\infty(\mathbb{R}^+, W^{2,\infty}(\mathbb{R}))$, which satisfies the energy inequalities*

$$\int (R^2 + S^2)(t, x)dx \leq \int (R_0^2 + S_0^2)(x)dx \tag{2.3}$$

and

$$\int_0^\infty \int_{\mathbb{R}} (c'(u^\varepsilon)^+ G_\varepsilon^+ + c'(u^\varepsilon)^- G_\varepsilon^-)\, dx\, dt \leq \int (R_0^2 + S_0^2)(x)dx, \tag{2.4}$$

where

$$G_\varepsilon^\pm := R(R^2 - 2Q_{\pm\varepsilon}(R)) + S(S^2 - 2Q_{\pm\varepsilon}(S))$$

and $G_\varepsilon^+ \geq 0$ and $G_\varepsilon^- \leq 0$. Moreover, if we introduce the plus and minus characteristics $\Phi_t^\pm(b)$ as

$$\begin{cases} \frac{d}{dt}\Phi_t^\pm = \pm c(u(t, \Phi_t^\pm)), \\ \Phi_t^\pm|_{t=0} = b, \end{cases} \tag{2.5}$$

then, we have the energy inequality in a characteristic cone

$$\int_a^d R^2(t_a^+(y), y)dy + \int_d^b S^2(t_b^-(y), y)dy \leq \frac{1}{2}\int_a^b (R_0^2 + S_0^2)(x)dx, \tag{2.6}$$

where $a < b$, and d is where the two characteristics $\Phi_t^+(a)$ and $\Phi_t^-(b)$ meet at some positive time, and $t = t_a^+(y)$ is the inverse of $y = \Phi_t^+(a)$, etc.

Sketch of proof. It can be proved exactly as that in the proof of Lemma 6 of Zhang and Zheng [10] that if T^* is the life span of the Lipschitz solution to (2.2) and $T^* < +\infty$ implies

$$\lim_{t \to T^*} (\|R(t,\cdot)\|_{L^\infty} + \|S(t,\cdot)\|_{L^\infty}) = +\infty. \tag{2.7}$$

□

Now we can see from Lions-Aubin's Lemma, see Lemma 3 of Zhang and Zheng [10] for details, that there exists a subsequence of the approximate solutions $\{u^\varepsilon\}$ which converges in the maximum norm on any compact domain of the upper half plane to a continuous function $u(t,x)$:

$$u^\varepsilon \to u(t,x). \tag{2.8}$$

We can use the continuity of $u(t,x)$ and $c'(u)$ to obtain uniform estimates on $(R^\varepsilon, S^\varepsilon)$ in $L^{2+\alpha}$ at any point (t,x) such that $c'(u(t,x)) \neq 0$.

Lemma 2.2 *(Local $L^{2+\alpha}$ estimate). Let $(R_0, S_0) \in L^2$. For solutions $\{(R^\varepsilon, S^\varepsilon, u^\varepsilon)\}_{\varepsilon>0}$ of (2.2) there hold*

$$\int\int_\Omega (R-S)^2(R^\alpha + S^\alpha)\,dx\,dt \le C_{\Omega,\alpha}, \tag{2.9}$$

where Ω is a small neighborhood of any point (t,x) at which $c'(u(t,x)) \neq 0$, $\alpha \in (0,1)$, and $C_{\Omega,\alpha}$ is independent of ε.

Sketch of proof. We take an $\alpha = \frac{d_2}{d_1} \in (0,1)$ where d_2 is an even positive integer and d_1 an odd positive integer. We then multiply the first equation of (2.2) with $R^\alpha(t,x)$ to yield

$$\begin{aligned} &\tfrac{1-\alpha}{1+\alpha}\tilde{c}'(u)(R-S)R^{1+\alpha} + \tilde{c}'(u)(R^\alpha S^2 - SR^{1+\alpha}) \\ &\quad = \tilde{c}'(u)^+ R^\alpha(2Q_\varepsilon(R) - R^2) + \tilde{c}'(u)^- R^\alpha(2Q_{-\varepsilon}(R) - R^2) \\ &\quad\quad - \tfrac{1}{1+\alpha}\{\partial_t R^{1+\alpha} - \partial_x(c(u)R^{1+\alpha})\}. \end{aligned} \tag{2.10}$$

Similarly we have an equation for S.

Let $\chi(t,x)$ be a smooth cut-off function around a point where $c'(u(t_0,x_0)) \neq 0$. We use integration by parts and the energy bounds in Lemma 2.1 to obtain

$$\int\int \chi[(R-S)(R^{1+\alpha} - S^{1+\alpha}) + R^\alpha S^\alpha (R-S)(R^{1-\alpha} - S^{1-\alpha})]\,dx\,dt \le C_{\alpha,\chi}. \tag{2.11}$$

Regrouping the integrand in (2.11), we obtain (2.9). □

To prove the precompactness of the approximate solutions $\{R^\varepsilon, S^\varepsilon\}$, we need the following type of entropy condition for $\{R^\varepsilon, S^\varepsilon\}$:

Lemma 2.3 *Let $(R_0, S_0) \in L^2(\mathbb{R})$. Let $t_0 > 0$ and (t_0, x_0) be any point at which $c'(u(t_0, x_0)) \neq 0$, then there exists a neighborhood $\mathcal{N}(t_0, x_0)$ of (t_0, x_0) and some nonnegative constant $M(t_0, x_0)$ which is independent of ε, such that*

$$sign(c'(u^\varepsilon))R^\varepsilon(t,x) \geq -M(t_0, x_0), \quad sign(c'(u^\varepsilon))S^\varepsilon(t,x) \geq -M(t_0, x_0), \tag{2.12}$$

hold for all $(t, x) \in \mathcal{N}(t_0, x_0)$.

Sketch of proof. Along the trajectory of $-c'(u^\varepsilon)$, there holds

$$\frac{dR^\varepsilon(t, \Phi_t^{\varepsilon,-}(y))}{dt} \geq -\widetilde{c}'(u^\varepsilon)(S^\epsilon)^2(t, \Phi_t^{\varepsilon,-}(y)), \quad t \in [t_1, t_2].$$

Integrating the above inequality over $[t_1, t]$ with $t \leq t_2$, we can use (2.6) to bound the resulting equation. □

In particular, a refinement of the above proof yields

Lemma 2.4 *Let $c'(\cdot) \geq 0$, $c'(u_0(\cdot)) > 0$, and $(R_0, S_0) \in L^2(\mathbb{R})$, let $t_0 > 0$ be any sufficiently small positive number. Then for any $\bar{x} \in \mathbb{R}$ and any $\bar{t} > t_0$, there exist a neighbourhood $\mathcal{N}(\bar{t}, \bar{x})$ and some positive constant $\bar{M}$, which depends only on the L^2 norm of $(R_0, S_0), t_0$, and $c'(u_0)$, such that*

$$R^\epsilon(t,x) \geq -\bar{M}, \qquad S^\epsilon(t,x) \geq -\bar{M} \tag{2.13}$$

for all $(t, x) \in \mathcal{N}(\bar{t}, \bar{x})$.

3. Precompactness

Let $(R_0, S_0) \in L^2(\mathbb{R})$. Let $j_\varepsilon(x)$ be the standard Friedrichs' mollifier, and $\chi_\epsilon(x) = \chi(\frac{x}{\epsilon})$ with $\chi(x) \in C_c^\infty(\mathbb{R})$ and $\chi(x) = 1$ around $x = 0$. We denote $R_0^\varepsilon = (R_0\chi_\epsilon) * j_\varepsilon$ and $S_0^\varepsilon = (S_0\chi_\epsilon) * j_\varepsilon$. Then by Lemma 2.1, problem (2.2) has a global smooth solution $(R^\varepsilon, S^\varepsilon, u^\varepsilon)$ with the initial data $(R_0^\varepsilon, S_0^\varepsilon)$. Moreover, we have

$$\int ((R^\varepsilon)^2 + (S^\varepsilon)^2)(t,x)dx \le \int (R_0^2 + S_0^2)(x)dx \quad (t \ge 0). \tag{3.1}$$

We shall also use energy estimate (2.6) and (2.9) in this new setting.

We establish the precompactness of $\{(R^\varepsilon, S^\varepsilon, u^\varepsilon)(t,x)\}$ in this section. For the convenience of the reader, we recall the following lemma (see Lemmas 9–10 of Zhang and Zheng [10]).

Lemma 3.1 *(Time-distinguished Young measure). There exist a subsequence of the solution sequence $\{R^\varepsilon(t,x), S^\varepsilon(t,x)\}$, which we still denote by $\{R^\varepsilon(t,x),\ S^\varepsilon(t,x)\}$ for convenience, and three families of Young measures $\nu_{tx}^1(\xi)$ on $\mathbb{R}$, $\nu_{tx}^2(\eta)$ on $\mathbb{R}$, and $\mu_{tx}(\xi,\eta)$ on $\mathbb{R}^2$, such that for all continuous functions $f(\lambda) \in C_c^\infty(\mathbb{R})$, $\psi(x) \in C_c^\infty(\mathbb{R})$, $g(\xi,\eta) \in C^\infty(\mathbb{R}^2)$ with $g(\xi,\eta) = o((|\xi| + |\eta|)^p)$ as $|\xi| + |\eta| \to \infty$ for some $p < 2$, and $\varphi(t,x) \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$, there hold*

$$\begin{aligned} \lim_{\varepsilon\to 0}\int_{\mathbb{R}} f(R^\varepsilon(t,x))\psi(x)dx &= \int\int_{\mathbb{R}\times\mathbb{R}} f(\xi)\psi(x)d\nu_{tx}^1(\xi)dx, \\ \lim_{\varepsilon\to 0}\int_{\mathbb{R}} f(S^\varepsilon(t,x))\psi(x)dx &= \int\int_{\mathbb{R}\times\mathbb{R}} f(\eta)\psi(x)d\nu_{tx}^2(\eta)dx, \end{aligned} \tag{3.2}$$

uniformly in every compact subset of $[0,\infty)$, and

$$\begin{aligned} \lim_{\varepsilon\to 0}&\int_0^\infty\int_{\mathbb{R}} g(R^\varepsilon(t,x), S^\varepsilon(t,x))\varphi(t,x)dxdt \\ &= \int_0^\infty\int_{\mathbb{R}}\int\int_{\mathbb{R}\times\mathbb{R}} g(\xi,\eta)\varphi(t,x)d\mu_{tx}(\xi,\eta)dxdt. \end{aligned} \tag{3.3}$$

Moreover,

$$\begin{aligned} &t \in [0,\infty) \mapsto \textstyle\int\int_{\mathbb{R}\times\mathbb{R}} f(\xi)\psi(x)d\nu_{tx}^1(\xi)dx \text{ is continuous}, \\ &t \in [0,\infty) \mapsto \textstyle\int\int_{\mathbb{R}\times\mathbb{R}} f(\eta)\psi(x)d\nu_{tx}^2(\eta)dx \text{ is continuous}, \end{aligned} \tag{3.4}$$

and

$$\mu_{tx}(\xi,\eta) = \nu_{tx}^1(\xi) \otimes \nu_{tx}^2(\eta). \tag{3.5}$$

Furthermore, by Proposition 3.1.3 of Joly, Métivier and Rauch [7] and Lemma 3.1, we find that

$$\begin{aligned} &\xi \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}\times\mathbb{R}, dx \otimes d\nu_{tx}^1(\xi))), \\ &\eta \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}\times\mathbb{R}, dx \otimes d\nu_{tx}^2(\eta))). \end{aligned} \tag{3.6}$$

We remark that (3.5) implies directly that

$$R^\epsilon S^\epsilon \rightharpoonup RS \qquad \text{as } \epsilon \to 0. \tag{3.7}$$

in the sense of distributions. □

We shall use the notation

$$\overline{g(R,S)} = \int_{\mathbb{R}} g(\xi,\eta)\, d\mu_{tx}(\xi,\eta).$$

Thus, $(\overline{R}, \overline{S})$ represents the weak star limit of $\{R^\varepsilon, S^\varepsilon\}$ in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))$ or the weak limit in $L^2((0,T)\times\mathbb{R})$ for all $T<\infty$.

With the above preparation, we can now establish the precompactness of $\{R^\varepsilon, S^\varepsilon\}$.

Lemma 3.2 *(Precompactness of $\{(R^\varepsilon, S^\varepsilon)\}$). Let $c'(\cdot) \geq 0$, $c'(u_0(\cdot)) > 0$, and $(R_0(x),\ S_0(x)) \in L^2(\mathbb{R})$. Then $\nu^1_{tx}(\xi) = \delta_{\overline{R}(t,x)}(\xi)$ and $\nu^2_{tx}(\eta) = \delta_{\overline{S}(t,x)}(\eta)$.*

Sketch of proof. Since the proof of $\nu^1_{tx}(\xi) = \delta_{\overline{R}(t,x)}(\xi)$ is the same as that of $\nu^2_{tx}(\eta) = \delta_{\overline{S}(t,x)}(\eta)$, we present only the proof for the former.

The idea is to derive an evolution equation(inequality) for the quantity $\overline{R^2} - \overline{R}^2$, so that it is zero for all positive time if it is zero at time zero which is true in our case. In the derivation of the evolution equation we need to cut off desired multipliers and mollify various equations that are true only in the weak sense.

Step 1. Derivation of the equation for $\overline{R}$.

$$\partial_t \overline{R} - \partial_x(c(u)\overline{R}) = -\tilde{c}'(u)\left(\overline{R^2} - 2\overline{R}\,\overline{S} + \overline{S^2}\right). \tag{3.8}$$

Step 2. Cut-off of $(R^\varepsilon)^2$.
Let us define for $\lambda > 0$

$$T_\lambda(\xi) = \begin{cases} \xi, & |\xi| \leq \lambda \\ \lambda, & \xi \geq \lambda \\ -\lambda, & \xi \leq -\lambda, \end{cases} \qquad S_\lambda(\xi) = \begin{cases} \frac{1}{2}\xi^2, & |\xi| \leq \lambda \\ \lambda(\xi - \frac{\lambda}{2}), & \xi \geq \lambda \\ -\lambda(\xi + \frac{\lambda}{2}), & \xi \leq -\lambda. \end{cases}$$

We multiply the first equation of (2.2) with $T_\lambda(R^\varepsilon)$ and let $\varepsilon \to 0$ to obtain

$$\begin{aligned} &\partial_t \overline{S_\lambda(R)} - \partial_x(c(u)\overline{S_\lambda(R)}) \\ &= \tilde{c}'(u)\{-2\overline{RS_\lambda(R)} + \overline{T_\lambda(R)R^2} + 2\overline{S}\ \overline{S_\lambda(R)} - \overline{T_\lambda(R)}\ \overline{S^2}\}. \end{aligned} \tag{3.9}$$

Step 3. Cut-off of $\overline{R}^2$.

Similar procedure as step 2 yields

$$\begin{aligned} \partial_t S_\lambda(\overline{R}) - \partial_x(c(u)S_\lambda(\overline{R})) = \tilde{c}'(u)\{&-2\overline{R}S_\lambda(\overline{R}) + T_\lambda(\overline{R})\overline{R}^2 + 2\overline{S}S_\lambda(\overline{R}) \\ &-T_\lambda(\overline{R})\overline{S^2} - T_\lambda(\overline{R})(\overline{R^2} - \overline{R}^2)\}. \end{aligned} \tag{3.10}$$

Step 4. Evolution equation for "$\overline{R^2} - \overline{R}^2$".

By substracting (3.10) from (3.9), we find that

$$
\begin{aligned}
&\partial_t(\overline{S_\lambda(R)} - S_\lambda(\overline{R})) - \partial_x(c(u)(\overline{S_\lambda(R)} - S_\lambda(\overline{R}))) \\
&= \tilde{c}'(u)\Big\{(T_\lambda(\overline{R}) - \lambda)\,\overline{(R-\lambda)^2 1_{R\geq\lambda}} + (T_\lambda(\overline{R}) + \lambda)\,\overline{(R+\lambda)^2 1_{R\leq-\lambda}} \\
&\qquad + (T_\lambda(\overline{R}) - \overline{T_\lambda(R)})(\overline{S^2} - \lambda^2) + 2(\overline{S} + T_\lambda(\overline{R}))(\overline{S_\lambda(R)} - S_\lambda(\overline{R}))\Big\}.
\end{aligned} \tag{3.11}
$$

We comment that the term $(T_\lambda(\overline{R}) - \overline{T_\lambda(R)})\overline{S^2}$ is difficult. We will use renormalization to handle it. The term

$$
\begin{aligned}
G_\lambda := \tilde{c}'(u)\Big\{&(T_\lambda(\overline{R}) - \lambda)\,\overline{(R-\lambda)^2 1_{R\geq\lambda}} + (T_\lambda(\overline{R}) + \lambda)\,\overline{(R+\lambda)^2 1_{R\leq-\lambda}} \\
&-(T_\lambda(\overline{R}) - \overline{T_\lambda(R)})\lambda^2\Big\}
\end{aligned} \tag{3.12}
$$

will be shown to be nonpositive. The remaining term of product in (3.11) is not hard. In step 5 we do some preparation for handling the two difficult terms.

Step 5a.

$$
\frac{1}{2}\left(\overline{T_\lambda(R)} - T_\lambda(\overline{R})\right)^2 \leq \overline{S_\lambda(R)} - S_\lambda(\overline{R}). \tag{3.13}
$$

Step 5b. A convergence:

$$
\overline{T_\lambda(R)} - T_\lambda(\bar{R}) \to 0 \tag{3.14}
$$

as $\lambda \to \infty$.

Step 5c. Another inequality. Let $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}$ such that $c'(u(t_0, x_0)) > 0$, we claim that there is a neighborhood $\mathcal{N}(t_0, x_0)$ of (t_0, x_0) and some positive constant $M(t_0, x_0)$ such that whenever $\lambda \geq M(t_0, x_0)$, the term G_λ defined in (3.12) satisfies

$$
G_\lambda \leq 0, \quad (t,x) \in \mathcal{N}(t_0, x_0). \tag{3.15}
$$

Step 6a. Renormalization.

Let $f(t,x) =: \frac{1}{2}(\overline{R^2} - \overline{R}^2), g(t,x) =: \sqrt{f(t,x)}$. Then, we find

$$
\partial_t g - \partial_x(c(u)g) = \frac{1}{2}G + 2\tilde{c}'(u)\overline{S}g. \tag{3.16}
$$

Step 6b. Nonpositivity of G. Now let us assume that $c'(\cdot) \geq 0$ and $c'(u_0(\cdot)) > 0$. By Lemma 2.4 and the local property of the distributions, we find that

$$
G|_{(t_0,\infty)\times\mathbb{R}} \quad \text{is a distribution, and} \quad G|_{(t_0,\infty)\times\mathbb{R}} \leq 0, \tag{3.17}
$$

for any $t_0 > 0$.

By summing up (3.16), (3.17), we find

$$\partial_t g - \partial_x(c(u)g) \le 2\tilde{c}'(u)\overline{S}g, \quad (t_0,\infty)\times\mathbb{R}. \tag{3.18}$$

Step 6c. Re-renormalization.

$$\partial_t(\frac{g}{\sqrt{c(u)}}) - \partial_x(\sqrt{c(u)}g) \le 0, \quad \text{ in } (t_0,\infty)\times\mathbb{R}. \tag{3.19}$$

Step 6d. The precompactness.

By the energy inequality (3.1) and the proof of (6.39) in Zhang and Zheng [10], which imply that

$$\lim_{t\to 0}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}(\xi-\bar{R})^2\,d\nu_{tx}^1(\xi)+\int_{\mathbb{R}}(\eta-\bar{S})^2\,d\nu_{tx}^2(\eta)\right)\,dx=0. \tag{3.20}$$

Then formally integrating (3.19) over the space variable, and taking $t_0\to 0$, we obtain

$$g(t,x)=0, \quad \text{a.e.} \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}. \tag{3.21}$$

Hence $f(t,x)=0$ a.e. $(t,x)\in\mathbb{R}^+\times\mathbb{R}$ and therefore $\nu_{tx}^1(\xi)=\delta_{\bar{R}(t,x)}(\xi)$. Similarly, we can prove that $\nu_{tx}^2(\eta)=\delta_{\bar{S}(t,x)}(\eta)$. This completes the proof of the Lemma. □

Remark 3.1 *The assumptions that $c'(\cdot)\ge 0$ and $c'(u_0)>0$ were used only to prove that the distribution $G \le 0$ around the set $\{(t,x) : c'(u(t,x))=0\}$. Therefore, if we can prove this without the restriction, we can actually improve Theorem 1.1 for general wave speed $c(u)$ with $c'(\cdot)$ changing sign.*

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. Firstly, by second equation of (2.2),

$$\overline{R}=\partial_t u + c(u)\partial_x u, \tag{3.22}$$

while it is not difficult to prove that

$$\partial_t(\overline{R}-\overline{S})-\partial_x(c(u)(\overline{R}+\overline{S}))=0.$$

Substituting (3.22) to the above, we achieve

$$\partial_x(c(u)(2u_t-(\overline{R}+\overline{S}))=0,$$

that is

$$u_t=\frac{1}{2}(\overline{R}+\overline{S}) \tag{3.23}$$

Secondly, by Lemma 3.2, we take $\epsilon\to 0$ in (2.2) to get

$$\begin{cases}\partial_t\overline{R}-\partial_x(c(u)\overline{R})=-\tilde{c}'(u)(\overline{R}-\overline{S})^2,\\ \partial_t\overline{S}+\partial_x(c(u)\overline{S})=-\tilde{c}'(u)(\overline{R}-\overline{S})^2\end{cases} \tag{3.24}$$

hold in the sense of distributions. Summing up the two equations of (3.24) and using (3.22), (3.23), we find that there holds (1.7). This completes the proof of Theorem 1.1. □

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