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Abstract

We explore symplectic techniques to obtain long time estimates for a purely magnetic confinement in two degrees of freedom. Using pseudo-differential calculus, the same techniques lead to microlocal normal forms for the magnetic Laplacian. In the case of a strong magnetic field, we prove a reduction to a 1D semiclassical pseudo-differential operator. This can be used to derive precise asymptotic expansions for the eigenvalues at any order.

1. Introduction

This work is an example where geometric tools that are standard in the symplectic community, but not so familiar to analysts, shed a new light on an old problem in spectral theory, which is known for its subtleties and technicalities. The proofs of the results presented here are published in a joint article with Nicolas Raymond [10], and have benefited from essential ideas from Frédéric Faure.

1.1. The magnetic Laplacian

The object under study is the following selfadjoint unbounded operator on $L^2(\mathbb{R}^n)$,

$$\hat{H} = \sum_{j=1}^{n} \left(\frac{\hbar}{i} \frac{\partial}{\partial q_j} - a_j(q) \right)^2 ,$$

where $a_j \in C^\infty(\mathbb{R}^n)$. The functions a_j should be viewed as the components of a 1-form

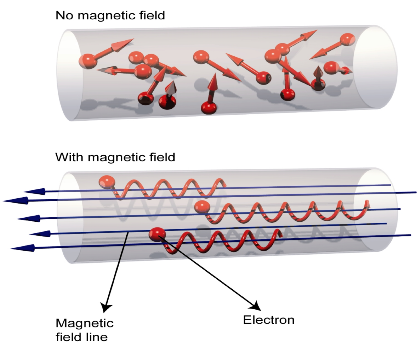
$$A = a_1 dq_1 + \cdots a_n dq_n,$$

and its exterior differential $B = dA$ is called the magnetic field. Thus it is a closed 2-form in $\mathbb{R}^n$. This framework can be extended to Riemannian manifolds, with the additional difficulty (in particular for quantization) that a closed 2-form is not necessarily exact. We hope to return to this question in a future research work.

Since B is supposed to be the relevant physical object, in some sense $\hat{H}$ should not depend on the choice of the "potential" A. This is reflected by the fact that, under unitary conjugation by $e^{if/\hbar}$ (where f is an arbitrary smooth, real-valued function), A is replaced by $A + df$ (hence this conjugation is just a gauge transformation).

1.2. Motivations

The need to motivate the study of magnetic fields may seem superfluous, especially from the author who is far from having a correct expertise in physics. However, what we might need to motivate is the semiclassical regime; indeed, the results that we are interested in here are valid only for strong magnetic fields. By this we mean that the norm $\|B\|$ is large or, equivalently, that the "semiclassical parameter" $\hbar$ is small. The operator $\hat{H}$ is the time-independent Schrödinger operator which describes the stationary states of a charged quantum particle, under the effect of a sole magnetic field (no electric field). This setting is appropriate for the study of systems where particles can be confined by a pure magnetic effect (the so-called magnetic bottles), and in general these devices need a very strong magnetic field. They can be found for instance in Tokamak experiments (Figure 1.2, right). The idea of confinement is intuitive when one thinks in terms of classical mechanics: a charged particle subject to a strong magnetic field is undergoing a very fast rotation with small radius which forces it to follow the magnetic field line (in 3D, the 2-form B can be identified with a vector field). See Figure 1.2, left. Thus, a natural way of confining is to ensure that magnetic lines are closed, as in the Tokamak torus. In two dimensions, as we shall see, this is much easier, since all field lines are closed, under a weak coercivity assumption on the norm of B.



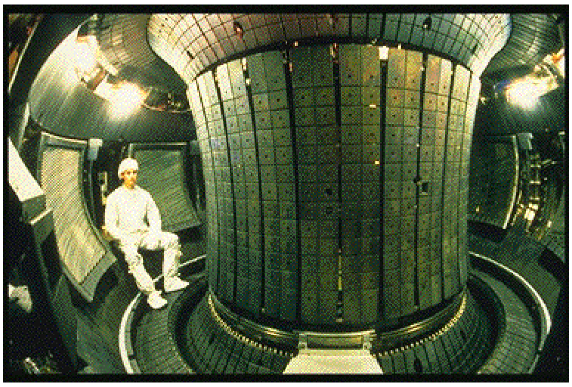


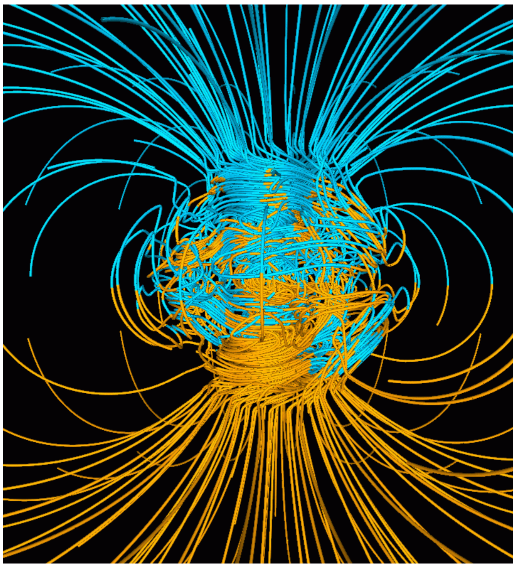
Figure 1.1: Quantum magnetic bottle (Magnetic confinement)

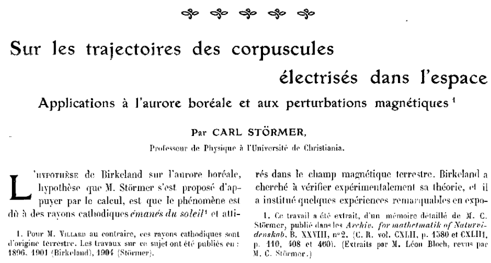
Magnetic bottles are not the only motivations for the study of $\hat{H}$. As is the case for many such simple operators, $\hat{H}$ appears in the linearization of more complicated equations, and in particular in the minimization problem for the Ginzburg-Landau functional for superconductors. This has been a strong incentive for many works on asymptotic expansions of small eigenvalues of $\hat{H}$, see [5].

Mathematics developments in semiclassical asymptotics have traditionally preferred the purely electric case $-\hbar^2\Delta+V$, because it has the nice and robust property that "mathematical confinement" (ie. discrete spectrum) is obtained directly at the level of the principal symbol $\xi^2 + V(x)$. This is never the case for pure magnetic

Laplacians, and in fact the latter have many (sometimes intriguing) similarities with hypoelliptic operators and problems with multiple characteristics, which have been intensively studied in the 1970's [11, 2, 7]. What's even more mysterious is that, in many cases, the study of magnetic Laplacian involves the norm of B as an *electric* field, living at the subprincipal level; this subprincipal term is responsible not only for discrete spectrum, but also for precise eigenvalue asymptotics. One of the goals of this work was to completely clarify this point in 2D for non-vanishing magnetic fields. For higher dimensions, or more general magnetic fields, the question is still widely open.

Let us end this introduction by mentioning that our techniques are actually also useful for another important question, namely the classical mechanical problem of the long time dynamics of charged particles. The classical Hamiltonian is simply $H(q,p) = \|p - A(q)\|^2$, where $(q,p) \in T^*X$, and X is a domain in $\mathbb{R}^n$, or a Riemannian manifold. The classical gauge transformations are the symplectic maps $p \mapsto p + df$. Motivated by the study of solar winds around the earth (in relation to the *Störmer problem* of *aurora borealis*: Figure 1.2), an important advance was made recently by Cheverry [4], where several scales of oscillations appear; we believe that they are related to our semiclassical parameter $\hbar$. It would certainly be instructive to see how both methods can profit to each other.





Sur les trajectoires des corpuscules électrisés dans l'espace

Applications à l'aurore boréale et aux perturbations magnétiques [1]

Par CARL STÖRMER,

Professeur de Physique à l'Université de Christiania.

L'hypothèse de Birkeland sur l'aurore boréale, hypothèse que M. Störmer s'est proposé d'appuyer par le calcul, est que le phénomène est dû à des rayons cathodiques *émanés du soleil* [1] et attirés dans le champ magnétique terrestre. Birkeland a cherché à vérifier expérimentalement sa théorie, et il a institué quelques expériences remarquables en expo-

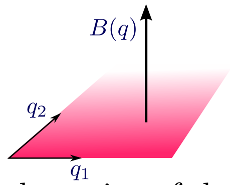
1. Pour M. Villard au contraire, ces rayons cathodiques sont d'origine terrestre. Les travaux sur ce sujet ont été publiés en : 1896, 1901 (Birkeland), 1904 (Störmer).

1. Ce travail a été extrait, d'un mémoire détaillé de M. C. Störmer, publié dans les *Archiv. for mathetmatik of Naturvidenskab*, B. XXVIII, n°2. (C. R. vol. CXLII, p. 1580 et CXLIII, p. 110, 408 et 460). (Extraits par M. Léon Bloch, revus par M. C. Störmer.)

Figure 1.2: Earth magnetic field lines (source: wikipedia) and Störmer's article

2. Results: classical mechanics

2.1. Classical dynamics for 2D magnetic fields



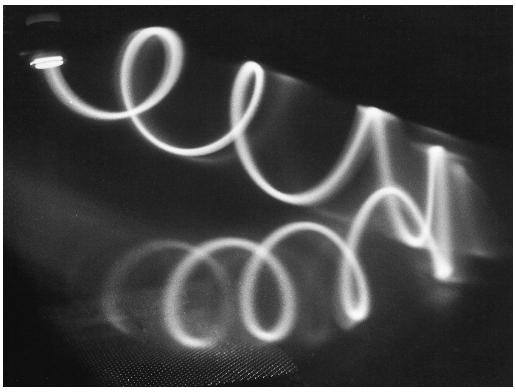
Let (e_1, e_2, e_3) be an orthonormal basis of $\mathbb{R}^3$. Our configuration space is $\mathbb{R}^2 = \{q_1 e_1 + q_2 e_2;\ (q_1, q_2) \in \mathbb{R}^2\}$, and the magnetic field is $\vec{B} = B(q_1, q_2)e_3$, $B \neq 0$.

We recall the well-known Newton equation for the particle under the action of the *Lorentz force*:

$$\ddot{q} = 2\dot{q} \wedge \vec{B}, \tag{2.1}$$

and it's standard to see that the *kinetic energy* $E = \frac{1}{4}\|\dot{q}\|^2$ is conserved.

If the speed $\dot{q}$ is small, we may 'linearize' the system, which amounts to have a *constant magnetic field*. In this case, the system can be solved explicitly and yields a circular motion of angular velocity $\dot{\theta} = -2B$ and radius $\|\dot{q}\|/2B$. Thus, even if the norm of the speed is small, the angular velocity may be very important. If B is in fact not constant, then after a while, the particle may leave the region where the linearization is meaningful. This suggests a *separation of scales, where the fast circular motion is superposed with a slow motion of the center*. This phenomenon is well known from physics, and is actually easy to observe, see Figure 2.1, or simulate numerically, see Figure 2.2. It will be the content of our first result, which we first state informally.



This photograph shows the motion of an electron beam in a non-uniform magnetic field. One can clearly see the fast rotation coupled with a drift. The turning point (here on the right) is called a *mirror point*. Credits: Prof. Reiner Stenzel, `http://www.physics.ucla.edu/plasma-exp/beam/BeamLoopyMirror.html`

Figure 2.1: Electron beam in a non-uniform magnetic field

Theorem 2.1 ([10]). *There exists a small energy $E_0 > 0$ such that, for all $E < E_0$, for times $t \leqslant T(E)$, the magnetic flow φ_H^t at kinetic energy $H = E$ is, up to an error of order $\mathcal{O}(E^\infty)$, the superposition of two commuting motions:*

- [fast rotating motion] *a periodic flow with frequency depending smoothly in E;*
- [slow drift] *a 1D-Hamiltonian flow in the "position variables" (q_1, q_2).*

Here the time $T(E)$ is, for regular starting point, of order $T(E) \sim 1/E^N$, for arbitrary $N > 0$. At a singular point, we have only the (more standard) estimate $T(E) \simeq |\ln E|$. Thus, we can informally describe the motion as a coupling between

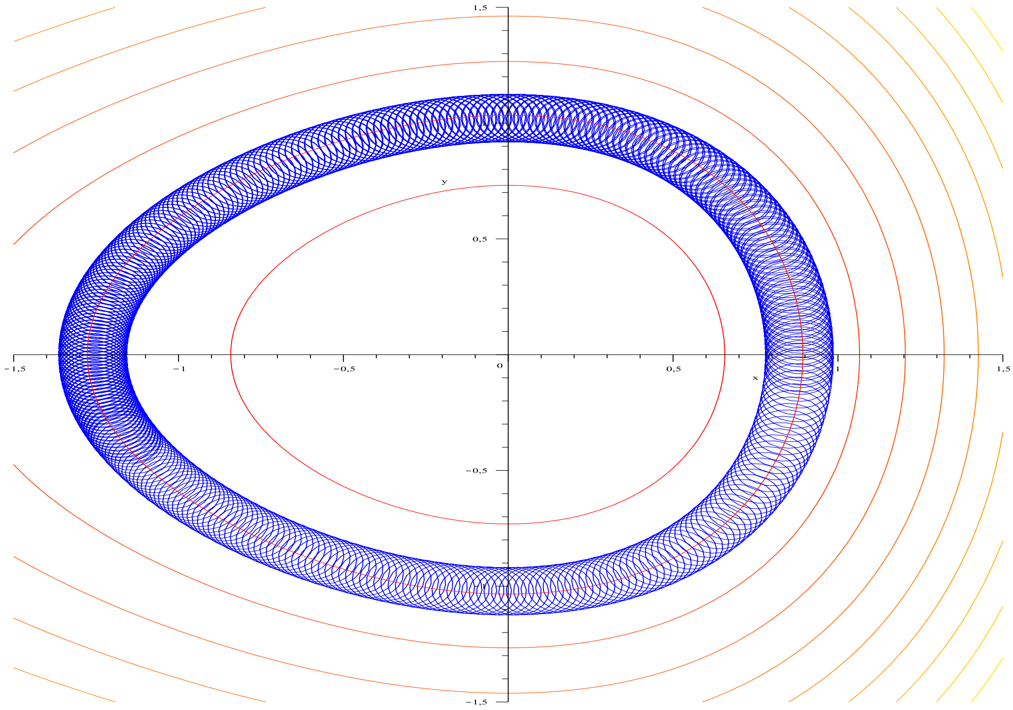


Figure 2.2: Numerical simulation of the flow of H when the magnetic field is given by $B(x,y) = 2+x^2+y^2+\frac{x^3}{3}+\frac{x^4}{20}$, and $\epsilon = 0.05$, $t \in [0, 500]$. The picture also displays in red some level sets of B.

a fast rotating motion around a center $c(t) \in H^{-1}(0)$ and a slow drift of the point $c(t)$. In the mathematical literature, this idea goes back (at least) to Arnol'd [1] where he used the averaging method to obtain times of order $1/E$. The idea to use formal Birkhoff normal forms to reach any order in E was proposed by Littlejohn [9], but he had to employ non-canonical transformations, which are difficult to leverage in the context of quantization. One of the key ideas of our work is to apply the Weinstein symplectic neighborhood theorem *first*, and *then* resort to standard Birkhoff normal form.

In order to make our result more precise, it is natural to split it into three steps:

- Theorem A: the normal form;
- Theorem B: the flow of the normal form, involving an effective (reduced) Hamiltonian;
- Theorem C: the justification of the approximation for long times.

Theorem 2.2 (A). *Let $\Omega \subset \mathbb{R}^2$ be an open set where B does not vanish. Then there exists a symplectic diffeomorphism Φ, defined in an open set $\tilde{\Omega} \subset \mathbb{C}_{z_1} \times \mathbb{R}^2_{z_2}$, with values in $T^*\mathbb{R}^2$, which sends the plane $\{z_1 = 0\}$ to the surface $\{H = 0\}$, and such that*

$$H \circ \Phi = |z_1|^2 f(z_2, |z_1|^2) + \mathcal{O}(|z_1|^\infty), \tag{2.2}$$

where $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is smooth. Moreover, the map

$$\varphi : \Omega \ni q \mapsto \Phi^{-1}(q, \mathbf{A}(q)) \in (\{0\} \times \mathbb{R}^2_{z_2}) \cap \tilde{\Omega} \tag{2.3}$$

is a local diffeomorphism and $f \circ (\varphi(q), 0) = |B(q)|$.

Once this is settled, the dynamics of the normal form is easy to compute:

Theorem 2.3 (B)**.** *The flow of the normal form* $K = |z_1|^2 f(z_2, |z_1|^2) \circ \Phi^{-1}$ *is completely integrable. It is the superposition of two motions:*

- [fast rotating motion in $z_1 \in \mathbb{R}^2$]*: rotation of angle* $\frac{\partial K}{\partial I} t$.
- [slow motion in $z_2 \in \mathbb{R}^2$] *the Hamiltonian flow of* $h_I(z_2) := K(z_2, I) = I f(z_2, I)$.

The quantity $I = |z_1|^2$ is called the *adiabatic invariant*. The effective Hamiltonian h_I can be seen as a (singular) symplectic reduction of H on $\Sigma := H{-}1(E)/S^1 = H^{-1}(0) = \mathbb{R}^2$. It remains to justify the approximations:

Theorem 2.4 (C)**.** *Assume that the magnetic field* $B > 0$ *is* confining*: there exists* $C > 0$ *and* $M > 0$ *such that* $B(q) \geqslant C$ *if* $\|q\| \geqslant M$. *Let* $C_0 < C$. *Then*

1. *The flow* φ_H^t *is uniformly bounded for all starting points* (q, p) *such that* $B(q) \leqslant C_0$ *and* $H(q, p) = \mathcal{O}(\epsilon)$ *and for times of order* $\mathcal{O}(1/\epsilon^N)$, *where* N *is arbitrary.*

2. *Up to a time of order* $T_\epsilon = \mathcal{O}(|\ln \epsilon|)$, *we have*
$$\left\|\varphi_H^t(q, p) - \varphi_K^t(q, p)\right\| = \mathcal{O}(\epsilon^\infty) \tag{2.4}$$
for all starting points (q, p) *such that* $B(q) \leqslant C_0$ *and* $H(q, p) = \mathcal{O}(\epsilon)$.

It is interesting to notice that, if one restricts to regular values of B, one obtains the same control for a much longer time, as stated below.

Theorem 2.5 (C bis)**.** *Under the same confinement hypothesis, let* $J \subset (0, C_0)$ *be a closed interval such that* dB *does not vanish on* $B^{-1}(J)$. *Then up to a time of order* $T = \mathcal{O}(1/\epsilon^N)$, *for an arbitrary* $N > 0$, *we have*
$$\left\|\varphi_H^t(q, p) - \varphi_K^t(q, p)\right\| = \mathcal{O}(\epsilon^\infty)$$
for all starting points (q, p) *such that* $B(q) \in J$ *and* $H(q, p) = \mathcal{O}(\epsilon)$.

The longer time $T = \mathcal{O}(1/\epsilon^N)$ perhaps also applies for some types of singularities of B; this seems to be an open question.

3. Results: quantum mechanics

We will proceed analogously to the classical case, by first stating an informal result, which we then split into three steps. The main idea is that, at first order, the spectral theory of $\hat{H} = \left\|\frac{\hbar}{i}\nabla - A(q)\right\|^2$ is governed by the magnetic field itself, viewed as a symbol on Σ. In order to use standard microlocal tools, we will assume that $\hat{H}$ belongs to a standard symbol class. In particular, A is smooth, which ensures that $\hat{H}$ is essentially self-adjoint.

Theorem 3.1 ([10])**.** *Assume the set $\{q \in \mathbb{R}^2; B(q) \leqslant C\}$ is compact. Then for any $C_1 < C$, the spectrum of $\hat{H}$ below $C_1\hbar$ is discrete and is given, modulo $\mathcal{O}(\hbar^\infty)$, by the union of the sets*

$$\sigma_n(\hbar) \subset \mathbb{R}, \qquad n \in \mathbb{N}$$

where for each n, σ_n is the spectrum of a 1D *semiclassical pseudo-differential operator with principal symbol $\hbar(2n+1)B(q)$*

The integer n is naturally called the 'quantum adiabatic invariant'; the numbers $\hbar(2n+1)$, $n \in \mathbb{N}$ form the eigenvalues of $\hat{I}$. The more precise statements below follow the same logic as the classical mechanical case:

- Theorem Â: microlocal quantum normal form
- Theorem $\hat{\text{B}}$: spectrum of the normal form, effective quantum Hamiltonian
- Theorem $\hat{\text{C}}$: justification of the approximation up to any order in $\hbar$.

3.1. Microlocal normal form

The result is a combination of Weinstein tubular neighborhood theorem and a microlocal Birkhoff normal form in the z_1 variable. Similar techniques can be found in [12], [3], and [8].

Theorem 3.2 (Â)**.** *For $\hbar$ small enough there exists a Fourier Integral Operator $U_\hbar$ such that*

$$U_\hbar^* U_\hbar = I + Z_\hbar, \qquad U_\hbar U_\hbar^* = I + Z'_\hbar,$$

where $Z_\hbar, Z'_\hbar$ are pseudo-differential operators that microlocally vanish in a neighborhood of $\tilde{\Omega} \cap \Sigma$, and

$$U_\hbar^* \hat{H} U_\hbar = \mathcal{N}_\hbar + \hat{\mathcal{O}}(\hbar^\infty), \tag{3.1}$$

where $\mathcal{N}_\hbar$ is a classical pseudo-differential operator in $S(m)$ that commutes with $\mathcal{I}_\hbar := -\hbar^2 \frac{\partial^2}{\partial x_1^2} + x_1^2$.

3.2. Effective Quantum Hamiltonian

We want to understand the normal form $\mathcal{N}_\hbar$. Geometers will recognize in the theorem below a variant of the "quantization commutes with reduction" precept, extended to the level of quantum operators.

Theorem 3.3 ($\hat{\text{B}}$)**.** 1. *For any Hermite function $h_n(x_1)$ such that $\mathcal{I}_\hbar h_n = \hbar(2n-1)h_n$, the operator $\mathcal{N}_\hbar^{(n)}$ acting on $L^2(\mathbb{R}_{x_2})$ by*

$$h_n \otimes \mathcal{N}_\hbar^{(n)}(u) = \mathcal{N}_\hbar(h_n \otimes u)$$

is a semiclassical pseudo-differential operator in $\mathbb{R}^2$ with principal symbol *$N^{(n)}(x_2, \xi_2) = \hbar(2n+1)B(q)$;*

2. *Therefore, the spectrum of $\mathcal{N}_\hbar$ is the union of the spectra $\sigma_n(\hbar)$ of all $\mathcal{N}_\hbar^{(n)}$, $n \in \mathbb{N}$.*

3.3. Quantum spectrum

Theorem 3.4. *Assume that the magnetic field B is confining and non vanishing. Then the spectrum $\lambda_1(\hbar) \leqslant \lambda_2(\hbar) \leqslant \cdots$ of $\hat{H}$ below $C_1\hbar$ is 'almost the same' as the spectrum $\mu_1(\hbar) \leqslant \mu_2(\hbar) \leqslant \cdots$ of the normal form $\mathcal{N}_\hbar$, i.e.:*

$$|\lambda_j(\hbar) - \mu_j(\hbar)| = O(\hbar^\infty),$$

for all j such that $\lambda_j(\hbar) \leqslant C_1\hbar$ or $\mu_j(\hbar) \leqslant C_1\hbar$.

This statement improves (and hopefully clarifies) several results in the literature (esp. recent works by Helffer-Kordyukov[6]). Indeed, by specializing to the case where B admits a non-degenerate minimum, we obtain a full asymptotic expansions of low eigenvalues in integer powers of $\hbar$:

Corollary 3.5. *Assume that B has a unique* non-degenerate minimum. *Then there exists a constant c_0 such that for any j, the eigenvalue $\lambda_j(\hbar)$ has a full asymptotic expansion in integral powers of $\hbar$ whose first terms have the following form:*

$$\lambda_j(\hbar) \sim \hbar \min B + \hbar^2(c_1(2j-1) + c_0) + O(\hbar^3),$$

with $c_1 = \frac{\sqrt{\det(B"\circ\varphi^{-1}(0))}}{2B\circ\varphi^{-1}(0)}$, where the minimum of B is reached at $\varphi^{-1}(0)$.

But, since the reduced Hamiltonian is 1D, we may also explore higher energies (Magnetic excited states):

Corollary 3.6. *Let c be a* regular value *of B, and assume that the level set $B^{-1}(c)$ is connected. Then there exists $\epsilon > 0$ such that the eigenvalues of the magnetic Laplacian in the interval $[\hbar(c-\epsilon), \hbar(c+\epsilon)]$ have the form*

$$\lambda_j(\hbar) = (2n-1)\hbar f_\hbar(\hbar n(j), \hbar k(j)) + O(\hbar^\infty), \quad (n(j), k(j)) \in \mathbb{Z}^2,$$

where $f_\hbar = f_0 + \hbar f_1 + \cdots$, $f_i \in C^\infty(\mathbb{R}^2;\mathbb{R})$ and $\partial_1 f_0 = 0$, $\partial_2 f_0 \neq 0$. Moreover, the corresponding eigenfunctions are microlocalized in the annulus $B^{-1}([c-\epsilon, c+\epsilon])$.

In particular, if $c \in (\min B, 3\min B)$, the eigenvalues of the magnetic Laplacian in the interval $[\hbar(c-\epsilon), \hbar(c+\epsilon)]$ have gaps of order $O(\hbar^2)$.

4. Ideas of the proof

In this last part of the paper, we try to give a flavor of the proofs. Of course, details can be found in [10].

4.1. The Hamiltonian setting

It is known that the Lorentz system (2.1) is Hamiltonian. In terms of canonical variables $(q,p) \in T^*\mathbb{R}^2 = \mathbb{R}^4$ the Hamiltonian is the kinetic energy:

$$H(q,p) = \|p - A(q)\|^2 . \tag{4.1}$$

We use here the Euclidean norm on $\mathbb{R}^2$, which allows the identification of $\mathbb{R}^2$ with $(\mathbb{R}^2)^*$ by

$$\forall (v,p) \in \mathbb{R}^2 \times (\mathbb{R}^2)^*, \qquad p(v) = \langle p, v\rangle. \tag{4.2}$$

Thus, the canonical symplectic structure ω on $T^*\mathbb{R}^2$ is given by

$$\omega((Q_1,P_1),(Q_2,P_2)) = \langle P_1, Q_2\rangle - \langle P_2, Q_1\rangle. \tag{4.3}$$

It is easy to check that Hamilton's equations for H imply Newton's equation (2.1). In particular, through the identification (4.2) we have $\dot{q} = 2(p - A)$.

4.2. A symplectic submanifold

Of particular importance for our study will be the submanifold of all particles at rest ($\dot{q} = 0$):

$$\Sigma := H^{-1}(0) = \{(q, p); \qquad p = A(q)\}.$$

Since it is a graph, it is an embedded submanifold of $\mathbb{R}^4$, parameterized by $q \in \mathbb{R}^2$.

Lemma 4.1. *Σ is a symplectic submanifold of $\mathbb{R}^4$. In fact,*

$$j^*\omega_{\restriction\Sigma} = dA \simeq B,$$

where $j : \mathbb{R}^2 \to \Sigma$ is the embedding $j(q) = (q, A(q))$.

Proof. We compute $j^*\omega = j^*(dp_1 \wedge dq_1 + dp_2 \wedge dq_2) = (-\frac{\partial A_1}{\partial q_2} + \frac{\partial A_2}{\partial q_1})dq_1 \wedge dq_2 \neq 0$. □

4.3. The symplectic normal bundle

We wish now to describe a small neighborhood of Σ in $\mathbb{R}^4$, which amounts to understanding the *symplectic normal bundle* of Σ. (Weinstein, 1971 [13]). We note that the *tangent bundle* is easy to find, because $\Sigma = \{(q, A(q)); \quad q \in \mathbb{R}^2\}$, which implies $T_{j(q)}\Sigma = \text{span}\{(Q, T_q\mathbf{A}(Q)); \quad Q \in \mathbb{R}^2\}$.

Lemma 4.2. *For any $q \in \Omega$, a symplectic basis of $T_{j(q)}\Sigma^\perp$ is:*

$$u_1 := \frac{1}{\sqrt{|B|}}(e_1, {}^tT_q\mathbf{A}(e_1)); \quad v_1 := \frac{\sqrt{|B|}}{B}(e_2, {}^tT_q\mathbf{A}(e_2))$$

Proof. Let $(Q_1, P_1) \in T_{j(q)}\Sigma$ and (Q_2, P_2) with $P_2 = {}^tT_q\mathbf{A}(Q_2)$. Then $\omega((Q_1, P_1), (Q_2, P_2)) = \langle T_q\mathbf{A}(Q_1), Q_2\rangle - \langle {}^tT_q\mathbf{A}(Q_2), Q_1\rangle = 0$.

The other terms are treated similarly. □

4.4. The transverse Hessian

Once we understand the linearized geometry near the surface of particles at rest, we may describe the Hamiltonian in this region. By its very nature, H is quadratic in the transverse variables; the transverse Hessian is the relevant well-defined object of study. A small calculation gives:

Lemma 4.3. *The transverse Hessian of H, as a quadratic form on $T_{j(q)}\Sigma^\perp$, is given by*

$$\forall q \in \Omega, \forall (Q, P) \in T_{j(q)}\Sigma^\perp, \quad d_q^2H((Q, P)^2) = 2\|Q \wedge \vec{B}\|^2.$$

It is remarkable that this Hessian is diagonal in the symplectic basis (u_1, v_1) given by the previous lemma:

$$d^2H_{\restriction T_{j(q)}\Sigma^\perp} = \begin{pmatrix} 2\,|B| & 0 \\ 0 & 2\,|B| \end{pmatrix}. \tag{4.4}$$

Indeed, $\|e_1 \wedge \vec{B}\|^2 = B^2$, and the off-diagonal term is $\frac{1}{B}\langle e_1 \wedge \vec{B}, e_2 \wedge \vec{B}\rangle = 0$.

4.5. The Weinstein theorem

We turn now to the "technical" geometric part, for which we rely on the Weinstein theorem. We endow $\mathbb{C}_{z_1} \times \mathbb{R}^2_{z_2}$ with canonical variables $z_1 = x_1 + i\xi_1$, $z_2 = (x_2, \xi_2)$, and symplectic form $\omega_0 := d\xi_1 \wedge dx_1 + d\xi_2 \wedge dx_2$. By Darboux theorem, there exists a diffeomorphism $g : \Omega \to g(\Omega) \subset \mathbb{R}^2_{z_2}$ such that $g(q_0) = 0$ and $g^*(d\xi_2 \wedge dx_2) = j^*\omega$. In other words, the new embedding $\tilde{j} := j \circ g^{-1} : \mathbb{R}^2 \to \Sigma$ is symplectic. Consider

$$\mathbb{C} \times \Omega \xrightarrow{\tilde{\Phi}} N\Sigma$$

$$(x_1 + i\xi_1, z_2) \mapsto x_1 u_1(z_2) + \xi_1 v_1(z_2),$$

where $q = g^{-1}(z_2)$. What we have constructed so far shows that this is an isomorphism between the normal symplectic bundle of $\{0\} \times \Omega$ and $N\Sigma$, the normal symplectic bundle of Σ (for fixed z_2, the map $z_1 \mapsto \tilde{\Phi}(z_1, z_2)$ is a linear symplectic map). Then the Weinstein theorem implies the existence of a symplectomorphism Φ from a neighborhood of $\{0\} \times \Omega$ to a neighborhood of $\tilde{j}(\Omega) \subset \Sigma$ whose differential at $\{0\} \times \Omega$ is equal to $\tilde{\Phi}$.

4.6. The transformed Hamiltonian

We may now transport the Hamiltonian into these new coordinates. The zero-set $\Sigma = H^{-1}(0)$ is now $\{0\} \times \Omega$, and the symplectic orthogonal $T_{\tilde{j}(0,z_2)}\Sigma^\perp$ is canonically equal to $\mathbb{C} \times \{z_2\}$. By (4.4), the matrix of the transverse Hessian of $H \circ \Phi$ in the canonical basis of $\mathbb{C}$ is simply $d^2(H \circ \Phi)_{\restriction \mathbb{C}\times\{z_2\}} =$

$$= d^2_{\Phi(0,z_2)} H \circ (d\Phi)^2 = \begin{pmatrix} 2\,|B(g^{-1}(z_2))| & 0 \\ 0 & 2\,|B(g^{-1}(z_2))| \end{pmatrix}. \tag{4.5}$$

Therefore, by Taylor's formula in the z_1 variable (locally uniformly with respect to the z_2 variable seen as a parameter), we get $H \circ \Phi(z_1, z_2) = = H \circ \Phi_{\restriction z_1=0} + dH \circ \Phi_{\restriction z_1=0}(z_1) + \frac{1}{2} d^2(H \circ \Phi)_{\restriction z_1=0}(z_1^2) + \mathcal{O}(|z_1|^3)$
$= 0 + 0 + |B(g^{-1}(z_2))|\,|z_1|^2 + \mathcal{O}(|z_1|^3)$.

4.7. Semiclassical Birkhoff normal form

We see that we are now in a perfect situation for performing a Birkhoff normal for in the z_1 variable. Of course, this would have been very delicate to try this *before* the non-linear transformation given by Weinstein coordinates.

We may actually write directly a *semiclassical* Birkhoff normal form, *ie* we add the formal parameter $\hbar$. Recall that

$$H(z_1, z_2) = H^0 + O(|z_1|^3), \text{ where } H^0 = B(g^{-1}(z_2))|z_1|^2.$$

Consider the space of the formal power series in $\hat{x}_1, \hat{\xi}_1, \hbar$ with coefficients smoothly depending on $(\hat{x}_2, \hat{\xi}_2)$: $\mathcal{E} = \mathcal{C}^\infty_{\hat{x}_2,\hat{\xi}_2}[\hat{x}_1, \hat{\xi}_1, \hbar]$. We endow $\mathcal{E}$ with the Moyal product (compatible with the Weyl quantization), and define the appropriate grading:

> The degree of $\hat{x}_1^\alpha \hat{\xi}_1^\beta \hbar^l$ is $\alpha + \beta + 2l$. $\mathcal{D}_N$ denotes the space of the monomials of degree N. $\mathcal{O}_N$ is the space of formal series with valuation at least N.

This leads to the following proposition:

Proposition 4.4. *Given $\gamma \in \mathcal{O}_3$, there exist formal power series $\tau, \kappa \in \mathcal{O}_3$ such that:*

$$e^{i\hbar^{-1}\mathrm{ad}_\tau}(H^0 + \gamma) = H^0 + \kappa,$$

with: $[\kappa, |z_1|^2] = 0$.

This essentially proves Theorem Â, using the famous Egorov theorem for quantizing canonical transformations into semiclassical Fourier integral operators. Of course, in order to derive Theorem Ĉ from Theorem B̂, some precise microlocal estimates that we skip here are needed, but they are not sufficient. We also need an a priori estimate on the number of eigenvalues in the magnetic well, which can be obtained by a Lieb-Thirring argument.

4.8. Open questions

The most pregnant problem is of course the extension to three dimensions. In 3D, magnetic confinement is much more subtle, because it involves the direction of the magnetic field B (recall that in 3D, B can be viewed as a vector field). Works in progress with Colin de Verdière, Helffer, Kordyukov and Raymond give some hope, but there are intrinsic difficulties with the mixing of different scales. This problem was solved by Cheverry [4] in the classical mechanical case using non-canonical transformations, which is a no-show at this moment for an extension to the quantum setting.

Other interesting problems should be investigated when the magnetic field vanishes (presymplectic case), and when the configuration space is a Riemannian manifold of even dimension (symplectic case). Concerning the latter, there is a work in progress by Faure, on which we expect to return in a future work.

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