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Abstract

In this note we describe recent results on semiclassical random walk associated to a probability density which may also concentrate as the semiclassical parameter goes to zero. The main result gives a spectral asymptotics of the close to 1 eigenvalues. This problem was studied in [1] and relies on a general factorization result for pseudo-differential operators. In this note we just sketch the proof of this second theorem. At the end of the note, using the factorization, we give a new proof of the spectral asymptotics based on some comparison argument.

1. Introduction

Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a smooth function and let $h \in]0, 1]$ denote a small parameter in all the paper. Under suitable assumptions specified later, the density $e^{-\phi(x)/h}$ is integrable and there exists $Z_h > 0$ such that $d\mu_h(x) = Z_h e^{-\phi(x)/h}dx$ defines a probability measure on $\mathbb{R}^d$. We can associate to μ_h the Markov kernel $t_h(x, dy)$ given by

$$t_h(x, dy) = \frac{1}{\mu_h(B(x, h))} \mathbb{1}_{|x-y|<h} d\mu_h(y). \tag{1.1}$$

From the point of view of random walks, this kernel can be understood as follows: assume at step n, the walk is in x_n, then the point x_{n+1} is choosen in the small ball $B(x_n, h)$, uniformly at random with respect to $d\mu_h$. The probability distribution at time $n \in \mathbb{N}$ of a walk starting from x is given by the kernel $t_h^n(x, dy)$. The long time behavior ($n \to \infty$) of the kernel $t_h^n(x, dy)$ carries informations on the ergodicity of the random walk, and has many practical applications (we refer to [11] for an overview of computational aspects). Observe that if ϕ is a Morse function, then the density $e^{-\phi/h}$ concentrates at scale $\sqrt{h}$ around minima of ϕ, whereas the moves of the random walk are at scale h.

Another point of view comes from statistical physics and can be described as follows. One can associate to the kernel $t_h(x, dy)$ an operator $\mathbf{T}_h$ acting on the

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space C_0 of continuous functions going to zero at infinity, by the formula

$$\mathbf{T}_h f(x) = \int_{\mathbb{R}^d} f(y) t_h(x, dy) = \frac{1}{\mu_h(B(x,h))} \int_{|x-y|<h} f(y) d\mu_h(y).$$

This defines a bounded operator on C_0, enjoying the Markov property ($\mathbf{T}_h(1) = 1$).

The transpose $\mathbf{T}_h^\star$ of $\mathbf{T}_h$ is defined by duality on the set of bounded positive measures $\mathbf{M}_b^+$ (resp. bounded measures $\mathbf{M}_b$). If $d\nu$ is a bounded measure we have

$$\mathbf{T}_h^\star(d\nu) = \left(\int_{\mathbb{R}^d} \mathbb{1}_{|x-y|<h} \mu_h(B(y,h))^{-1} d\nu(y) \right) d\mu_h. \tag{1.2}$$

Assume that a particle in $\mathbb{R}^d$ is distributed according to a probability measure $d\nu$, then $\mathbf{T}_h^\star(d\nu)$ represents its distribution after a move according to $t_h(x, dy)$, and the distribution after n steps is then given by $(\mathbf{T}_h^\star)^n(d\nu)$. The existence of a limit distribution is strongly related to the existence of an invariant measure. In the present context, one can easily see that $\mathbf{T}_h^\star$ admits the following invariant measure

$$d\nu_{h,\infty}(x) = \widetilde{Z}_h \mu_h(B(x,h)) d\mu_h(x),$$

where $\widetilde{Z}_h$ is chosen so that $d\nu_{h,\infty}$ is a probability. The aim of the present paper will be to prove the convergence of $(\mathbf{T}_h^\star)^n(d\nu)$ towards $d\nu_{h,\infty}$ when n goes to infinity, for any probability measure $d\nu$, and to get precise informations on the speed of convergence.

Before going further, let us recall some elementary properties of $\mathbf{T}_h$ that will be usefull in the sequel. First, we can see easily from its definition that the operator $\mathbf{T}_h$ can be extended as a bounded operator both on $L^\infty(d\nu_{h,\infty})$ and $L^1(d\nu_{h,\infty})$. From the Markov property and the fact that $d\nu_{h,\infty}$ is stationary it is clear that

$$\|\mathbf{T}_h\|_{L^\infty(d\nu_{h,\infty}) \to L^\infty(d\nu_{h,\infty})} = \|\mathbf{T}_h\|_{L^1(d\nu_{h,\infty}) \to L^1(d\nu_{h,\infty})} = 1.$$

Hence, by interpolation $\mathbf{T}_h$ defines also a bounded operator of norm 1 on $L^2(\mathbb{R}^d, d\nu_{h,\infty})$. Finally, observe that $\mathbf{T}_h$ is selfadjoint on $L^2(d\nu_{h,\infty})$ (thanks again to Markov property).

Let us go back to the study of the sequence $(\mathbf{T}_h^\star)^n$ and explain the topology we use to study the convergence of this sequence. Instead of looking at this evolution on the full set of bounded measures, we restrict the analysis by introducing the following stable Hilbert space

$$\mathcal{H}_h = L^2(d\nu_{h,\infty}) = \left\{ f \text{ measurable on } \mathbb{R}^d \text{ such that } \int |f(x)|^2 \, d\nu_{h,\infty} < \infty \right\}. \tag{1.3}$$

for which we have a natural injection with norm 1, $\mathcal{J} : \mathcal{H}_h \hookrightarrow \mathbf{M}_b$, when identifying an absolutely continuous measure $d\nu_h = f(x) d\nu_{h,\infty}$ with its density f. Using (1.2), we can see easily that $\mathbf{T}_h^\star \circ \mathcal{J} = \mathcal{J} \circ \mathbf{T}_h$. From this identification $\mathbf{T}_h^\star$ (acting on $\mathcal{H}_h$) inherits the properties of $\mathbf{T}_h$:

$$\mathbf{T}_h^\star : \mathcal{H}_h \longrightarrow \mathcal{H}_h \text{ is selfadjoint and continuous with operator norm 1.} \tag{1.4}$$

Hence, its spectrum is contained in the interval $[-1, 1]$. Moreover, we will see later that -1 is sufficiently far from the spectrum. Since we are interested in the convergence of $(\mathbf{T}_h^\star)^n$ in L^2 topology, it is then sufficient for our purpose to give a precise description of the spectrum of $\mathbf{T}_h$ near 1.

Let us now make some precise assumptions on the function ϕ.

Hypothesis 1. *We suppose that ϕ is a smooth function and that there exists $c, R > 0$ and some constants $C_\alpha > 0$, $\alpha \in \mathbb{N}^d$ such that for all $|x| \geq R$, we have*

$$\forall \alpha \in \mathbb{N}^d \setminus \{0\}, \qquad |\partial_x^\alpha \phi(x)| \leq C_\alpha, \quad |\nabla \phi(x)| \geq c \quad \textit{and} \quad |\phi(x)| \geq c|x|.$$

Observe that the above assumption insures that $d\mu_h(x) = Z_h e^{-\phi(x)/h} dx$ is a probability measure.

As we will see later, the spectral analysis of the operator T_h has many common points with the study of semiclassical Witten Laplacien on functions $P^{W,(0)} = -h^2\Delta + |\nabla\phi|^2 - h\Delta\phi$. Under the above assumptions it is wellknown (see [7] and references given there) that $P^{W,(0)}$ has compact resolvant. In the following we will denote by $(\mu_k(h))$ the increasing sequence of eigenvavalues of $P^{W,(0)}$. In the case where ϕ is a Morse function one can show easily that its eigenvalues in any interval of the form $[0, o(h)]$ are in fact exponentially small (see [5]). More recently, under some generic additional assumption a complete asymptotic expansion was proved [2], [6]. In the same situation we studied in [1] the operator T_h. Here we would like to give a less precise statement in a more general situation. The following result holds true without assuming that ϕ is a Morse function:

Theorem 1.1. *Assume that Hypothesis 1 holds true. There exist $\delta, h_0 > 0$ such that for $h \in]0, h_0]$ we have $\sigma(\mathbf{T}_h^\star) \subset [-1+\delta, 1]$, $\sigma_{ess}(\mathbf{T}_h^\star) \subset [-1+\delta, 1-\delta]$ and 1 is a simple eigenvalue for the eigenstate $\nu_{h,\infty} \in \mathcal{H}_h$.*

Moreover, denoting $(\lambda_k(h))$ the decreasing sequence of eigenvalues of T_h and given $\nu(h)$ such that $0 < \nu(h) \to 0$ when $h \to 0$, we have

$$\lambda_k(h) = \frac{\mu_k(h)}{2d+4}(1 + o_{h\to 0}(1)), \tag{1.5}$$

uniformly with respect to k such that $|\mu_k(h)| \leq \nu(h)$

Let us now give some corollary of this result when ϕ satisfies additional assumptions. In the following, we will denote by $\mathcal{U}$ the set of critical points of ϕ.

Hypothesis 2. *We suppose that ϕ is a Morse function.*

When Hypotheses 1 and 2 are satisfied, the set $\mathcal{U}$ is finite. We denote by $\mathcal{U}^{(0)}$ the set of minima of ϕ and $\mathcal{U}^{(1)}$ the set of saddle points, i.e. the critical points with index 1 (note that this set may be empty). We also introduce $n_j = \sharp\mathcal{U}^{(j)}$, $j = 0, 1$, the number of elements of $\mathcal{U}^{(j)}$.

Hypothesis 3. *We suppose that the values $\phi(\mathbf{s}) - \phi(\mathbf{m})$ are distinct for any $\mathbf{s} \in \mathcal{U}^{(1)}$ and $\mathbf{m} \in \mathcal{U}^{(0)}$.*

Let us recall that under the above assumptions, there exists a labeling of minima and saddle points: $\mathcal{U}^{(0)} = \{\mathbf{m}_k;\ k = 1, \ldots, n_0\}$ and $\mathcal{U}^{(1)} = \{\mathbf{s}_j;\ j = 2, \ldots, n_1 + 1\}$ which permits to describe the low liying eigenvalues of the Witten Laplacian (see [6], [9] for instance). Observe that the enumeration of $\mathcal{U}^{(1)}$ starts with $j = 2$ since we will need a fictive saddle point $\mathbf{s}_1 = +\infty$.

From Theorem 1.1 and the asymptotic expansion of the $\mu_k(h)$ proved in Theorem 5.1 of [6], we deduce the following

Theorem 1.2. *Under Hypotheses 1, 2 and 3, there exist some constants $\alpha, h_0 > 0$ such that, for all $k = 2, \dots, n_0$ and for any $h \in]0, h_0]$,*

$$1 - \lambda_k(h) = \frac{h}{(2d+4)\pi}\theta_k \sqrt{\left|\frac{\det \phi''(\mathbf{m}_k)}{\det \phi''(\mathbf{s}_k)}\right|} e^{-2S_k/h}(1 + o(1)),$$

where $S_k := \phi(\mathbf{s}_k) - \phi(\mathbf{m}_k)$ (Aarhenius number) and $-\theta_k$ denotes the unique negative eigenvalue of ϕ'' at $\mathbf{s}_k$.

Observe that this theorem is very close to Theorem 1.2 in [1]. The only difference is that the error term here is $o(1)$ whereas it is $\mathcal{O}(h)$ in [1]. Since the proof to get the $o(1)$ error term is completely different and easyer, we decided to state and prove here the weakest version. However, a proof of asymptotics with error term equal to $\mathcal{O}(h)$ can be found in [1].

As an immediate consequence of Theorem and of the spectral theorem, we get that the convergence to equilibrium holds slowly and that the system has a metastable regime.

Corollary 1.3. *Let $d\nu_h$ be probability measure in $\mathcal{H}_h$ and assume first that ϕ has a unique minimum. Then, using that $\sigma(\mathbf{T}_h^\star) \subset [-1 + \delta, 1 - \delta h]$, it yields*

$$\left\|(\mathbf{T}_h^\star)^n(d\nu_h) - d\nu_{h,\infty}\right\|_{\mathcal{H}_h} = \mathcal{O}(h)\|d\nu_h\|_{\mathcal{H}_h}. \tag{1.6}$$

for all $n \gtrsim |\ln h| h^{-1}$ which corresponds to the Ehrenfest time. But, if ϕ has now several minima, we can write

$$(\mathbf{T}_h^\star)^n(d\nu_h) = \Pi d\nu_h + \mathcal{O}(h)\|d\nu_h\|_{\mathcal{H}_h}, \tag{1.7}$$

for all $h^{-1}|\ln h| \lesssim n \lesssim e^{2S_{n_0}/h}$. Here, Π can be taken as the orthogonal projector on the n_0 functions $\chi_k(x)e^{-(\phi(x)-\phi(\mathbf{m}_k))/h}$ where χ_k is any cutoff function near $\mathbf{m}_k$.

On the other hand, we have, for any $n \in \mathbb{N}$,

$$\left\|(\mathbf{T}_h^\star)^n(d\nu_h) - d\nu_{h,\infty}\right\|_{\mathcal{H}_h} \leq (\lambda_2(h))^n\|d\nu_h\|_{\mathcal{H}_h}, \tag{1.8}$$

where $\lambda_2(h)$ is described in Theorem 1.2. Note that this inequality is optimal. In particular, for $n \gtrsim |\ln h| h^{-1} e^{2S_2/h}$, the right hand side of (1.8) *is of order $\mathcal{O}(h)\|d\nu_h\|_{\mathcal{H}_h}$.*

Thus, for a reasonable number of iterations (which guaranties (1.6)*), 1 seems to be an eigenvalue of multiplicity n_0; whereas, for a very large number of iterations, the system returns to equilibrium. Then,* (1.7) *is a metastable regime.*

Let us now explain how Theorem 1.1 can be used to get some information in the case where ϕ is not necessary a Morse function. For instance, suppose that the space dimension d is equal to 1. Assume that ϕ has a unique degenerate critical point (say in $x = 0$) and that near the origin we have $\phi(x) = \frac{\alpha}{a}x^a + \mathcal{O}(x^{a+1})$ for some $a \in \mathbb{N}^*$ (observe that a is necessarily even since $e^{-\phi/h}$ is integrable). Using localization technics as in [13] we can prove that the spectral gap is asymptotically equal to $\frac{1}{6}h^{2-\frac{2}{a}}\nu_1$, where ν_1 denotes the first non-zero eigenvalue of the operator

$$N_{a,\alpha} = -\partial_y^2 + \alpha^2 y^{2a-2} - \alpha(a-1)y^{a-2}.$$

In the case case where ϕ has several critical points and that only one of them is degenerate and of the preceding form, we could also get an asymptotics of the exponentially small eigenvalues.

Throughout this paper, we use semiclassical analysis (see [4], [12], or [15] for expository books of this theory). Let us recall that a function $m : \mathbb{R}^d \to \mathbb{R}^+$ is an order function if there exists $N_0 \in \mathbb{N}$ and a constant $C > 0$ such that for all $x, y \in \mathbb{R}^d$, $m(x) \leq C\langle x-y\rangle^{N_0} m(y)$. This definition can be extended to functions $m : \mathbb{R}^d \times \mathbb{C}^{d'} \to \mathbb{R}^+$ by identifying $\mathbb{R}^d \times \mathbb{C}^{d'}$ with $\mathbb{R}^{d+2d'}$. Given on order function m on $T^*\mathbb{R}^d \simeq \mathbb{R}^{2d}$, we will denote by $S^0(m)$ the space of semiclassical symbols on $T^*\mathbb{R}^d$ whose all derivatives are bounded by m and $\Psi^0(m)$ the set of corresponding pseudodifferential operators. For any $\tau \in]0, \infty]$, and any order function m on $\mathbb{R}^d \times \mathbb{C}^d$ we will denote by $S^0_\tau(m)$ the set of symbols which are analytic with respect to ξ in the strip $|\operatorname{Im}\xi| < \tau$ and bounded by some constant times $m(x,\xi)$ in this strip. We will denote by $S^0_\infty(m)$ the intersection for $\tau > 0$ of $S^0_\tau(m)$. We denote by $\Psi^0_\tau(m)$ the set of corresponding operators. Eventually, we say that a symbol p is classical if it admits an asymptotic expansion $p(x,\xi;h) \sim \sum_{j\geq 0} h^j p_j(x,\xi)$. We will denote by $S^0_{\tau,cl}(m)$, $S^0_{cl}(m)$ the corresponding class of symbols.

We will also need some matrix valued pseudodifferential operators. Let $\mathscr{M}_{p,q}$ denote the set of real valued matrices with p rows and q columns and $\mathscr{M}_p = \mathscr{M}_{p,p}$. Let $\mathcal{A} : T^*\mathbb{R}^d \to \mathscr{M}_{p,q}$ be a smooth function. We will say that $\mathcal{A}$ is a (p,q)-matrix-weight if $\mathcal{A}(x,\xi) = (a_{i,j}(x,\xi))_{i,j}$ and for any $i = 1\ldots,p$ and $j = 1,\ldots,q$, $a_{i,j}$ is an order function. If $p = q$, we will simply say that $\mathcal{A}$ is q-matrix-weight.

Given a (p,q)-matrix-weight $\mathcal{A}$, we will denote by $S^0(\mathcal{A})$ the set of symbols $p(x,\xi) = (p_{i,j}(x,\xi))_{i,j}$ defined on $T^*\mathbb{R}^d$ with values in $\mathscr{M}_{p,q}$ such that for all i,j, $p_{i,j} \in S^0(a_{i,j})$ and $\Psi^0(\mathscr{M}_{p,q})$ the set of corresponding pseudodifferential operators. Obvious extensions of this definitions leads to the definition of matrix valued symbol analytic w.r.t. to ξ and the corresponding operators: $S^0_\tau(\mathcal{A})$ and $\Psi^0_\tau(\mathcal{A})$. In the following, we shall mainly use the Weyl semiclassical quantization of symbols, defined by

$$\operatorname{Op}(p)u(x) = (2\pi h)^{-d} \int_{T^*\mathbb{R}^d} e^{ih^{-1}(x-y)\xi} p(\frac{x+y}{2}, \xi) u(y) dy d\xi \tag{1.9}$$

for $p \in S^0(\mathcal{A})$. We shall also use the following notations all along the paper. Given two pseudo differential operators A and B, we shall write $A = B + \Psi^k(m)$ if the difference $A - B$ belongs to $\Psi^k(m)$. At the level of symbols, we shall write $a = b + S^k(m)$ instead of $a - b \in S^k(m)$.

Let us introduce the d-matrix-weight, $\Xi, \mathcal{A} : T^*\mathbb{R}^d \to \mathscr{M}_d$ given by $\mathcal{A}_{i,j}(x,\xi) = (\langle\xi_i\rangle\langle\xi_j\rangle)^{-1}$, $\Xi_{i,j} = \delta_{i,j}\langle\xi_i\rangle$ and observe that $(\Xi\mathcal{A})_{i,j} = \langle\xi_j\rangle^{-1}$. In the following theorem, we state an exact factorization result which will be the key point in our approach.

Theorem 1.4. *Let $p(x,\xi;h) \in S^0_\infty(1)$ be a real valued symbol such that $p(x,\xi;h) = p_0(x,\xi) + S^0_\infty(h)$ and let $P_h = \operatorname{Op}(p)$. Let ϕ satisfy Hypothesis 1 and assume that the following assumptions hold true:*

i) $P_h(e^{-\phi/h}) = 0$,

ii) *for all* $x \in \mathbb{R}^d$, *the function* $\xi \in \mathbb{R}^d \mapsto p(x,\xi;h)$ *is even,*

iii) $\forall \delta > 0,\ \exists \alpha > 0,\ \forall (x,\xi) \in T^*\mathbb{R}^d, \quad (d(x,\mathcal{U})^2 + |\xi|^2 \geq \delta \Longrightarrow p_0(x,\xi) \geq \alpha)$,

iv) *for any critical point* $\mathbf{u} \in \mathcal{U}$ *we have*

$$p_0(x,\xi) = |\xi|^2 + |\nabla\phi(x)|^2 + r(x,\xi),$$

with $r(x,\xi)=\mathcal{O}(|(x-\mathbf{u},\xi)|^3)$ *near* $(\mathbf{u},0)$.

Then, for $h>0$ *small enough, there exists a symbol* $q\in S^0(\Xi\mathcal{A})$ *such that* $P_h=d^*_{\phi,h}Q^*Qd_{\phi,h}$ *with* $Q=\mathrm{Op}(q)$. *Moreover,* $q(x,\xi;h)=q_0(x,\xi)+S^0(h\Xi\mathcal{A})$ *and for any critical point* $\mathbf{u}\in\mathcal{U}$*, we have* $q_0(\mathbf{u},0)=\mathrm{Id}$. *Eventually, if* $p\in S^0_{cl}(1)$ *then* $q\in S^0_{cl}(\Xi\mathcal{A})$.

Let us now make some comments on the above theorem. As already mentioned, we decided in this paper not to give results in the most general case so that technical aspects do not hide the main ideas. Nevertheless, we would like to mention here some possible generalizations of the preceding result (more can be found [1]).

First, it should certainly be possible to use more general order functions and to prove a factorization results for symbols in other classes (for instance $S^0(\langle(x,\xi)\rangle^2)$. This should allow to see the supersymmetric structure of the Witten Laplacian as a special case of our result. In other words, the symbol $p(x,\xi;h)=|\xi|^2+|\nabla\phi(x)|^2-h\Delta\phi(x)$ would satisfy Assumptions i) to iv) above.

A more delicate question should be to get rid of the parity assumption ii). It is clear that this assumption is not necessary (take $q(x,\xi)=\langle\xi\rangle^{-2}(\mathrm{Id}+\mathrm{diag}(\xi_i/\langle\xi\rangle))$ in the conclusion) but it seems difficult to prove a factorization result without it. For instance, the operator hD_x in dimension 1 can not been factorized smoothly both left and right simultaneously.

In [10], Hérau, Hitrik and Sjöstrand proved that semiclassical differential operator P_h of order 2 satisfying $P_h(e^{-\varphi/h})=P_h^*(e^{-\psi/h})=0$ for some suitable φ,ψ admit a supersymmetric structure $P_h=d^*_{\psi,h}A_h(x)d_{\varphi,h}$, where $A_h(x)$ is a $d\times d$ matrix. Nevertheless, as it is constructed A_h can grow exponentially with respect to h. In Lemma 3.2 below, we show that if the parity assumption is fulfilled, then the operator P_h can be factorized with a pseudo-differential operator $\widehat{Q}$ which is bounded with respect to h (no exponential growth). However, getting some control on A_h in a general setting is still an open (an interesting) question.

As it will be seen in the proof below, the operator Q (as well as Q^*Q) above is not unique. Trying to characterize the set of all possible Q should be also a question of interrest.

The plan of the note is the following. In the next section we analyse the structure of operator $\mathbf{T}_h^\star$ and prove the first results on the spectrum stated in Theorem 1.1. In section 3 we prove Theorem 1.4 and apply it to the case of the random walk operator. In section 4, we prove Theorem 1.2.

2. Structure of the operator and first spectral results

In this section, we analyse the structure of the spectrum of the operator $\mathbf{T}_h^\star$ on the space $\mathcal{H}_h=L^2(d\nu_{h,\infty})$. Introduce the Maxwellian $\mathcal{M}_h$ defined by

$$d\nu_{h,\infty}=\mathcal{M}_h(x)\,dx \qquad \text{so that} \qquad \mathcal{M}_h=\widetilde{Z}_h\mu_h(B_h(x))Z_he^{-\phi(x)/h}, \tag{2.1}$$

and make the following change of function

$$\mathcal{U}_hu(x):=\mathcal{M}_h^{-1/2}(x)u(x),$$

where $\mathcal{U}_h$ is unitary from $L^2(\mathbb{R}^d)=L^2(\mathbb{R}^d,dx)$ to $\mathcal{H}_h$. Denoting

$$T_h:=\mathcal{U}_h^*\mathbf{T}_h^\star\mathcal{U}_h, \tag{2.2}$$

the conjugated operator acting in $L^2(\mathbb{R}^d)$, we have

$$T_h u(x) = Z_h \mathcal{M}_h^{-1/2}(x) e^{-\phi(x)/h} \int_{\mathbb{R}^d} \mathbb{1}_{|x-y|<h} \mathcal{M}_h^{1/2}(y) \mu_h(B(y,h))^{-1} u(y)\, dy$$
$$= \left(\frac{Z_h e^{-\phi(x)/h}}{\mu_h(B(x,h))} \right)^{1/2} \int_{|x-y|<h} u(y) \left(\frac{Z_h e^{-\phi(y)/h}}{\mu_h(B(y,h))} \right)^{1/2} dy.$$

We pose for the following

$$a_h(x) = (\alpha_d h^d)^{1/2} \left(\frac{Z_h e^{-\phi(x)/h}}{\mu_h(B(x,h))} \right)^{1/2},$$

and define the operator $\mathbb{G}$ by

$$\mathbb{G}u(x) = \frac{1}{\alpha_d h^d} \int_{|x-y|<h} u(y) dy \tag{2.3}$$

where $\alpha_d = \mathrm{vol}(B(0,1))$ denotes the euclidean volume of the unit ball, so that with these notations, operator T_h reads

$$T_h = a_h \circ \mathbb{G} \circ a_h, \tag{2.4}$$

i.e.

$$T_h u(x) = a_h(x) \mathbb{G}(a_h u)(x).$$

We note that

$$a_h^{-2}(x) = \frac{\mu_h(B(x,h)) e^{\phi(x)/h}}{\alpha_d h^d Z_h} = \frac{1}{\alpha_d h^d} \int_{|x-y|<h} e^{(\phi(x)-\phi(y))/h} dy = e^{\phi(x)/h} \mathbb{G}(e^{-\phi/h})(x). \tag{2.5}$$

We now collect some properties on $\mathbb{G}$ and a_h.

One very simple but fundamental observation is that $\mathbb{G}$ is a semiclassical Fourier multiplier $\mathbb{G} = G(hD) = \mathrm{Op}(G)$ where

$$\forall \xi \in \mathbb{R}^d, \qquad G(\xi) = \frac{1}{\alpha_d} \int_{|z|<1} e^{iz\cdot\xi} dz. \tag{2.6}$$

The following lemmas are easy to prove (see [1] for details).

Lemma 2.1. *The function G is analytic on $\mathbb{C}^d$ and enjoys the following properties:*

i) $G : \mathbb{R}^d \longrightarrow \mathbb{R}$.

ii) There exists $\delta > 0$ such that $G(\mathbb{R}^d) \subset [-1+\delta, 1]$. Near $\xi = 0$, we have
$$G(\xi) = 1 - \beta_d |\xi|^2 + \mathcal{O}(|\xi|^4),$$
where $\beta_d = (2d+4)^{-1}$. For any $r > 0$, $\sup_{|\xi|\geq r} |G(\xi)| < 1$ and $\lim_{|\xi|\to\infty} G(\xi) = 0$.

iii) For all $\tau \in \mathbb{R}^d$, $G(i\tau) \in \mathbb{R}$, $G(i\tau) \geq 1$ and, for any $r > 0$, $\inf_{|\tau|\geq r} G(i\tau) > 1$.

iv) For all $\xi, \tau \in \mathbb{R}^d$ we have $|G(\xi + i\tau)| \leq G(i\tau)$.

Lemma 2.2. *There exist $c_1, c_2 > 0$ such that $c_1 < a_h(x) < c_2$ for all $x \in \mathbb{R}^d$ and $h \in]0,1]$. Moreover, the functions a_h and a_h^{-2} belong to $S^0(1)$ and have classical expansions $a_h = a_0 + ha_1 + \cdots$ and $a_h^{-2} = a_0^{-2} + \cdots$, and*

$$a_0(x) = G(i\nabla\phi(x))^{-1/2}.$$

Eventually, there exist $c_0, R > 0$ such that for all $|x| \geq R$, $a_h^{-2}(x) \geq 1 + c_0$ for $h > 0$ small enough.

Since we want to study the spectrum near 1, it will be convenient to introduce

$$P_h := 1 - T_h. \tag{2.7}$$

Using (2.4) and (2.5), we get

$$P_h = a_h(V_h(x) - G(hD_x))a_h \tag{2.8}$$

with $V_h(x) = a_h^{-2}(x) = e^{\phi/h}G(hD_x)(e^{-\phi/h})$. As a consequence of the previous lemmas, we get the following proposition for P_h.

Proposition 2.3. *The operator P_h is a semiclassical pseudodifferential operator whose symbol $p(x,\xi;h) \in S^0_\infty(1)$ admits a classical expansion which reads $p = p_0 + hp_1 + \cdots$ with*

$$p_0(x,\xi) = 1 - G(i\nabla\phi(x))^{-1}G(\xi) \geq 0.$$

We finish this subsection with the following proposition which is a part of Theorem 1.1.

Proposition 2.4. *There exist $\delta, h_0 > 0$ such that the following assertions hold true for $h \in]0, h_0]$. First, $\sigma(T_h) \subset [-1+\delta, 1]$ and $\sigma_{ess}(T_h) \subset [-1+\delta, 1-\delta]$. Eventually, 1 is a simple eigenvalue for the eigenfunction $\mathcal{M}_h^{1/2}$.*

Proof. We start by proving $\sigma(T_h) \subset [-1+\delta, 1]$. From (1.4), we already know that $\sigma(T_h) \subset [-1, 1]$. Moreover, Lemma 2.1 *ii)* and *iii)* imply $0 \leq a_0(x) \leq 1$ and $G(\mathbb{R}^d) \subset [-1+\nu, 1]$ for some $\nu > 0$. Thus, we deduce that the symbol $\tau_h(x,\xi)$ of the pseudodifferential operator $T_h \in \Psi^0(1)$ satisfies

$$\tau_h(x,\xi) \geq -1 + \nu + \mathcal{O}(h).$$

Then, Gårding's inequality yields

$$T_h \geq -1 + \nu/2,$$

for h small enough. Summing up, we obtain $\sigma(T_h) \subset [-1+\delta, 1]$.

Let us prove the assertion about the essential spectrum. Let $\chi \in C_0^\infty(\mathbb{R}^d; [0,1])$ be equal to 1 on $B(0,R)$, where $R > 0$ is as in Lemma 2.2. Since $\mathbb{G} = G(hD) \in \Psi^0(1)$ and $\lim_{|\xi|\to\infty} G(\xi) = 0$, the operator

$$T_h - (1-\chi)T_h(1-\chi) = \chi T_h + T_h\chi - \chi T_h \chi,$$

is compact. Hence, $\sigma_{ess}(T_h) = \sigma_{ess}((1-\chi)T_h(1-\chi))$. Now, for all $u \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}\Big\langle (1-\chi)T_h(1-\chi)u, u\Big\rangle &= \Big\langle \mathbb{G}a_h(1-\chi)u, a_h(1-\chi)u\Big\rangle \\ &\leq \|a_h(1-\chi)u\|^2 \leq (1+c_0)^{-1}\|u\|^2,\end{aligned}$$

since $\|\mathbb{G}\|_{L^2\to L^2} \leq 1$ and $|a_h(1-\chi)| \leq (1+c_0)^{-1/2}$ thanks to Lemma 2.1 *ii)* and Lemma 2.2. As a consequence, there exists $\delta > 0$ such that $\sigma_{ess}(T_h) \subset [-1+\delta, 1-\delta]$.

To finish the proof, it remains to show that 1 is a simple eigenvalue. First, observe that the distribution kernel $k_h^{(n)}(x,y)$ of T_h^n satisfies

$$k_h^{(n)}(x,y) \geq \varepsilon_n h^{-d}\mathbb{1}_{|x-y|<nh},$$

for some $\varepsilon_n > 0$. Thus, we can conclude by using Krein-Rutman theorem (see Theorem XIII.43 of [14]). More details can be found in [1]. □

3. Supersymmetric structure

In this section, we prove Theorem 1.4 and deduce that the operator $\mathrm{Id} - \mathbf{T}_h^\star$ admits a supersymmetric structure. We showed in the preceding section that $\mathrm{Id} - \mathbf{T}_h^\star = \mathcal{U}_h P_h \mathcal{U}_h^*$ and before proving Theorem 1.4, we state and prove as a corollary the main result on the operator P_h. Recall here that $\beta_d = (2d+4)^{-1}$ and $\Xi\mathcal{A}$ is the matrix symbol defined by $\Xi\mathcal{A}_{i,j} = \langle \xi_j \rangle^{-1}$, for all $i, j = 1, \dots, d$.

Corollary 3.1. *There exists a classical symbol $q \in S^0_{cl}(\Xi\mathcal{A})$ such that the following holds true. First $P_h = L_\phi^* L_\phi$ with $L_\phi = Q d_{\phi,h} a_h$ and $Q = \mathrm{Op}(q)$. Next, $q = q_0 + \Psi^0(h\Xi\mathcal{A})$ with $q_0(\mathbf{u}, 0) = \beta_d^{1/2}\,\mathrm{Id}$ for any critical point $\mathbf{u} \in \mathcal{U}$.*

Proof. Since we know that $P_h = a_h(V_h(x) - \mathbb{G})a_h$, we only have to prove that $\beta_d^{-1}\widetilde{P}_h$ satisfies the asumptions of Theorem 1.4, where

$$\widetilde{P}_h = V_h(x) - G(hD). \tag{3.1}$$

Assumption i) is satisfied by construction.

Observe that thanks to Proposition 2.3, it is a pseudodifferential operator and since variable x and ξ are separated, its symbol in any quantization is given by $\tilde{p}_h(x,\xi) = V_h(x) - G(\xi)$. Moreover, Lemma 2.2 and Proposition 2.3 show that $\tilde{p}_h$ admits a classical expansion $\tilde{p} = \sum_{j=0}^\infty h^j \tilde{p}_j$ with $\tilde{p}_j$, $j \geq 1$ depending only on x and $\tilde{p}_0(x,\xi) = G(i\nabla\phi(x)) - G(\xi)$. Hence, it follows from Lemma 2.1 that $\tilde{p}$ satisfies assumptions ii) and iii).

Finally, it follows from ii) of Lemma 2.1 that near $(\mathbf{u}, 0)$ (for any $\mathbf{u} \in \mathcal{U}$) we have

$$\tilde{p}(x,\xi) = \beta_d(|\xi|^2 + |\nabla\phi(x)|^2) + \mathcal{O}(|(x-\mathbf{u},\xi)|^3) + S^0(h),$$

so that we can apply Theorem 1.4 . Taking into account the multiplication part a_h completes the proof for P_h. ☐

Now we can sketch the proof of Theorem 1.4. It goes in two steps. First we prove that there exists a symbol $\hat{q} \in S^0_\infty(\mathcal{A})$ such that $P_h = d_{\phi,h}^* \widehat{Q} d_{\phi,h}$w where $\widehat{Q} = \mathrm{Op}(\hat{q})$. In a second time we shall prove that the operator $\widehat{Q}$ can be chosen so that $\widehat{Q} = Q^*Q$ for some pseudodifferential operator Q satisfying some nice properties.

Let us start with the first step. For this purpose we need the following lemma whose proof can be found in [1].

Lemma 3.2. *Let $p \in S^0_\infty(1)$ and $P_h = \mathrm{Op}(p)$. Assume that for all $x \in \mathbb{R}$, the function $\xi \mapsto p(x,\xi;h)$ is even. Suppose also that $P_h(e^{-\phi/h}) = 0$. Then there exists $\hat{q} \in S^0_\infty(\mathcal{A})$ such that $P_h = d_{\phi,h}^* \widehat{Q} d_{\phi,h}$ with $\widehat{Q} = \mathrm{Op}(\hat{q})$. Moreover, if p has a principal symbol, then so does $\hat{q}$ and if $p \in S^0_{\infty,cl}(1)$ then $\hat{q} \in S^0_{\infty,cl}(\mathcal{A})$.*

Remark 3.3. *Since $P_h(e^{-\phi/h}) = 0$, it is quite clear that P_h can be factorized by $d_{\phi,h}$ on the right. On the other hand, the fact that P_h can be factorized by $d_{\phi,h}^*$ on the left necessarily implies that $P_h^*(e^{-\phi/h}) = 0$. At a first glance, there is no reason for this identity to hold true since we don't suppose in the above lemma that P_h is self-adjoint. This is actually verified for the following reason. Start from $\mathrm{Op}(p)(e^{-\phi/h}) = 0$, then taking the conjugate and using the fact that ϕ is real we get*

$$\mathrm{Op}(\overline{p}(x,-\xi))(e^{-\phi/h}) = 0.$$

Hence, the parity assumption on p implies that $\mathrm{Op}(p)^*(e^{-\phi/h}) = 0$.

Let us apply Lemma 3.2 to $P_h = \mathrm{Op}(p)$. Then, there exists a symbol $\widehat{q} \in S^0_\infty(\mathcal{A})$ such that

$$P_h = d^*_{\phi,h}\widehat{Q}d_{\phi,h},$$

with $\widehat{Q} = \mathrm{Op}(\widehat{q})$ and $\hat{q} = \hat{q}_0 + S^0(h)$. Now the strategy is the following. We will modify the operator $\widehat{Q}$ so that the new $\widehat{Q}$ is selfadjoint, non-negative and $\widehat{Q}$ can be written as the square of a pseudodifferential operator $\widehat{Q} = Q^*Q$.

First observe that since P_h is selfadjoint,

$$P_h = \frac{1}{2}(P_h + P_h^*) = d^*_{\phi,h}\frac{\widehat{Q} + \widehat{Q}^*}{2}d_{\phi,h},$$

so that we can assume in the following that $\widehat{Q}$ is selfadjoint. This means that the partial operators $\widehat{Q}_{j,k} = \mathrm{Op}(\widehat{q}_{j,k})$ verify $\widehat{Q}^*_{j,k} = \widehat{Q}_{k,j}$ (or at the level of symbols $\hat{q}_{k,j} = \overline{\hat{q}_{j,k}}$). For $k = 1, \dots, d$, let us denote $d^k_{\phi,h} = h\partial_k + \partial_k\phi(x)$. Then

$$P_h = \sum_{j,k=1}^{d} (d^j_{\phi,h})^*\widehat{Q}_{j,k}d^k_{\phi,h}. \tag{3.2}$$

We would like to take the square root of $\widehat{Q}$ and show that it is still a pseudodifferential operator. The problem is that we don't even know if $\widehat{Q}$ is non-negative. Nevertheless, we can use the non-uniqueness of operators $\widehat{Q}$ such that (3.2) holds to go to a situation where $\widehat{Q}$ is close to a diagonal operator with non-negative partial operators on the diagonal. The starting point of this strategy is the following commutation relation

$$\forall j,k \in \{1,\dots,d\}, \qquad \left[d^j_{\phi,h}, d^k_{\phi,h}\right] = 0, \tag{3.3}$$

which holds true since $d^j_{\phi,h} = e^{-\phi/h}h\partial_j e^{\phi/h}$ and thanks to Schwarz Theorem. Hence, for any bounded operator B on $L^2(\mathbb{R}^d)$, we have

$$P_h = d^*_{\phi,h}\widehat{Q}^{mod}d_{\phi,h} = \sum_{j,k=1}^{d} (d^j_{\phi,h})^*\widehat{Q}^{mod}_{j,k}d^k_{\phi,h}, \tag{3.4}$$

with $\widehat{Q}^{mod} = \widehat{Q} + \mathcal{B}$ for some $\mathcal{B}$ being of the following form : For any $j_0, k_0, n \in \{1,\dots,d\}$, the operator $\mathcal{B}(j_0,k_0,n;B) = (\mathcal{B}_{j,k})_{j,k=1,\dots,d}$ is defined by

$$\begin{aligned} &\mathcal{B}_{j,k} = 0 \qquad \text{if } (j,k) \notin \{(n,n),(j_0,k_0),(k_0,j_0)\} \\ &\mathcal{B}_{j_0,k_0} = -(d^n_{\phi,h})^*Bd^n_{\phi,h} \qquad \text{and} \qquad \mathcal{B}_{k_0,j_0} = (\mathcal{B}_{j_0,k_0})^* \\ &\mathcal{B}_{n,n} = (d^{j_0}_{\phi,h})^*Bd^{k_0}_{\phi,h} + (d^{k_0}_{\phi,h})^*B^*d^{j_0}_{\phi,h}. \end{aligned} \tag{3.5}$$

When $j_0 = k_0$, we use the convention that $\mathcal{B}_{j_0,j_0} = -(d^n_{\phi,h})^*(B + B^*)d^n_{\phi,h}$.

Recall that the d-matrix-weights $\mathcal{A}$ and $\Xi\mathcal{A}$ are given by $\mathcal{A}_{j,k} = \langle\xi_j\rangle^{-1}\langle\xi_k\rangle^{-1}$ and $(\Xi\mathcal{A})_{j,k} = \langle\xi_k\rangle^{-1}$. Using the preceding remark, we can prove the following

Lemma 3.4. *Let $\widehat{Q} = \mathrm{Op}(\hat{q})$ where $\hat{q} \in S^0(\mathcal{A})$ is a Hermitian symbol such that $\hat{q}(x,\xi;h) = \hat{q}_0(x,\xi) + S^0(h\mathcal{A})$. We denote $P = d^*_{\phi,h}\widehat{Q}d_{\phi,h}$ and $p(x,\xi;h) = p_0(x,\xi) + S^0(h) \in S^0(1)$ its symbol. Assume that the following assumptions hold:*

(A1) $\forall\delta > 0,\ \exists\alpha > 0,\ \forall(x,\xi) \in T^*\mathbb{R}^d, \quad (|\xi|^2 + d(x,\mathcal{U})^2 \geq \delta \Longrightarrow p_0(x,\xi) \geq \alpha).$

(A2) *Near* $(\mathbf{u},0)$ *for any critical point* $\mathbf{u}\in\mathcal{U}$*, we have*

$$p_0(x,\xi)=|\xi|^2+|\nabla\phi(x)|^2+r(x,\xi), \tag{3.6}$$

with $r(x,\xi)=\mathcal{O}(|(x-\mathbf{u},\xi)|^3)$.

Then, for h small enough, there exists a symbol $q\in S^0(\Xi\mathcal{A})$ *such that*

$$P_h=d^*_{\phi,h}Q^*Qd_{\phi,h},$$

with $Q=\mathrm{Op}(q)$*,* $q=q_0+S^0(h)$ *and* $q_0(\mathbf{u},0)=\mathrm{Id}$ *for any* $\mathbf{u}\in\mathcal{U}$*. Moreover,* $Q=F\,\mathrm{Op}(\Xi^{-1})$ *for some* $F\in\Psi^0(1)$ *invertible and self-adjoint with* $F^{-1}\in\Psi^0(1)$*. Eventually, if* $\widehat{q}\in S^0_{cl}(\mathcal{A})$ *then* $q\in S^0_{cl}(\Xi\mathcal{A})$*.*

Proof. We just sketch the proof and refer to [1] for details. Given $\varepsilon>0$, let $w_0,w_1,\dots,w_d\in S^0(1)$ be non-negative functions such that

$$w_0+w_1+\cdots+w_d=1, \tag{3.7}$$

whose support satisfies

$$\mathrm{supp}(w_0)\subset\Big\{|\xi|^2+|\nabla\phi(x)|^2\le 2\varepsilon\Big\},$$

and, for all $\ell\geq1$,

$$\mathrm{supp}(w_\ell)\subset\Big\{|\xi|^2+|\nabla\phi(x)|^2\ge\varepsilon\text{ and }|\xi_\ell|^2+|\partial_\ell\phi(x)|^2\ge\frac{1}{2d}\Big(|\xi|^2+|\nabla\phi(x)|^2\Big)\Big\}.$$

Let us decompose $\widehat{Q}$ according to these truncations

$$\widehat{Q}=\sum_{\ell=0}^{d}\widehat{Q}^\ell, \tag{3.8}$$

with $\widehat{Q}^\ell:=\mathrm{Op}(w_\ell\widehat{q})$ for all $\ell\geq0$. We will modify each of the operators $\widehat{Q}^\ell$ separately, using the modifiers

$$\mathcal{B}(j_0,k_0,n;\beta):=\mathcal{B}(j_0,k_0,n;\mathrm{Op}(\beta)),$$

where for $j_0,k_0,n\in\{1,\dots,d\}$ and $\beta\in S^0(\langle\xi_{j_0}\rangle^{-1}\langle\xi_{k_0}\rangle^{-1}\langle\xi_n\rangle^{-2})$ the right hand side is defined by (3.5). Let $\mathscr{M}(\mathcal{A})\subset\Psi^0(\mathcal{A})$ be the vector space of bounded operators on $L^2(\mathbb{R}^d)^d$ generated by such operators. Then, (3.4) says exactly that

$$P_h=d^*_{\phi,h}(\widehat{Q}+\mathcal{M})d_{\phi,h}, \tag{3.9}$$

for any $\mathcal{M}\in\mathscr{M}(\mathcal{A})$.

Step 1. We first observe that near $\mathcal{U}\times\{0\}$, there is no modification needed. Indeed, writing $\hat{q}=\hat{q}^0+S^0(h\mathcal{A})$ and using (3.2), (3.6) together with Taylor expansion we see easily that for all $\mathbf{u}\in\mathcal{U}$, $\hat{q}^0(\mathbf{u},0)=\mathrm{Id}$. Hence

$$\breve{Q}^0:=\widehat{Q}^0=\mathrm{Op}(\breve{q}^0)+\Psi^0(h\mathcal{A}), \tag{3.10}$$

where $\breve{q}^0\in S^0(\mathcal{A})$ satisfies

$$\breve{q}^0(x,\xi)=w_0(x,\xi)\Big(\mathrm{Id}+\rho(x,\xi)\Big), \tag{3.11}$$

with $\rho\in S^0(\mathcal{A})$ such that $\rho(x,\xi)=\mathcal{O}(|(d(x,\mathcal{U}),\xi)|)$.

Step 2. We remove the antidiagonal terms away from the origin. More precisely, we show that there exist some $\mathcal{M}^\ell\in\mathscr{M}(\mathcal{A})$ and some diagonal symbols $\widetilde{q}^\ell\in S^0(\mathcal{A})$ such that

$$\widetilde{Q}^\ell:=\widehat{Q}^\ell+\mathcal{M}^\ell=\mathrm{Op}(w_\ell\widetilde{q}^\ell)+\Psi^0(h\mathcal{A}), \tag{3.12}$$

for any $\ell \in \{1,\dots,d\}$. In order to do that we define

$$\beta_{j_0,k_0,\ell}(x,\xi) := \frac{w_\ell(x,\xi)\widehat{q}_{j_0,k_0}(x,\xi)}{|\xi_\ell|^2 + |\partial_\ell \phi(x)|^2}$$

for any $j_0,k_0,\ell \in \{1,\dots,d\}$ with $j_0 \neq k_0$, let $\beta_{j_0,k_0,\ell}$. Thanks to the support properties of w_ℓ, we have $\beta_{j_0,k_0,\ell} \in S^0(\langle\xi_{j_0}\rangle^{-1}\langle\xi_{k_0}\rangle^{-1}\langle\xi_\ell\rangle^{-2})$ so that $\mathcal{B}^\infty(j_0,k_0,\ell;\beta_{j_0,k_0,\ell})$ belongs to $\mathscr{M}(\mathcal{A})$. Then, it follows from symbolic calculus that

$$\mathcal{M}^\ell := \sum_{j_0\neq k_0} \mathcal{B}^\infty(j_0,k_0,\ell;\beta_{j_0,k_0,\ell}),$$

enjoys the required properties.

Step 3. We prove that we can modify each $\widetilde{Q}^\ell$ in order that its diagonal coefficients are suitably bounded from below. More precisely, we claim that there exist $c>0$ and $\widetilde{\mathcal{M}}^\ell \in \mathscr{M}(\mathcal{A})$ such that

$$\breve{Q}^\ell := \widetilde{Q}^\ell + \widetilde{\mathcal{M}}^\ell = \mathrm{Op}(\breve{q}^\ell) + \Psi^0(h\mathcal{A}), \tag{3.13}$$

with $\breve{q}^\ell$ diagonal and $\breve{q}^\ell_{i_0,i_0}(x,\xi) \geq c w_\ell(x,\xi)\langle\xi_{i_0}\rangle^{-2}$ for all $i_0 \in \{1,\dots,d\}$.

For $\ell, i_0 \in \{1,\dots,d\}$, let $\beta_{i_0,\ell}$ be defined by

$$\beta_{i_0,\ell}(x,\xi) := \frac{w_\ell(x,\xi)}{2\Big(|\xi_\ell|^2 + |\partial_\ell\phi(x)|^2\Big)}\bigg(\widetilde{q}^\ell_{i_0,i_0}(x,\xi) - \frac{\gamma}{1+|\xi_{i_0}|^2+|\partial_{i_0}\phi(x)|^2}\bigg),$$

where $\gamma>0$ will be specified later. The symbol $\beta_{i_0,\ell}$ belongs to $S^0(\langle\xi_{i_0}\rangle^{-2}\langle\xi_\ell\rangle^{-2})$ so that $\mathcal{B}^\infty(i_0,i_0,\ell;\beta_{i_0,\ell}) \in \mathscr{M}(\mathcal{A})$. Then, pseudo differential calculus shows that

$$\widetilde{\mathcal{M}}^\ell := \sum_{i_0\neq\ell} \mathcal{B}^\infty(i_0,i_0,\ell;\beta_{i_0,\ell}),$$

satisfies (3.13).

Step 4. Consider $\breve{Q} = \sum_\ell \breve{Q}_l$ and let $E = \mathrm{Op}(\Xi)\breve{Q}\,\mathrm{Op}(\Xi)$. Then $\breve{Q} = Q^*Q$ with $Q = E^{1/2}\,\mathrm{Op}(\Xi^{-1})$. From the above construction we can show that $E = \mathrm{Op}(e)$ with

$$e(x,\xi;h) \geq c\,\mathrm{Id}\,. \tag{3.14}$$

Thus, it follows from Theorem 4.8 in [7] that both $E^{1/2}$ and Q belongs to $S^0(1)$ which completes the proof of Theorem 1.4. ☐

4. From P to the usual Witten Laplacian

In this section, we give a detailed proof of Theorem 1.1. Here P_h denotes again $\mathcal{U}_h^*(I-T_h)\mathcal{U}_h$. As already mentioned, this proof is an original piece of work.

On the total De Rham complex, we define

$$P^W = d^*_{\phi,h}d_{\phi,h} + d_{\phi,h}d^*_{\phi,h},$$

the semiclassical Witten Laplacian, and $P^{W,(k)}$ its restriction to the k-forms. These operators have been intensively studied (see e.g. [3], [8], [6], ...), and a lot is known concerning their spectral properties. In particular, from (14.9) in [7], we know that

$$P^{W,(1)} = P^{W,(0)} \otimes \mathrm{Id} + \Psi^0(h).$$

In this section, we compare the small eigenvalues of P_h with those of $P^{W,(0)}$. This idea is natural since $a_0(\mathbf{u}) = 1$ and $q_0(\mathbf{u}, 0) = \mathrm{Id}$ for all critical points $\mathbf{u} \in \mathcal{U}$, and $P^{W,(0)}$ is then the operator P_h with the coefficients a_h and Q frozen at 1 and Id.

Let $f \in C_0^\infty(\mathbb{R}; [0,1])$ equal to 1 on $[-1,1]$. For $\varepsilon > 0$, we define $f_\varepsilon(\lambda) = f(\lambda/\varepsilon)$. In the sequel, $\delta > 0$ and $C > 1$ will design constants which may change from line to line but do not depend on ε and h. On the other hand, the subscript ε (as in $C_\varepsilon > 1$) will point out that the quantity may depend on ε (but is independent or uniform in h). Finally, to shorten the equations, we will sometimes use the notation $g^{(\bullet)} = g(P^{W,(\bullet)})$ for a function g on $\mathbb{R}$ and for $k \in \mathbb{N}$, $f^{k(\bullet)}$ will denote $f^{(\bullet)}$ to the power k.

Lemma 4.1. *We have*

$$a_h F_-(P^{W,(0)}) a_h \leq P_h \leq a_h F_+(P^{W,(0)}) a_h,$$

where

$$F_-(\lambda) = \lambda\Big(f_\varepsilon^2(\lambda)(1 - C\sqrt{\varepsilon} - C_\varepsilon h) + (\lambda+1)^{-1}(1 - f_\varepsilon(\lambda))^2(\delta - C\sqrt{\varepsilon} - C_\varepsilon h)\Big),$$
$$F_+(\lambda) = \lambda\Big(f_\varepsilon^2(\lambda)(1 + C\sqrt{\varepsilon} + C_\varepsilon h) + 2 f_\varepsilon(\lambda)(1 - f_\varepsilon(\lambda)) + C_\varepsilon(1 - f_\varepsilon(\lambda))^2\Big).$$

Proof. We can decompose

$$\begin{aligned} Q^*Q &= f_\varepsilon^{(1)} Q^*Q f_\varepsilon^{(1)} + f_\varepsilon^{(1)} Q^*Q(1 - f_\varepsilon^{(1)}) + (1 - f_\varepsilon^{(1)}) Q^*Q f_\varepsilon^{(1)} + (1 - f_\varepsilon^{(1)}) Q^*Q (1 - f_\varepsilon^{(1)}) \\ &= I + II + III + IV. \end{aligned} \tag{4.1}$$

We first estimate I. Let $f \prec g \in C_0^\infty(\mathbb{R}; [0,1])$. Using $P^{W,(1)} = P^{W,(0)} \otimes \mathrm{Id} + \Psi^0(h)$, the formula of the functional calculus and the functional calculus of pseudodifferential operators, we can write

$$\begin{aligned} g_\varepsilon(P^{W,(1)}) &= g_\varepsilon(P^{W,(0)}) \otimes \mathrm{Id} + \int \overline{\partial} \widetilde{g}_\varepsilon(z)(P^{W,(1)} - z)^{-1} \Psi^0(h)(P^{W,(0)} \otimes \mathrm{Id} - z)^{-1} dz \\ &= \mathrm{Op}(g_\varepsilon(p^{W,(0)})) \otimes \mathrm{Id} + R, \end{aligned} \tag{4.2}$$

where $\|R\| \leq C_\varepsilon h$ as an operator from $H_h^{-2}(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ and $p^{W,(0)}$ denotes the principal symbol of $P^{W,(0)}$. Moreover, from Theorem 1.4, we have $Q^*Q = \mathrm{Id} + \widetilde{Q} + \Psi^0(h)$ where the remainder term $\widetilde{Q}$ is a pseudodifferential operator in $\Psi^0(1)$ whose symbol $\widetilde{q}$ vanishes at $(\mathbf{u}, 0)$, $\mathbf{u} \in \mathcal{U}$. Then,

$$g_\varepsilon(P^{W,(1)}) Q^*Q = g_\varepsilon(P^{W,(1)}) + \mathrm{Op}(g_\varepsilon(p^{W,(0)})\widetilde{q}) + \mathcal{O}_\varepsilon(h). \tag{4.3}$$

Recall now that, for $a \in S^0(1)$,

$$\big\| \mathrm{Op}(a) \big\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} = \|a\|_{L^\infty(\mathbb{R}^{2d})} + \mathcal{O}(h),$$

(see e.g. [15, Theorem 13.13]). Thus, using that $g_\varepsilon(p^{W,(0)})$ is supported in a neighborhood of size $\sqrt{\varepsilon}$ of $(\mathbf{c}, 0)$ at which $\widetilde{q}$ vanishes, it yields $\| \mathrm{Op}(g_\varepsilon(p^{W,(0)})\widetilde{q})\| \leq C\sqrt{\varepsilon} + C_\varepsilon h$ and (4.3) implies

$$\Big\| g_\varepsilon(P^{W,(1)})(Q^*Q - \mathrm{Id}) \Big\| \leq C\sqrt{\varepsilon} + C_\varepsilon h. \tag{4.4}$$

Using that $f_\varepsilon^{(1)} Q^*Q f_\varepsilon^{(1)} = f_\varepsilon^{(1)} g_\varepsilon^{(1)} Q^*Q f_\varepsilon^{(1)}$, this estimate gives

$$f_\varepsilon^{2(1)}(1 - C\sqrt{\varepsilon} - C_\varepsilon h) \leq I \leq f_\varepsilon^{2(1)}(1 + C\sqrt{\varepsilon} + C_\varepsilon h). \tag{4.5}$$

We now treat II and III. Writing

$$f_\varepsilon^{(1)} Q^* Q(1-f_\varepsilon^{(1)}) = f_\varepsilon^{(1)}(1-f_\varepsilon^{(1)}) \\ + f_\varepsilon^{(1)} g_\varepsilon(P^{W,(1)})\Big(Q^*Q - \mathrm{Id}\Big)(P^{W,(1)}+1)^{1/2}(P^{W,(1)}+1)^{-1/2}(1-f_\varepsilon^{(1)}),$$

the Cauchy–Schwartz inequality gives

$$\Big|\Big\langle \Big(f_\varepsilon^{(1)}(Q^*Q-1)(1-f_\varepsilon^{(1)})\Big)u, u\Big\rangle\Big| \\ \leq \Big\| g_\varepsilon(P^{W,(1)})\Big(Q^*Q-\mathrm{Id}\Big)(P^{W,(1)}+1)^{1/2}\Big\| \\ \Big(\Big\|f_\varepsilon^{(1)}u\Big\|^2 + \Big\|(P^{W,(1)}+1)^{-1/2}(1-f_\varepsilon^{(1)})u\Big\|^2\Big). \tag{4.6}$$

Proceeding as in (4.4) and using that $(P^{W,(1)}+1)^{1/2}$ is bounded as operator from $L^2(\mathbb{R}^d)$ to $H_h^{-1}(\mathbb{R}^d)$, we get

$$\Big\| g_\varepsilon(P^{W,(1)})\Big(Q^*Q-\mathrm{Id}\Big)(P^{W,(1)}+1)^{1/2}\Big\| \leq C\sqrt{\varepsilon} + C_\varepsilon h.$$

Thus, (4.6) implies

$$2f_\varepsilon^{(1)}(1-f_\varepsilon^{(1)}) - \Big(f_\varepsilon^{2(1)} + (P^{W,(1)}+1)^{-1}(1-f_\varepsilon^{(1)})^2\Big)(C\sqrt{\varepsilon}+C_\varepsilon h) \\ \leq II + III \leq 2f_\varepsilon^{(1)}(1-f_\varepsilon^{(1)}) + \Big(f_\varepsilon^{2(1)} + (P^{W,(1)}+1)^{-1}(1-f_\varepsilon^{(1)})^2\Big)(C\sqrt{\varepsilon}+C_\varepsilon h)$$

and then, using $f(1-f)\geq 0$,

$$-f_\varepsilon^{2(1)}(C\sqrt{\varepsilon}+C_\varepsilon h) - (P^{W,(1)}+1)^{-1}(1-f_\varepsilon^{(1)})^2(C\sqrt{\varepsilon}+C_\varepsilon h) \\ \leq II + III \leq 2f_\varepsilon^{(1)}(1-f_\varepsilon^{(1)}) + f_\varepsilon^{2(1)}(C\sqrt{\varepsilon}+C_\varepsilon h) + C_\varepsilon(1-f_\varepsilon^{(1)})^2. \tag{4.7}$$

It remains to study IV. From Lemma 3.4, we have $Q = F\,\mathrm{Op}(\Xi^{-1})$ and then

$$\langle Q^*Qu, u\rangle = \Big\|F\,\mathrm{Op}(\Xi^{-1})u\Big\|^2 \geq \delta\|\,\mathrm{Op}(\Xi^{-1})u\|^2, \tag{4.8}$$

for some $\delta > 0$. For the last inequality, we have used that F^{-1} is uniformly bounded since it belongs to $\Psi^0(1)$. Moreover, using $0 \leq P^{W,(1)} = -h^2\Delta\otimes\mathrm{Id} + \mathcal{O}(1)$, we get

$$\mathrm{Op}(\Xi^{-2}) \geq (-h^2\Delta+1)^{-1}\otimes\mathrm{Id} \geq \delta(P^{W,(1)}+1)^{-1}.$$

Thus, (4.8) yields

$$Q^*Q \geq \delta(P^{W,(1)}+1)^{-1}$$

On the other hand, using $Q\in\Psi^0(1)$, we deduce $Q^*Q \leq C$. These two estimates imply

$$\delta(P^{W,(1)}+1)^{-1}(1-f_\varepsilon^{(1)})^2 \leq IV \leq C(1-f_\varepsilon^{(1)})^2. \tag{4.9}$$

Combining (4.1) with (4.5), (4.7) and (4.9), we obtain

$$d_{\phi,h}^*\Big(F_-(\lambda)/\lambda\Big)(P^{W,(1)})d_{\phi,h} \leq d_{\phi,h}^*Q^*Qd_{\phi,h} \leq d_{\phi,h}^*\Big(F_+(\lambda)/\lambda\Big)(P^{W,(1)})d_{\phi,h}.$$

Using now the classical intertwining relation $P^{W,(1)}d_{\phi,h} = d_{\phi,h}P^{W,(0)}$ and $P^{W,(0)} = d_{\phi,h}^*d_{\phi,h}$, this estimate eventually implies

$$F_-(P^{W,(0)}) \leq d_{\phi,h}^*Q^*Qd_{\phi,h} \leq F_+(P^{W,(0)}),$$

and the lemma follows. □

In Lemma 4.1, we have "removed" Q^*Q from the operator P. We will now "remove" a_h using the same strategy. We define the self-adjoint operators

$$\widetilde{P}_\pm := F_\pm^{1/2}(P^{W,(0)}) a_h^2 F_\pm^{1/2}(P^{W,(0)}),$$

and we have the following standard result.

Lemma 4.2. *The operators $a_h F_\pm(P^{W,(0)}) a_h$ and $\widetilde{P}_\pm$ have same spectrum. Moreover, their eigenvalues have the same multiplicity.*

Proof. The fact that they have the same spectrum outside of 0 is a consequence of the classical relation $(1-BA)^{-1} = 1 + B(1-AB)^{-1}A$. Moreover, if u is an eigenvector of AB for the eigenvalue $\lambda \neq 0$, then $Bu \neq 0$ is an eigenvector of BA for the same eigenvalue. Thus, the multiplicity of the non-zero eigenvalues of $a_h F_\pm(P^{W,(0)}) a_h$ and $\widetilde{P}_\pm$ are the same. Finally, using that $a_h, a_h^{-1} \in L^\infty(\mathbb{R}^d)$, the vector spaces $\ker(a_h F_\pm(P^{W,(0)}) a_h) = \ker(F_\pm^{1/2}(P^{W,(0)}) a_h) = a_h^{-1} \ker(F_\pm^{1/2}(P^{W,(0)}))$ and $\ker(\widetilde{P}_\pm) = \ker(a_h F_\pm^{1/2}(P^{W,(0)})) = \ker(F_\pm^{1/2}(P^{W,(0)}))$ have the same dimension. □

As in Lemma 4.1, we can control a_h^2 using the following

Lemma 4.3. *We have*

$$\begin{aligned} f_\varepsilon^{2(0)}(\lambda)(1 - C\sqrt{\varepsilon} - C_\varepsilon h) + (1 - f_\varepsilon^{(0)}(\lambda))^2(\delta - C\sqrt{\varepsilon} - C_\varepsilon h) \\ \leq a_h^2 \leq f_\varepsilon^{2(0)}(1 + C\sqrt{\varepsilon} + C_\varepsilon h) + 2 f_\varepsilon^{(0)}(1 - f_\varepsilon^{(0)}) + C_\varepsilon(1 - f_\varepsilon^{(0)})^2. \end{aligned} \tag{4.10}$$

In particular,

$$G_-(P^{W,(0)}) \leq \widetilde{P}_- \qquad \text{and} \qquad \widetilde{P}_+ \leq G_+(P^{(0)}), \tag{4.11}$$

with

$$\begin{aligned} G_-(\lambda) &= F_-(\lambda)\Big(f_\varepsilon^2(\lambda)(1 - C\sqrt{\varepsilon} - C_\varepsilon h) + (1 - f_\varepsilon(\lambda))^2(\delta - C\sqrt{\varepsilon} - C_\varepsilon h)\Big), \\ G_+(\lambda) &= F_+(\lambda)\Big(f_\varepsilon^2(\lambda)(1 + C\sqrt{\varepsilon} + C_\varepsilon h) + 2 f_\varepsilon(1 - f_\varepsilon(\lambda)) + C(1 - f_\varepsilon(\lambda))^2\Big). \end{aligned}$$

Proof. As in (4.1), we write

$$\begin{aligned} a_h^2 &= f_\varepsilon^{(0)} a_h^2 f_\varepsilon^{(0)} + f_\varepsilon^{(0)} a_h^2 (1 - f_\varepsilon^{(0)}) + (1 - f_\varepsilon^{(0)}) a_h^2 f_\varepsilon^{(0)} + (1 - f_\varepsilon^{(0)}) a_h^2 (1 - f_\varepsilon^{(0)}) \\ &= I + II + III + IV. \end{aligned} \tag{4.12}$$

Working as in (4.5) and using that the function $a_h^2 \in \Psi^0(1)$ satisfies $a_h^2 = 1 + \widetilde{a} + \Psi^0(h)$ where $\widetilde{a}(\mathbf{u}) = 0$ for all $\mathbf{u} \in \mathcal{U}$, we obtain

$$f_\varepsilon^{2(0)}(1 - C\sqrt{\varepsilon} - C_\varepsilon h) \leq I \leq f_\varepsilon^{2(0)}(1 + C\sqrt{\varepsilon} + C_\varepsilon h). \tag{4.13}$$

The same way, as in (4.7), we get

$$\begin{aligned} -f_\varepsilon^{2(0)}(C\sqrt{\varepsilon} + C_\varepsilon h) - (1 - f_\varepsilon^{(0)})^2(C\sqrt{\varepsilon} + C_\varepsilon h) \\ \leq II + III \leq 2 f_\varepsilon^{(0)}(1 - f_\varepsilon^{(0)}) + f_\varepsilon^{2(0)}(C\sqrt{\varepsilon} + C_\varepsilon h) + C_\varepsilon(1 - f_\varepsilon^{(0)})^2. \end{aligned} \tag{4.14}$$

Eventually, since $\delta \leq a_h^2 \leq C$, we directly obtain

$$\delta(1 - f_\varepsilon^{(0)})^2 \leq IV \leq C(1 - f_\varepsilon^{(0)})^2. \tag{4.15}$$

Combining (4.12) with (4.13)–(4.15), we obtain (4.10). Finally, (4.11) follows directly from the definition of $\widetilde{P}_\pm$ and (4.10). □

Combining the previous lemmas, we obtain the following proposition which is exactly the second part of Theorem 1.2.

Proposition 4.4. *Let $0 < \nu(h) \to 0$ as $h \to 0$. We have*

$$\lambda_j(P_h) = \lambda_j(P^{W,(0)})(1 + o_{h\to 0}(1)),$$

uniformly for j such that $\lambda_j(P^{W,(0)}) \leq \nu(h)$.

Proof. We first recall the maxi-min principle (see [14, Theorem XIII.1]). For selfadjoint operators A bounded from below, we have

$$\lambda_j(A) = \sup_{\dim E = j-1} \inf_{u\in E^\perp,\ \|u\|=1} \langle Au, u\rangle, \tag{4.16}$$

where $\lambda_j(A)$ is either the j-th eigenvalue (counted with the multiplicity) or the bottom of the essential spectrum. Thus, Lemma 4.1, Lemma 4.2 and Lemma 4.3 give

$$\begin{aligned} \lambda_j\Big(G_-(P^{W,(0)})\Big) &\leq \lambda_j(\widetilde{P}_-) = \lambda_j\Big(a_h F_-(P^{W,(0)}) a_h\Big) \\ &\leq \lambda_j(P_h) \leq \lambda_j\Big(a_h F_+(P^{W,(0)}) a_h\Big) = \lambda_j(\widetilde{P}_+) \leq \lambda_j\Big(G_+(P^{W,(0)})\Big). \end{aligned} \tag{4.17}$$

for all $j \in \mathbb{N} \setminus \{0\}$.

Looking now to the particular form of $G_\pm$, one can verify that there exists $\delta > 0$ with the following properties. For ε and then h small enough, $G_\pm(0) = 0$, $G_\pm$ is strictly increasing in $[0, \varepsilon]$ and $G_\pm \geq \delta\varepsilon$ on $[\varepsilon, +\infty[$. This implies that, for ε and then h small enough,

$$\lambda_j\Big(G_\pm(P^{W,(0)})\Big) = G_\pm\Big(\lambda_j(P^{W,(0)})\Big)$$

for all j such that $\lambda_j(P^{W,(0)}) \leq \nu(h)$. Using then that $f_\varepsilon(\lambda_j(P^{W,(0)})) = 1$, we then deduce

$$\lambda_j\Big(G_\pm(P^{W,(0)})\Big) = \lambda_j(P^{W,(0)})(1 \pm C\sqrt{\varepsilon} \pm C_\varepsilon h)^2. \tag{4.18}$$

Thus, (4.17) and (4.18) imply that, for ε and then h small enough,

$$\lambda_j(P^{W,(0)})(1 - C\sqrt{\varepsilon} - C_\varepsilon h)^2 \leq \lambda_j(P) \leq \lambda_j(P^{W,(0)})(1 + C\sqrt{\varepsilon} + C_\varepsilon h)^2,$$

for all j such that $\lambda_j(P^{W,(0)}) \leq \nu(h)$. Since the quantities $\lambda_j(P^{W,(0)})$ and $\lambda_j(P_h)$ do not depend on ε, this estimate implies the proposition. □

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