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**Transport equation and Cauchy problem for $BV$ vector fields and applications**

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# Transport equation and Cauchy problem for $BV$ vector fields and applications

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## Introduction

In this talk I am going to describe the main results of [7], where the DiPerna–Lions theory is extended to the case of a $BV$ dependence of the vector field with respect to the spatial variables. I will also illustrate some differences between my approach and the DiPerna–Lions one in the treatment of the uniqueness of the flow, and some applications obtained in [8], [9], [25] to PDE's. Finally, I will also mention some open problems and some work in progress.

## 1. Description and short history of the problem

### 1.1. The Lagrangian side

Given $b(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, we would like to find conditions ensuring the "generic uniqueness" of solutions of the ODE

$$\begin{cases} \dot\gamma(t) = b\left(t, \gamma(t)\right) \\ \gamma(0) \ = \ x. \end{cases}$$

Even though uniqueness is lacking, we would like to have a kind of selection principle relative to the approximation of $b$ by smooth vector fields $b_h$:

$$\exists \lim_{h\to\infty} \gamma_h(x,t) \qquad \text{in } C([0,T];\mathbb{R}^d) \text{ for } \mathscr{L}^d\text{-a.e. } x,$$

where $\gamma_h(x,\cdot)$ are the solutions of the approximating Cauchy problems.
We don't assume that $b(t,\cdot)$ is Lipschitz but, to prevent blow-up in finite time, we assume from now on that $|b|$ is globally bounded (more general conditions are considered in [28], [11]). In order to simplify even more, we assume that $b$ is autonomous, i.e. $b = b(x)$.

## 1.2. The Eulerian side

We consider the Cauchy problem for the transport equation in conservative form:

$$\frac{d}{dt}\mu_t + D_x \cdot (b\mu_t) = 0, \qquad \mu_0 = \mathscr{L}^d \llcorner A, \qquad t \in [0,T].$$

Here $\mu_t$ is a time-dependent family of positive measures. We want to study the well posedness of this problem and obtain a comparison principle for solutions.
Particular classes of solutions (e.g. $\mu_t = w_t \mathscr{L}^d$ with $w_t$ locally uniformly bounded) can be considered.

## 1.3. Essential bibliography

DiPerna–Lions '89. [28] Sobolev regularity for $b(t,\cdot)$.
Cellina '95, Cellina–Vornicescu '98. [18], [19] $\gamma'(t) \in A(\gamma(t))$, with $A$ maximal monotone.
Lions '96. [35] Extension to the case when $b(t,\cdot)$ is "piecewise $W^{1,1}$".
Bouchut '02. [15] II order ODE's $\gamma''(t) = b(t,\gamma(t))$ and equations of Vlasov type in the Hamiltonian space, with a $BV$ dependence.
Colombini–Lerner '03. [22] Co-normal $BV$ fields, with singularities.
A. '03. [7] The general case $b(t,\cdot) \in BV$.
A.–De Lellis '03, A.–Bouchut–De Lellis '03. [8], [9] Applications to Keyfitz–Kranzer system of PDE.

# 2. Renormalized solutions of the transport equation with $BV$ vectorfields

## 2.1. Sobolev case: the DiPerna–Lions result

**Theorem 2.1 (Renormalization)** *Let $B \in W^{1,1}_{\rm loc}(\mathbb{R}^m;\mathbb{R}^m)$ and let $w \in L^\infty_{\rm loc}(\mathbb{R}^m)$ be satisfying the transport equation*

$$B \cdot \nabla w := D\cdot(Bw) - wD\cdot B = c\mathscr{L}^m.$$

*Then*

$$B \cdot \nabla(h(w)) = ch'(w)\mathscr{L}^m \qquad \forall h \in C^1(\mathbb{R}).$$

Choosing $m = d+1$, $B = (1,b)$ and $c = ew$ we obtain that

$$w_t + b \cdot \nabla_x w = ew \qquad \text{in } \mathbb{R}^+_t \times \mathbb{R}^d_x$$

implies

$$h(w)_t + b\cdot\nabla_x(h(w)) = ewh'(w) \qquad \text{in } \mathbb{R}^+_t \times \mathbb{R}^d_x.$$

Choosing $h(t) = t^\pm$, via Gronwall Lemma one obtains, under some natural boundary conditions at infinity and $L^\infty$ bounds on $e$ and $D\cdot b$, uniqueness and comparison results for the PDE.

**Sketch of proof.** Mollifying both sides we get

$$(B \cdot \nabla w_t) * \rho_\epsilon = c * \rho_\epsilon \mathscr{L}^m$$

and therefore

$$B \cdot \nabla(w_t * \rho_\epsilon) = c * \rho_\epsilon \mathscr{L}^m + r_\epsilon$$

with

$$r_\epsilon := B \cdot \nabla(w_t * \rho_\epsilon) - (B \cdot \nabla w_t) * \rho_\epsilon.$$

Multiplying both sides by $h'(w_t * \rho_\epsilon)$ we get

$$B \cdot \nabla h(w_t * \rho_\epsilon) = h'(w_t * \rho_\epsilon) \left[c * \rho_\epsilon \mathscr{L}^m + r_\epsilon\right]$$

and therefore the renormalization property follows as $\epsilon \downarrow 0$, using the fact that $r_\epsilon \to 0$ in the *strong* topology of $L^1_{\rm loc}$.
Why $r_\epsilon \to 0$ strongly? We have indeed

$$r_\epsilon(t,x) = \int w_t(x - \epsilon y) \frac{B(x - \epsilon y) - B(x)}{\epsilon} \cdot \nabla \rho(y)\, dy - (w_t {\rm div}\, B) * \rho_\epsilon(x)$$

and the strong convergence of difference quotients of Sobolev functions gives

$$r_\epsilon(t,x) \sim -w_t(x) \int \sum_{i,j=1}^{d} \frac{\partial B^i}{\partial x_j}(x) y_j \frac{\partial \rho}{\partial y_i}(y)\, dy - w_t(x) {\rm div}\, B(x) = 0$$

since

$$\int y_j \frac{\partial \rho}{\partial y_i}(y)\, dy = -\int \rho \frac{\partial y_i}{\partial y_j}\, dy = -\delta_{ij}.$$

When $B \notin W^{1,1}_{\rm loc}$ the behaviour of $r_\epsilon$ is *very* sensitive to the choice of $\rho$: for instance when $\rho$ is *radial* we have (see [17])

$$D_i B_j + D_j B_i \in L^1_{\rm loc} \quad \Longrightarrow \quad r_\epsilon \to 0 \ \text{ in } L^1_{\rm loc}.$$

In general, however, $r_\epsilon$ *do not* converge strongly to 0 in $L^1_{\rm loc}$ when $B \in BV_{\rm loc}$.

## 2.2. The $BV$ case

**Notation.** We split $|DB|$ as $|D^a B| + |D^s B|$, with $|D^a B| \ll \mathscr{L}^m$ and $|D^s B| \perp \mathscr{L}^m$, and we write $D^s B = M |D^s B|$.
Given a convolution kernel $\rho$ and a $d \times d$ matrix $M$, we define

$$I(\rho) := \int |y| |\nabla \rho(y)|\, dy, \qquad I(M, \rho) := \int |\langle M(y), \nabla \rho(y) \rangle|\, dy.$$

Since

$$D \cdot B = {\rm trace}(M) |D^s B| + \sum_i D^a_i B_i$$

we have that ${\rm trace}(M) = 0$ $|D^s B|$-a.e. if and only if $D \cdot B \ll \mathscr{L}^m$.

**Optimal commutator estimates.** *If $K \subset \mathbb{R}^m$ is any compact set, then*

$$\limsup_{\epsilon\downarrow 0} \int_K |r_\epsilon|\, dx \le \|w\|_\infty I(\rho) |D^s B|(K)$$

*and*

$$\limsup_{\epsilon\downarrow 0} \int_K |r_\epsilon|\, dx \le \|w\|_\infty \int_K I(M(x),\rho)\, d|D^s B|(x) + \|w\|_\infty \left(m + I(\rho)\right) |D^a B|(K).$$

The proof of the first estimate, the more delicate one, requires a splitting of the difference quotient into a strongly converging part and a weakly converging one, the latter controlled by the singular part of derivative only.
Roughly speaking, the first estimate is useful in the regions $K$ where $|D^a B|$ is dominant (so that $|D^s B|(K) << 1$), while the second estimate is useful in the regions $K$ where $|D^s B|$ is dominant (so that $|D^a B|(K) << 1$).

**Theorem 2.2** *Let $B \in BV_{\text{loc}}(\mathbb{R}^m;\mathbb{R}^m)$ with $D\cdot B \ll \mathscr{L}^m$ and let $w \in L^\infty_{\text{loc}}(\mathbb{R}^m;\mathbb{R}^k)$ be satisfying*

$$B\cdot\nabla w_i = c_i \mathscr{L}^m, \qquad i = 1,\ldots,k.$$

*Then*

$$B\cdot\nabla h(w) = \sum_{i=1}^k \frac{\partial h}{\partial z_i}(w) c_i \mathscr{L}^m \qquad \forall h \in C^1(\mathbb{R}^k).$$

**Sketch of proof.** It is a refinement of the anisotropic smoothing argument devised by Bouchut [15] and improved by Colombini–Lerner [22]: repeating the same smoothing scheme of the Sobolev case, the first estimate gives that

$$\sigma := B\cdot\nabla h(w) - \sum_{i=1}^k \frac{\partial h}{\partial z_i}(w) c_i \mathscr{L}^m$$

is a measure absolutely continuous with respect to $|D^s B|$.
But we can use the second estimate to obtain

$$|\sigma| \le C(h,w) I(M,\rho)|D^s B| + C(h,w,\rho)|D^a B|,$$

and therefore the two informations together give

$$|\sigma| \le C(h,w) I(M,\rho)|D^s B|.$$

Now, notice that $\sigma$ does not depend on $\rho$, therefore we can improve the estimate above just varying the convolution kernel:

$$|\sigma| \le C(h,w) \inf_\rho I(M,\rho)|D^s B|.$$

Recalling the definition of $I$, we are led to the *pointwise* minimization problem:

$$\inf\left\{ \int_{\mathbb{R}^m} |\langle Mz, \nabla\rho(z)\rangle|\, dz : \ \rho \text{ convolution kernel} \right\}.$$

One can show ([4], see the proof in [11]) that the infimum above is the modulus of the trace of $M$, therefore $\inf_\rho I(M(x),\rho) = 0$ for $|D^s B|$-a.e. $x$ because $D\cdot B \ll \mathscr{L}^m$. If $M = \eta\otimes\xi$ the "optimal" choice of mollifiers results in a smoothing in the $\xi$ direction much faster than in the $\eta$ direction, or in all other directions, i.e.

$$\rho_\epsilon(x) = \rho^1_\epsilon(x - \langle x,\xi\rangle\xi)\,\rho^2_{\epsilon'}(\langle x,\xi\rangle) \quad \text{with} \quad \epsilon' = o(\epsilon).$$

This is the procedure used in the previous papers on the subject.
The information that $M$ has rank one, i.e. that $M$ is representable as $\eta\otimes\xi$, is provided by Alberti's rank one theorem [2], but this information is not strictly necessary in the kernel optimization argument outlined before.

# 3. Applications

## 3.1. Uniqueness and stability of Lagrangian flows

**Definition.** A *Lagrangian flow* associated to $b$ is a map $\Gamma(t,x) : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ such that:
• $\Gamma(\cdot,x)$ is an integral solution of the ODE $\gamma(t) = x + \int_0^t b(\gamma(\tau))\,d\tau$ in $[0,T]$;
• The image measures $\lambda_t := \Gamma(t,\cdot)_\# \mathscr{L}^d$ satisfy $\lambda_t \le C\mathscr{L}^d$ for any $t\in[0,T]$.
The following theorem is proved in [7] for the case of bounded vector fields and in [11] for the general $L^1 + L^\infty$ assumptions considered in the DiPerna–Lions paper [28].

**Theorem 3.1** *Assume that $b \in L^\infty \cap BV_{\mathrm{loc}}$ and $D\cdot b = \alpha\mathscr{L}^d$, with $\alpha^- \in L^\infty_{\mathrm{loc}}$. Then there exists a unique, up to sets whose projection on the second factor is $\mathscr{L}^d$-negligible, Lagrangian flow $\Gamma$.*
*If $b_h$ are smooth, equi-bounded and converge to $b$ in $L^1_{\mathrm{loc}}$, then*

$$\lim_{h\to\infty}\int_{B_R}\max_{[0,T]}|\Gamma^h(\cdot,x) - \Gamma(\cdot,x)|\,dx = 0 \qquad \forall R>0,$$

*where $\Gamma^h$ are the classical flows associated to $b_h$.*

This definition of Lagrangian flow is slightly different from the DiPerna–Lions one, as it does not involve the semigroup property. It is however equivalent, and the semigroup property comes as a consequence.
**Strategy of the proof:** [1] Look at the behaviour of the measures $\eta_h$ in $B_R \times C([0,T];\mathbb{R}^d)$ defined by

$$\int_{B_R\times C([0,T];\mathbb{R}^d)} \varphi(x,\gamma)\,d\eta_h := \int_{B_R}\varphi(x,\Gamma^h(x,\cdot))\,dx. \tag{1}$$

[2] Show that the family $\eta_h$ is *tight*, i.e. for any $\delta>0$ there is a compact set $K$ such that $\eta_h(K)\ge 1-\delta$ for any $h$. This comes from the apriori bound

$$\int\left(\int_0^T \frac{|\dot\gamma|}{1+|\gamma|}\,dt\right)d\eta_h(x,\gamma) \le C.$$

[3] Show that any weak limit point $\eta$ is concentrated on pairs $(x, \gamma)$ solving the ODE. Then, using the comparison principle, show that $\eta$ has still the same structure in (1) for some limit flow $\Gamma$. This immediately leads to the uniqueness of $\Gamma$ and to the stability property.
This strategy involves the comparison principle *only* for positive and bounded distributional solutions of the PDE and it does not require the concept of renormalized solutions that are not distributional solutions.
Both methods work under the same growth and regularity conditions on $b$, but fail to give a *quantitative* order of convergence (in mean) of the trajectories, e.g. a polynomial order of convergence in $\|b_h - b\|$.

## 3.2. Bressan's conjecture

In connection with the Keyfitz–Kranzer system of PDE, Bressan recently made this conjecture:
*Let $b_h(t,x)$ be smooth, equi-bounded and such that*

$$\int_0^T \int_{\mathbb{R}^d} |\nabla_{t,x} b_h| \, dxdt + \sup J_h + \sup \frac{1}{J_h} \leq C < +\infty,$$

*where $J_h$ are the Jacobians of the classical flows $\Gamma^h$ induced by $b_h$. Then $\Gamma^h$ is strongly relatively compact in $L^1_{\rm loc}\left([0,T] \times \mathbb{R}^d\right)$.*
By applying the theory of renormalized solutions to vector fields $B$ of the form

$$B = (\rho, \rho b), \quad D \cdot B = 0, \quad b \in BV_{\rm loc}, \quad \rho + \frac{1}{\rho} \in L^\infty, \quad D_x \cdot b \ll \mathscr{L}^{d+1},$$

in a joint work [9] with Bouchut and De Lellis we proved the conjecture under the *additional* assumption that some limit point $b$ of $b_h$ satisfies $D_x \cdot b \ll \mathscr{L}^{d+1}$ (in this case the statement is true even with no bounds on $\partial_t b_h$). The general case is still open.

## 3.3. The Keyfitz–Kranzer system of conservation laws

We consider the system of conservation laws

$$(*) \qquad u_t + \sum_{i=1}^d \frac{\partial}{\partial x_i} \left(f_i(|u|)u\right) = 0, \qquad u : \mathbb{R}^d \times (0, +\infty) \to \mathbb{R}^k$$

with $f : \mathbb{R}^+ \to \mathbb{R}^k$ smooth. Bressan has recently shown in [16] that the problem is ill posed (at least in the stability sense) for $L^\infty$ initial data $\bar{u}$. We proved in [9] the following result.

**Theorem 3.2** *If $|\bar{u}| \in L^\infty \cap BV_{\rm loc}$ then (*) has a unique distributional solution in the class of u's whose modulus $\rho$ is an entropy solution of the scalar conservation law*

$$\rho_t + D_x \cdot (\rho f(\rho)) = 0, \qquad \rho(0, \cdot) = |\bar{u}(\cdot)|.$$

**Sketch of proof.** (Existence) Given $\rho$ by Kruzkhov's theory, we know that the vector field $B := (\rho, \rho f(\rho))$ is bounded, divergence-free and $BV_{\mathrm{loc}}$. Therefore the Lagrangian flow associated to $\rho$ is well defined. A reparameterization w.r.t. time then defines a flow $\Gamma$ for the vector field $f(\rho)$, i.e.

$$\dot{\Gamma}(t,x) = f\left(\rho(t,\Gamma(t,x))\right), \qquad \Gamma(0,x) = x.$$

Setting $\bar{u} = \bar{\theta}|\bar{u}|$ and $u = \theta\rho$ the PDE can be (formally) decoupled, writing the system

$$\theta_t + f(\rho)\cdot\nabla_x\theta = 0, \qquad \theta(0,\cdot) = \bar{\theta}(\cdot).$$

A solution of the system above is given by $\theta(t,x) = \bar{\theta}\left(\Gamma^{-1}(t,x)\right)$.
Is this formal solution a distributional one ? Yes, because the whole theory is *stable* w.r.t. smooth approximations of $B$.
(Uniqueness) Uniqueness of $\rho = |u|$ follows by Kruzhkov's theory. Uniqueness of $\theta$ is based on the observation that if a vector valued map $w$ solves

$$\partial_t(\rho w) + D_x\cdot(\rho f(\rho) w) = 0,$$

then $|w|^2$ solves the same PDE.
On the other hand, uniqueness for the scalar problem

$$\partial_t(\rho z) + D_x\cdot(\rho f(\rho) z) = 0, \qquad z(0,\cdot) = 0$$

can be proved regardless of boundary conditions (under the assumptions $z \geq 0$, $z \in L^\infty$) by integration on suitable cones, thanks to the condition $|\rho f(\rho)| \leq C\rho$.
We apply the uniqueness result to $z = |\theta^1 - \theta^2|^2$, with $u^i = \rho\theta^i$ solutions of the system.

## 3.4. Solutions in physical space of the semi-geostrophic system

We consider the *semigeostrophic* system arising in metereology (here $D_t = \partial_t + u\cdot\nabla$ is the Eulerian derivative)

$$\begin{cases} D_t(v_1^g, v_2^g) + (\partial_1 p, \partial_2 p) = (u_2, -u_1), \quad (v_1^g, v_2^g) = (-\partial_2 p, \partial_1 p) \\ D_t\rho = 0, \quad D\cdot u = 0, \quad \partial_3 p + \rho = 0 \end{cases}$$

Recently Cullen and Feldman proved in [25] an existence result for the system in the original physical variables, using previous existence results by Benamou–Brenier [13] and Cullen–Gangbo [24] in the so-called dual coordinates.
Since the passage from physical to dual coordinates requires a non smooth but $BV$ vector field (more precisely, the gradient of a convex function), the stability theorem provides a natural way to justify some formal calculations, going back to a "physical" solution.

## 4. Beyond $BV$ vectorfields ?

Let us discuss the sharpness of the two assumptions in our main result, namely that $B \in BV_{\mathrm{loc}}$ and that $D \cdot B \ll \mathscr{L}^m$.
We know, by the Capuzzo Dolcetta–Perthame result [17], that denoting by $EB$ the symmetric part of the distributional derivative, the renormalization lemma holds provided $EB \in L^1$.
Recall that the space $BD$ of *functions of bounded deformation* (Matthies–Kristiansen–Strang, Suquet, Temam–Strang) consists of all functions $B \in L^1$ such that $EB$ is a symmetric matrix of measures.
Therefore it is natural to guess that the whole theory extends to $BD$. This is still open, but in a recent work [10] with Crippa and Maniglia we prove the result for $SBD$ fields (i.e. such that $|E^s B|$ is concentrated on the union of countably many $C^1$ hypersurfaces). The kernel optimization argument cannot be used and a new one is needed.
Another interesting class of vector fields to be considered is

$$B(x, v) := (b(x), c(x, v)), \quad \text{with } c \text{ measurable w.r.t. } v$$

in connection with the linearized theory of Lagrangian flows, initiated in the Sobolev context by Le Bris–Lions [33] (in that context $c(x, v) = \nabla b(x)v$, see also the recent work of Lerner [34]).
If we wish to linearize even Lagrangian flows of $BV$ vector fields, we should even consider vector fields of the form

$$B(x, v) := (b(x), Db(x)v) \qquad b \in BV_{\mathrm{loc}}, \ \ D \cdot b = 0$$

This is a work in progress with Lecumberry and Maniglia. In connection with conservation laws, see also the recent work by Chen–Frid [20] on divergence measure fields.
In the joint work with Bouchut and De Lellis we considered the following conjecture.
**Conjecture.** *Let $B = (\rho, \rho b)$ be divergence-free, with $\rho > 0$, $\rho + 1/\rho \in L^\infty$ and $b \in BV_{\mathrm{loc}}$. Then any distributional solution $w$ of*

$$\partial_t(\rho w) + D_x \cdot (\rho b w) = 0$$

*is a renormalized solution.*
We proved that this conjecture implies Bressan's one. Notice that the vector field $B$ in the conjecture *is not $BV$*, but divergence-free. On the other hand, the field $b$ *is $BV_{\mathrm{loc}}$*, but we have no control on its divergence.
The study of this conjecture leads in a natural way to the following problem:
*Assume that $B \in BV_{\mathrm{loc}}$ and $w$ is a scalar function such that $D \cdot (wB)$ is a measure. Given $h \in C^1(\mathbb{R})$ we know, by the commutator argument, that $D \cdot (h(w)B)$ is still a measure. Can we compute this measure ?*
If we replace "distributional divergence" by "distributional derivative" this problem is exactly the problem of writing a chain rule in $BV$, solved by Vol'pert in the '60 (still in connection with scalar conservation laws). The problem is non trivial and interesting even if one assumes that $wB$ is divergence-free.

Recall that any measure $\sigma$ can be uniquely written as $\sigma^a + \sigma^j + \sigma^c$, where $\sigma^a \ll \mathscr{L}^d$, $\sigma^j$ is concentrated on a set $\sigma$-finite with respect to $\mathscr{H}^{d-1}$ and $\sigma^c$, the so-called Cantor part, is singular with respect to $\mathscr{L}^d$ and vanishing on any set with finite $\mathscr{H}^{d-1}$-measure.
In a work in progress with De Lellis we proved that:

$$D^a \cdot (h(w)B) = [h(w) - wh'(w)]D^a \cdot B + h'(w)D^a \cdot (wB)$$

and that

$$D^j \cdot (wB) = 0 \quad \Longrightarrow \quad D^j \cdot (h(w)B) = 0.$$

However the rule for the computation of $D^c \cdot (h(w)B)$ is still missing, and therefore a complete solution to the conjectures above.

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